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Control Performance Improvements due to Fluctuations in Dynamics of Stochastic Control Systems

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Abstract— **Human motor control mechanisms are distinguished by two remarkable properties: time delays and random fluctuations. In this work, we focus on these two properties and investigate the effects of fluctuations on the stability and behavior of a stochastic linear time-delay system. The control performance improvements due to fluctuation are presented using numerical simulations as well as its theoretical analyses. In conventional controller design, a random component such as fluctuation is targeted for removal from the system as a factor causing performance reduction. However, the results presented in this paper suggest that an appropriate fluctuation component in control systems can help to achieve better control performances.**

I. Introduction

Human motor control mechanisms are distinguished by two remarkable properties – *time delays* and *random fluctuations* – in closed-loop systems including sensory organs, the musculoskeletal system, and the brain. Delay and fluctuation components appear to impact control performances negatively. Nevertheless, with their help, the quick, smooth and precise voluntary movements can be achieved by human beings.

Recently, the mechanism responsible for the high control performance achieved in organisms including human beings received considerable attention from many researchers. In fact, various vital activities in organisms are supported by extremely elaborate control mechanisms. These activities include not only human motor control but also the control of internal organs to achieve homeostasis and gene control at the cell level. In this context, *fluctuations* are considered to play an important role in such activities in organisms, e.g., an improvement in the equilibrium ability via stochastic resonance[1] and a constancy in the ventricular rate and blood pressure of an able-bodied person[2], [3].

In this work, we focus on the two properties and investigate the effects of fluctuation on the stability and the behavior of a stochastic linear time-delay system. In particular, in conjunction with robot vision systems, we come across the use of an image-based inverted pendulum control system, which corresponds to the stick balancing task for human beings[4], [5], [6]. The control performance improvements due to fluctuation are presented using numerical simulations as well as theoretical analyses (Fig. 1). First, under the assumption that the system has no fluctuation, we derive a stability condition for the time-delayed control system (Result 1). The stability condition for deterministic systems is compared with the stability condition for the stochastic system with fluctuation at a later point in the document. Moreover, by using our numerical simulator, fluctuation that is generated in the proportional controller gain yields an improvement in the system stability and maneuverability, as can be seen from the time responses of the pendulum (Result 2). Finally, theoretical analyses for the stochastic system with no time delay that is based on the famous Ito-type stochastic differential equation reveal the mechanism causing improvements in the stability region with respect to the controller gain and adjustable pendulum length (Result 3). Note that in this analyses, both the definition of the stability of the stochastic system and the definition of the derivative of stochastic state variables differ from those of the deterministic system[7], [8].

In conventional controller design, a random component such as fluctuation is targeted for removal from a system as a factor causing performance reduction. However, the results presented in this paper suggest that an appropriate fluctuation in control systems can help to achieve better control performances.

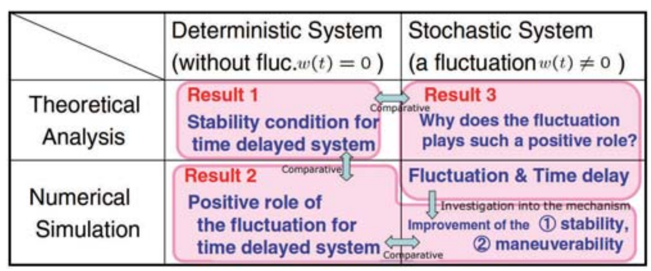

Fig. 1. Approaches to the time-delayed control system with fluctuations.

II. Image-Based Inverted Pendulum/Robot Arm Control Systems and its Stochastic model

In this section, we analyze the pendulum behavior of artificial visual servoing robotic systems.

Recently, in the field of control engineering and control theory, *biological motor control*and *bio-mimetic control*, which elucidate and mimic a control mechanism for various organisms, respectively, are the topics of interest for many researchers. In related investigations, a human-like motion, in particular, a vision-based motor control for stick balancing, has attracted attention owing to not only the similarity of appearances between a camera/robot arm and an eye/arm of a human beings but also its mechanism or intra-dynamics.

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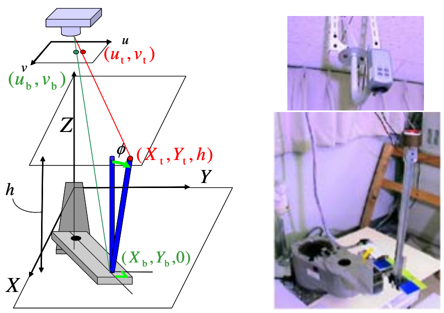
Let us consider a two-link, nonlinear direct-drive arm (SICE DD arm) and a pendulum at the end of the DD arm (Fig. 2). Using a camera fixed to the ceiling of our experimental room, we can calculate the pendulum angle through simple real-time image processing.



