2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Stabilization of Multirate Networked Control Systems

Wei Chen and Li Qiu

Abstract— **In this paper, we study the stabilization of multirate networked control systems with norm bounded uncertainties in the input channels. The key idea is to use the channel resource allocation, i.e., given the overall capacity of the transmission network, we do have the freedom to allocate the capacities among different input channels. With this idea, we successfully show that a multirate networked control system could be stabilized by state feedback under an appropriate resource allocation if and only if the overall network capacity is larger than the topological entropy of the plant. We also apply the result to multirate quantized networked control systems. A sufficient condition for stabilization is obtained which involves a trade-off between the densities of time quantization and spatial quantization.**

I. Introduction

Arising from the cross-pollination of control, network and information theories, the networked control systems (NCSs) have attracted great attention nowadays. They are feedback systems where communications between plants and controllers occur through shared communication networks. Applications of NCSs have been found in more and more areas. Examples include mobile sensor networks [17], multi-agent systems [16] and ariel space technologies [19], *etc.*. In special issues [1], [2], much information of the current status of NCSs research has been presented.

A lot of work has been done on the networked control stabilization problem. In the NCSs, due to the unperfect communication networks, different kinds of information constraints and uncertainties appear frequently, such as quantization [7], [8], packet drop [6] and data rate [14], *etc.*. There are numerous results reported in the literature studying the stabilization of NCSs under these uncertainties. For discrete-time single-input NCSs, in [8], logarithmic quantization of the control inputs was considered as a sector uncertainty. It was shown that the largest uncertainty bound which renders stabilization possible is given in terms of the Mahler measure of the system, i.e., the absolute product of the unstable poles. In [6], multiplicative stochastic input channel has been taken into consideration. There it stated that the networked feedback system could be mean-square stabilized by state feedback if and only if the mean-square capacity of the multiplicative channel exceeds the topological entropy of the plant which is the logarithm of the Mahler measure.

This work is supported by Hong Kong PhD Fellowship, Hong Kong Research Grants Council under project GRF 618608 and the National Science Foundation of China under grant 60834003.

W. Chen is with the Department of Electronic and Computer Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China wchenust@ust.hk

L. Qiu is with the Department of Electronic and Computer Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China eeqiu@ust.hk

For discrete-time multi-input NCSs, [9] has assumed that the information constraint in the input channels is determined by the total network recourse available to the channels that can be allocated by the controller designer. Thanks to the additional design freedom gained by the resource allocation, an analytical solution has been obtained which states that the largest overall uncertainty bound ensuring closed-loop stabilizability is given in terms of the Mahler measure. Different from the setting of multiple input channels in [9], [14] studied NCSs with multiple sensor channels each of which may have a different average data rate. By using a sequential design, they showed that the NCS is exponentially stabilizable if and only if the sum of the average data rates is larger than the topological entropy of the plant. Although not stated explicitly by the authors, the resource allocation was in fact embodied in their work. In light of these results, we see the significance and role of the channel resource allocation which entails the idea of channel-controller co-design, i.e., the control designer should also participate in the channel design rather than passively take the given channels. This idea would bring us much more convenience and flexibility in designing and is envisioned to be commonly used in real applications. Another example demonstrating the advantage of channel-controller co-design can be found in [22], where the stabilizing condition in [6] for NCSs with multiplicative stochastic channels was generalized to the multi-input case. Later one can see, in this paper, our main result can be obtained by allocating the channel resource judiciously.

Researchers have also devoted much effort to the stabilization of continuous-time NCSs. [4] studies stabilization of a distributed control system where a central controller communicates sequentially with the subsystems through one shared communication network under some periodic communication pattern. Both the communication pattern and the control law are to be designed, leading to a channel-controller co-design for multiple periodic linear systems. With the lifting technique [5], the author showed that the stabilization problem is equivalent to finding stable elements in a certain subspace spanned by a set of matrices derived from the system to be controlled. This problem is further studied in [10] which develops a simulated annealing algorithm to find stable elements in the aforementioned subspace.

Another line of work which is pertinent to our work in this paper studies the trade-off between the required densities of time quantization and spatial quantization for stabilization of NCSs. For single-input case, [7] considered the situation of uniform sampling and infinite-level logarithmic spatial quantization leading to a trade-off between the densities in terms of the Mahler measure. In the case when a finite-level

978-1-61284-799-3/11/$26.00 ©2011 IEEE

spatial quantizer was used, the trade-off was studied in [11], [12]. There it was concluded that the minimum data rate for stabilization could only be achieved by binary control. Unfortunately, so far, no efficient result has been reported on the trade-off for the multi-input case. Motivated from this, we in this paper consider the trade-off under a more general setup. We adopt a network model which not only can characterize quantization but also has the capability to address other network features. Moreover, in our model, different sampling rates are allowed for different input channels leading to a multirate NCS. Both our work and [4], [10] involve periodic multirate sampled-data systems. However, it is worth noting that we consider multiple parallel input