(a) World coordinates and camera coordinates

(b) Actual experimental machine

Fig. 2. Image-based inverted pendulum control system

In our experiments, the direction of the pendulum movement was restricted to the tangential plane for the trajectory of the end point of the second link. The first link was fixed to the ground and only the second link could be used for the control of the pendulum. The experimental setup consisted of a camera, DD arm, and pendulum, as shown in Fig. 2. We used two angle sensors for the pendulum, a high-speed, high-precision rotary encoder, and a low-frame-rate camera with a large time delay that could be switched as needed (Fig. 3). In Fig. 4, we compare the performance between the rotary encoder and the camera. The camera frame rate was 30Hz and the time delay was 33ms that was almost always constant.

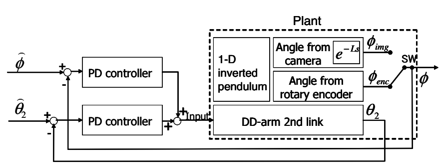


Fig. 3. Control for stabilizing the inverted pendulum

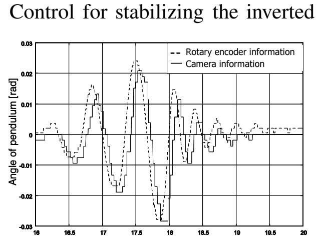


Fig. 4. Performance comparison of rotary encoder and camera

In our experiments, we stabilized this system by adjusting the gain $(P_\mathrm{p}, D_\mathrm{p}, P_2, D_2)$, where P_p and D_p are the proportional and differential gains for the pendulum, respectively; and P_2 and D_2 are the proportional and differential gains of the second of the DD arm, respectively. Moreover, we made the following experimental observations:

(1) We stabilized the system using a rotary encoder and adjusting the PD gains easily.
(2) The rotary encoder frame rate was 30Hz, and we stabilized it using the PD gain in (1).
(3) We artificially delayed the rotary encoder information by 50ms, and we could not asymptotically stabilize the system, whose behavior was embedded in certain regular nondivergent movements.

A. The stochastic robot arm control model

For such an inverted pendulum/robot arm control system, we consider a further simplified inverted pendulum control model as shown in Fig. 5. Here, $\theta(t)$ is pendulum angle in 2D space and its actual observation by using a camera is $\theta(t-\tau)$, where τ denotes a time delay due to capturing image and image processing time to calculate the actual 3D position from the image. The control input $u(t)$ is torque generated by swinging the robot arm. The calculation of $u(t)$ is based on the feedback control signal $\theta(t-\tau)$.

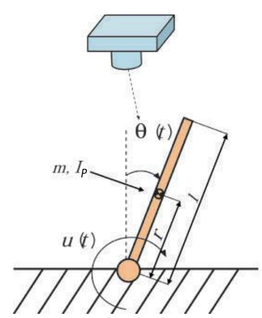


Fig. 5. Camera-robot arm-inverted pendulum control system

We can consider linearized deterministic model of the inverted pendulum. Dynamical equation is given by the following two-order differential equation:

$$\ddot{\theta}(t) + \Gamma\dot{\theta}(t) - q\theta(t) = u(t), \tag{1}$$

where $\theta(t)$ is the pendulum angle which is measured by a vision system and $\Gamma = \frac{3\gamma}{4m}$, $q = \frac{3g}{2l}$. The parameters m, γ, l mean the mass, the viscosity coefficient, the length of the pendulum, respectively. Γ and q are defined by this. g denotes the gravity constant. $u(t)$ is the torque input from the actuator to the pendulum mentioned above.

We introduce the fluctuation in the following simple proportional feedback controller given by

$$u(t) = -\left(P + \xi(t)\right)\theta(t-\tau), \tag{2}$$

where P denotes a constant feedback gain. The feedback gain in the control law in (2) consists of a constant gain P and Gaussian white noise process $\xi(t)$. The structure of the system (1) and (2) is not additive [9] but so-called parametric or multiplicative for the variable θ. The state-depended noise system is also investigated in [10].

The block diagram of the deterministic system is shown in Fig. 6.

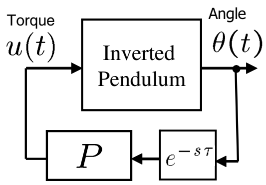


Fig. 6. Block diagram of the vision-based inverted pendulum stabilizing control systems

III. A STABILITY CONDITION FOR THE TIME-DELAYED CONTROL SYSTEMS (RESULT 1)

Under the assumption that the system has no fluctuation ($\xi(t) = 0$), we derive a stability condition for the time-delayed control systems. The condition for the deterministic systems is compared with the stability condition for the stochastic systems with a fluctuation.

Due to $\xi(t) = 0$, we have the following time-delayed deterministic control system:

$$\ddot{\theta}(t) + \Gamma\dot{\theta}(t) - q\theta(t) = -P\theta(t-\tau). \tag{3}$$

The corresponding characteristic equation $f(\lambda)$ is given by

$$f(\lambda) = \lambda^2 + \Gamma\lambda - q + Pe^{-\lambda\tau}, \tag{4}$$

which is an infinite dimensional system including a delay term $e^{-\lambda\tau}$. Here, we derive a stability condition for the system in (3) based on so-called direct method.

First, we introduce a notion of stability for infinite dimensional systems. An infinite dimensional system with a characteristic equation $f(\lambda) := \sum_{k=0}^{n} a_k(\lambda)\lambda^k$ is said to be *stable* if

$$\left\{\lambda \in \mathbb{C} : \mathrm{Re}[\lambda] \geq 0, \quad \sum_{k=0}^{n} a_k(\lambda)\lambda^k = 0\right\} = \emptyset. \tag{5}$$

This implies that a stable time delayed system has all infinite roots of the characteristic equation in the open left half plane. In this case, a stability condition for the system in (3) is as follows[11]:

Theorem 1: Consider a time-delayed inverted pendulum stabilizing control system with no fluctuation given in (3). In this case, the system is stable if and only if

$$P_{\min} := q < P < \frac{\Gamma\omega_0}{\tau\sin\omega_0} =: P_{\max}, \tag{6}$$

where $\omega_0 > 0$ is minimal root of $\omega_0 + q = P\cos\tau\omega_0$. △
The proof is found in [11]. The key idea of the proof is a proposition by Stépán et al. [12][13] and the condition in (6) is equivalent to non-existence of unstable zeros of the characteristic equation in (4).

In the next section, we verify a validity of the theoretical stability condition by comparing with numerical simulation results.

IV. IMPROVEMENT OF CONTROL PERFORMANCES VIA DYNAMICS FLUCTUATION (RESULT 2)

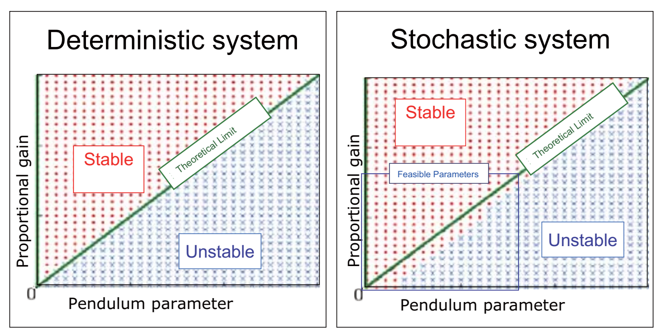
A. Simulation settings and results

The model parameters were determined as shown in Table I which is based on our actual robot arm systems mentioned in previous section. For various lengths of the pendulum and feedback gains (q, P), we simulated the time response of the of pendulum behavior. Changing $q = \frac{3g}{2l}$ and P corresponds to changing the pendulum length and the feedback gains, respectively. Simulation interval is $300\mathrm{ms}$ and its sampling time is $0.01\mathrm{ms}$.

TABLE I
LIST OF THE MODEL PARAMETERS.

$m = 0.2\mathrm{kg}$	mass of pendulum
$\tau = 0.033\mathrm{s}$	time delay of vision system
$\gamma = 0.5, \frac{\mathrm{N \cdot s}^2}{\mathrm{m}}$	viscosity constant of pendulum
$\mathcal{E}\left\{\xi^2(t)\right\} = 100$	power of fluctuation

The simulation results are shown in Fig. 7 which is just a comparison of the stability charts between our numerical simulation and theoretical bound of Result 1 for both deterministic and stochastic model. A determination of stable/unstable for the time response of the pendulum behavior is based on $|\theta(t)| < \epsilon$ and $|\dot{\theta}(t)| < \epsilon$ for a small threshold parameter $\epsilon > 0$ as sufficiently large simulation time. In



(a)Deterministic case (b)Stochastic case

Fig. 7. Simulation results for the stability chart

Fig. 7, we plot red or blue mark for each parameter setting P and q. Red mark means stable system for sufficiently large time interval and blue means unstability. For the deterministic case, the simulation results are quite well reproduced by the theories. It is extremely important to call your attention to the fact that the stable region of stochastic case spreads beyond the theoretical boundary.

B. Improvement of stability region

Comparing the stable region in Fig. 7(b) with theoretical boundary, we have the following results:

1) The stable region for time delay system robustly maintains against dynamics fluctuation with appropriate magnitude.

2) For a practical feasible gain P and length of pendulum q, the fluctuation yields spread of the stable region for stochastic case beyond the theoretical boundary.
3) For large q, i.e., short pendulum, the fluctuation yields negative effects for the stable region. Even if (q, P) is set in theoretical stable region, the stability is not necessarily guaranteed.
4) In the case with the dynamics fluctuation, the stability near the boundary $P_{\max}$ is lost.

We can intuitively interpret the above result 3) in which the increase of parameter q $(= \frac{3g}{2l})$ shows a difficulty of control of a short pendulum. Due to the result 4), setting the feedback gain P at neighborhood of the boundary occurs large oscillation of pendulum behavior. Hence, we investigate in detail for relatively small (q, P). For a feasible parameter region which corresponds to long pendulum and small gain, we repeated 8 trials at each parameter setting P and q as shown in Fig. 8. The color gradation in Fig. 8 denotes

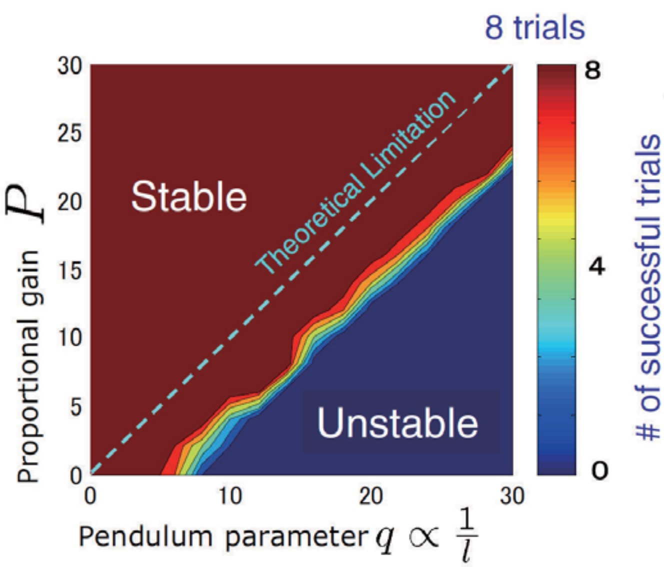


Fig. 8. Simulation results for the stability chart with fluctuation (8 trials for each point)

the number of stable trials. Obviously, the stable region spreads beyond the boundary. Actually, a time response at a theoretically unstable parameter shows a convergence due to the fluctuation. This results implies the positive effects of the fluctuation for the improvement of the stability.

C. Improvement of the maneuverability

Furthermore, near the boundary in Fig. 8, we can also found the rapidly convergence of the pendulum behaviour as shown in Fig. 9. Therefore, the improvement of the maneuverability of the pendulum behaviour can be achieved due to the dynamics fluctuation.

D. Relationship between the fluctuation and time delay

In the previous subsection, a power of fluctuation $\xi(t)$ and the time delay added to the stochastic system in (1) and (2) was always fixed 100 and $\tau = 0.033\mathrm{sec}$, respectively. In this

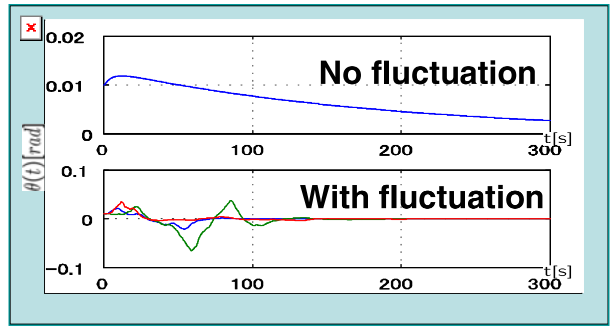


Fig. 9. Effect of fluctuation for the system maneuvarability ($q = 80,\ P = 82$)

subsection, we show the simulation results in Fig. 10 for varying both the fluctuation power and time delay.

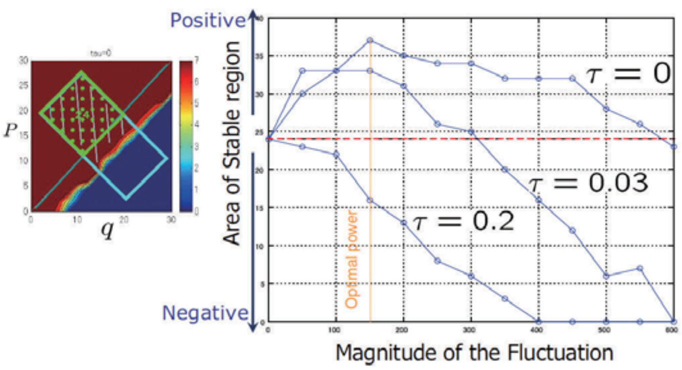


Fig. 10. The fluctuation power vs the time delay

1) The appropriate power of fluctuation can take part in improvement in a sense of spread of stability region.
2) Greater fluctuation power than some critical value destabilises the pendulum behavior. In particular, the critical value close to 0 as increasing time delay.
3) For large time delay system, positive effects for control performance due to fluctuation does not happen.

In the case of a small time delay, the structure of power v.s. delay seems to be similar to no time delay case. Therefore, we will concentrate on a stochastic system with no time delay and give a stability condition based on stochasitic control theory.

V. Theoretical Stability Analyses for Stochastic System (Result 3)

A. Representation via Ito-type Stochastic Differential Equation

In this section, to ascertain a validity of our numerical simulation results above we discuss theoretical stability condition for Ito-type stochastic differential equation. Our inverted pendulum control system given in (1) and (2) can be rewritten as

$$dx = \begin{bmatrix} 0 & 1 \\ q-P & -\Gamma \end{bmatrix} x\ dt + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x\ dw, \tag{7}$$

where, $x = \begin{bmatrix} \theta(t) & \dot{\theta}(t) \end{bmatrix}^T$ and $w(t)$ is Wiener process which is given by integrating $\xi(t)$.

First, we give conditions ensuring the stability of a particular solution $x(t)$ of (7). Even in the deterministic case the concept of stability can be given various meanings, e. g., one distinguishes between local and global stability, also between asymptotic and nonasymptotic stability. The diversity of stochastic equation is even greater in the presence of randomness[7]. However, in this section, we give only a concept of stability with probability 1, which guarantees a convergence for all sample processes.

Definition

Consider a system in (1) and (2). Assume that $\theta(t_0) = \theta_0$. A solution $\theta(t)$ is said to be *stable with probability 1* if

$$\Pr\left\{\lim_{\|\theta_0\|\to 0} \sup_{t\geq t_0} \|\theta(t;t_0,\theta_0)\| = 0\right\} = 1. \tag{8}$$

Furthermore, a solution $\theta(t)$ is said to be *asymptotically stable with probability 1* if the condition (8) holds, and for all $\epsilon > 0$, if$\|\theta_0\| < r$ then there exists a number $r > 0$ such that

$$\lim_{t\to\infty} \Pr\left\{\sup_{t\geq T} \|\theta(t;t_0,\theta_0)\| > \epsilon\right\} = 0. \tag{9}$$

$\triangle$

For a linear stochastic system in (7), we can also give a equivalent condition for the stability [8] as follows:

Theorem 2: A solution $x(t)$ of a linear stochastic system

$$dx(t,w) = A(t)x(t)dt + G(t)x(t)dw(t)$$
$$x(t_0) = x_0$$

is asymptotically stable with probability 1, if and only if

$$\lim_{t\to\infty} \frac{\ln|x(t)| - \ln|x_0|}{t} < 0 \tag{10}$$

holds. $\triangle$

B. Derivation of stability condition

We derive a necessary and sufficient condition for a stability w. p. 1 in (7). Applying a similarity transformation $T = [v_1 \quad v_2]$ to (7) we have

$$dz = \begin{bmatrix} \frac{-\Gamma+\sqrt{D}}{2} & 0 \\ 0 & \frac{-\Gamma-\sqrt{D}}{2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} dt - \frac{1}{\sqrt{D}} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} dw, \tag{11}$$

where $z := T^{-1}x$, $D := \Gamma^2 + 4(q - P)$,

$$\lambda_1 = \frac{-\Gamma+\sqrt{D}}{2}, \qquad \lambda_2 = \frac{-\Gamma-\sqrt{D}}{2}$$
$$v_1 = \begin{bmatrix} 1 \\ \frac{-\Gamma+\sqrt{D}}{2} \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ \frac{-\Gamma-\sqrt{D}}{2} \end{bmatrix}.$$

Note that the condition $D > 0$ covers a first quadrant of (q, P)-space in a natural way. The derivation of stability condition for (11) is going the following order: **Step 1:** For the system in (11), calculate a Ito's differential operator $\mathcal{L}$. **Step 2:** Applying a polar coordinate transformation $z_1 = r\cos\phi, z_2 = r\sin\phi$ and using the calculated $\mathcal{L}$, derive differential equations for $\ln|x(t)| = \ln r(t)$ and $\phi(t)$. **Step 3:** Based on a probability density function p_ϕ of ϕ, calculate a stability condition in (10) in details.

Step 1 The Ito's differential operator $\mathcal{L}$ in (11) is given as

$$\mathcal{L}(\cdot) = \frac{-\Gamma+\sqrt{D}}{2} z_1 \frac{\partial(\cdot)}{\partial z_2} + \frac{-\Gamma-\sqrt{D}}{2} z_2 \frac{\partial(\cdot)}{\partial z_2} + \frac{\sigma^2}{2}\left\{\frac{1}{D}(z_1+z_2)^2\left(\frac{\partial^2(\cdot)}{\partial z_1^2} + \frac{\partial^2(\cdot)}{\partial z_2^2} - 2\frac{\partial^2(\cdot)}{\partial z_1 \partial z_2}\right)\right\}. \tag{12}$$

Step 2 Wd apply a polar coordinate transformation $z_1 = r\cos\phi, z_2 = r\sin\phi$. From $r = \sqrt{z_1^2+z_2^2}$, $\phi = \tan^{-1}(z_2/z_1)$ we have

$$\frac{\partial \ln r}{\partial z_1} = \frac{\cos\phi}{r}, \quad \frac{\partial \ln r}{\partial z_2} = \frac{\sin\phi}{r} \tag{13}$$

$$\frac{\partial^2 \ln r}{\partial {z_1}^2} = -\frac{\cos 2\phi r^2}{r^2}, \quad \frac{\partial^2 \ln r}{\partial {z_2}^2} = \frac{\cos 2\phi}{r^2},$$
$$\frac{\partial^2 \ln r}{\partial z_1 \partial z_2} = -\frac{\sin 2\phi}{r^2} \tag{14}$$

$$\frac{\partial \phi}{\partial z_1} = -\frac{\sin\phi}{r}, \quad \frac{\partial \phi}{\partial z_2} = \frac{\cos\phi}{r} \tag{15}$$

$$\frac{\partial^2 \phi}{\partial {z_1}^2} = \frac{\sin 2\phi}{r^2}, \frac{\partial^2 \phi}{\partial {z_2}^2} = -\frac{\sin 2\phi}{r^2}, \frac{\partial^2 \phi}{\partial z_1 \partial z_2} = -\frac{\cos 2\phi}{r^2}. \tag{16}$$

For $V = \ln|z(t)| = \ln r$ and ϕ we apply a famous Ito's lemma. Using $\mathcal{L}$ in (12) and substituting (13)-(16) to Ito's lemma, it follows that

$$\begin{aligned} d\ln|z(t)| &= \mathcal{L}\ln|z(t)|\ dt + \left(\frac{\partial \ln|z(t)|}{\partial z}\right)^T \left(-\frac{1}{\sqrt{D}}\right) \begin{bmatrix} z_1+z_2 \\ -(z_1+z_2) \end{bmatrix} dw \\ &= \left[-\frac{\Gamma}{2} + \frac{\sqrt{D}}{2}\cos 2\phi + \frac{\sigma^2}{D}(1+\sin 2\phi)\sin 2\phi\right] dt \\ &\quad - \frac{1}{\sqrt{D}}\cos 2\phi\ dw =: f_{\ln|z|}\ dt + g_{\ln|z|}\ dw \end{aligned} \tag{17}$$

$$\begin{aligned} d\phi &= \mathcal{L}\phi\ dt + \left(\frac{\partial\phi}{\partial z}\right)^T \left(-\frac{1}{\sqrt{D}}\right) \begin{bmatrix} z_1+z_2 \\ -(z_1+z_2) \end{bmatrix} dw \\ &= \left[-\frac{\sqrt{D}}{2}\sin 2\phi + \frac{\sigma^2}{D}(1+\sin 2\phi)\cos 2\phi\right] dt \\ &\quad + \frac{1}{\sqrt{D}}(1+\sin 2\phi)\ dw =: f_\phi\ dt + g_\phi\ dw. \end{aligned} \tag{18}$$

We calculate the integral of both side of (17) and moreover multiplying $1/t$ and $t \to \infty$ we finally obtain

$$\lim_{t\to\infty} \frac{\ln|z(t)| - \ln|z_0|}{t} = \lim_{t\to\infty}\left(\int_0^t f_{\ln|z|}\ dt + \int_{w(0)}^{w(t)} g_{\ln|z|}\ dw\right). \tag{19}$$

From theorem 2, a negativeness of (19) gives the asymptotical stability w. p. 1 of the system in (11). In the first term of a right hand side in (19) ensemble mean equals to time one based on ergodicity condition. Furthermore, the second term reduce to zero as $t \to \infty$ from a characteristics of Wiener process. Hence, from (19) it follows that

$$\lim_{t\to\infty} \frac{\ln|z(t)| - \ln|z_0|}{t} = \lim_{t\to\infty} \int_0^t f_{\ln|z|}\, dt$$
$$= \mathcal{E}\{f_{\ln|z|}(\phi)\} = \int_{-\infty}^{\infty} f_{\ln|z|}(\phi) p_\phi \, d\phi$$
$$= \int_0^{2\pi} \left[-\frac{\Gamma}{2} + \frac{\sqrt{D}}{2}\cos 2\phi + \frac{\sigma^2}{D}(1+\sin 2\phi)\sin 2\phi \right] p_\phi \, d\phi < 0. \quad (20)$$

Step 3 Since p_ϕ in (20) is a probability density function, it is from its nonstationary property that

$$0 = -\frac{\partial(p_\phi f_\phi)}{\partial\phi} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial\phi^2}(p_\phi g_\phi {g_\phi}^T)$$
$$= -\left[-\frac{\sqrt{D}}{2}\sin 2\phi + \frac{\sigma}{D}(1+\sin 2\phi)\cos 2\phi \right] p_\phi + \frac{\partial}{\partial\phi}\left\{ \frac{\sigma^2}{2D}(1+\sin 2\phi)^2 p_\phi \right\}. \quad (21)$$

Integrating this we calculate p_ϕ and we have

$$p_\phi = \bar{c}\exp\left\{ \frac{-D\sqrt{D}}{\sigma^2}\int \frac{\sin 2\phi}{(1+\sin 2\phi)^2} d\phi - 2\int \frac{\cos 2\phi}{1+\sin 2\phi} d\phi \right\}$$
$$= \bar{c}(1+\sin 2\phi)^{-1} \exp\left[-\frac{D\sqrt{D}}{4\sigma^2}\left\{ \tan\left(\phi - \frac{\pi}{4}\right) - \frac{1}{3}\tan^3\left(\phi - \frac{\pi}{x}\right) \right\} \right], \quad (22)$$

where $\bar{c}$ is a constant for a normalization. Substituting it to (20), we have

$$\int_0^{2\pi} \left[-\frac{\Gamma}{2}\frac{1}{1+\sin 2\pi} + \frac{\sqrt{D}}{2}\frac{\cos 2\phi}{1+\sin 2\pi} + \frac{\sigma^2}{D}\sin 2\pi \right] \cdot \exp\left\{ -\frac{D\sqrt{D}}{4\sigma^2}\left\{ \tan\left(\phi - \frac{\pi}{4}\right) - \frac{1}{3}\tan^3\left(\phi - \frac{\pi}{4}\right) \right\} \right\} d\phi =: I < 0. \quad (23)$$

We are ready to give our main theorem:

Theorem 3: For a second order stochastic system with an appropriate magnitude σ^2 of Gaussian fluctuation $\xi(t)$ given as

$$\ddot{\theta}(t) + \Gamma\dot{\theta}(t) - q\theta(t) = -\{P + \xi(t)\}\theta(t),$$

the stable region with respect to a feedback gain and adjustable design parameter (P, q) spreads beyond a boundary $P > q$ which is for the corresponding deterministic system with no fluctuation($\xi(t) = 0$).

$$\Longleftrightarrow \quad {}^{\exists}P_{\text{fl}} := q - P > 0 \quad \text{s.t.}$$
$$(\Gamma^2 + 4P_{\text{fl}})I := I_1(P_{\text{fl}}) + I_2(P_{\text{fl}}) < 0, \quad (24)$$

where, $I_1(P_{\text{fl}})$ and $I_2(P_{\text{fl}})$ are given by the following forms:

$$I_1(P_{\text{fl}}) := -\frac{\Gamma}{2}(\Gamma^2 + 4P_{\text{fl}})\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin 2\phi} \cdot \exp\left\{ \frac{-(\Gamma^2+4P_{\text{fl}})^{\frac{3}{2}}}{4\sigma^2}\left\{ \tan\left(\phi - \frac{\pi}{4}\right) - \frac{1}{3}\tan^3\left(\phi - \frac{\pi}{4}\right) \right\} \right\} d\phi,$$

$$I_2(P_{\text{fl}}) = \int_0^{\frac{\pi}{2}} \left[\frac{(\Gamma^2+4P_{\text{fl}})^{\frac{3}{2}}}{2}\frac{\cos 2\phi}{1+\sin 2\phi} + \sigma\sin 2\phi \right] \cdot \exp\left\{ -\frac{(\Gamma^2+4P_{\text{fl}})^{\frac{3}{2}}}{4\sigma^2}\left\{ \tan\left(\phi - \frac{\pi}{4}\right) - \frac{1}{3}\tan^3\left(\phi - \frac{\pi}{4}\right) \right\} \right\} d\phi.$$

△

The function I_1 and I_2 consist of definite integral. This integral is NOT complicated although the expression is long somewhat. Because a sign, in particular, negativity, depends only on that of the function $\frac{1}{1+\sin 2\phi}$ in I_1.

Our theorem allows us to depict this stability chart. The simulation results for $0 < q < 30$ is shown as Fig. 11. Our theoretical boundary $I = 0$ in $q - P$ stability chart was quite well reproduced by our simulation results in previous section, e.g., see Fig. 8.

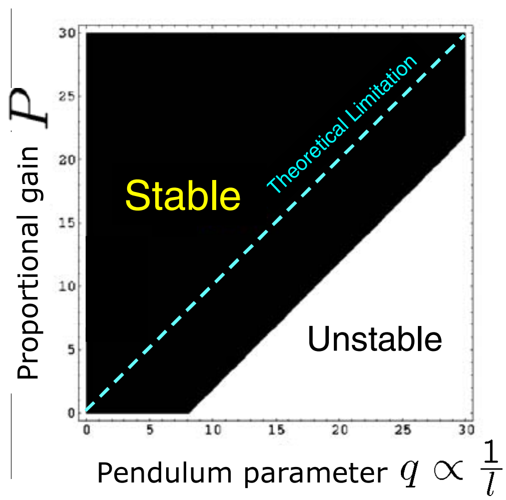
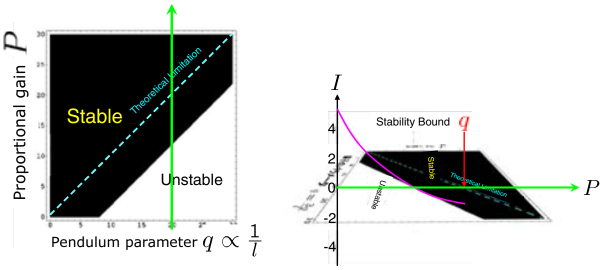


Fig. 11. The theoretical boundary for stochastic control system

In Fig. 11, the black-painted region means stable w. p. 1 which holds the inequality in (23) and the white region means unstability. We summarize how to depict the boundary from our theorem. For example in Fig. 12, when q is fixed to 20, I can calculate the value of the function I increasing P. The negativity of the function I means stable. The boundary is that point, approximately $P \sim 12$ in this case. This procedure is repeated for all q. This figure shows our theorem is useful for investigating the mechanism of improvement of control performances due to the dynamics fluctuation in Section IV. From our theorem it is theoretically proved for the system in (7) that the dynamics fluctuation can improve the control performances.

Furthermore, our theorem yields the following corollary.



(a) For example, first fix $q = 20$ and increase P (green line).

(b) for each P we calculate the definite integral I (pink line). $I = 0$ means the stability limit which is the theoretical boundary. Repeat this for all q.

Fig. 12. How to depict the theoretical boundary for a case of $q = 20$

Corollary 1: The inequalities

$$
\begin{aligned}
I_1(P_{\text{fl}}) &< 0 && \text{if} \quad \gamma \neq 0 \\
I_2(P_{\text{fl}}) &> 0 && \text{for a large } \sigma^2
\end{aligned}
$$

always holds. Since $I_1(P_{\text{fl}})$ vanishes in a case of no viscosity constant of pendulum $\gamma = 0$, the performance improvement due to the fluctuation, in other word, the negativeness of I in (23) essentially caused viscosity constant of pendulum $\triangle$

VI. Conclusion

In this work, we focused on the two properties of time delay and random fluctuations and investigated the effects of fluctuation on the stability and behavior of a stochastic linear time-delay system. The control performance improvements due to fluctuation were presented using numerical simulations, its theoretical analyses as well as experimental results. In conventional controller design, a random component such as fluctuation is targeted for removal from a system as a factor causing performance reduction. However, the results presented in this paper suggest that an appropriate fluctuation in control systems can help to achieve better control performances.

In the future, we plan to resolve the problem faced during the implementation of the pendulum control system with a random fluctuation component. Furthermore, finding other stochastic applications is very interesting because it gives us a deep insight into implications pertaining to fluctuations observed in human motor control activities.

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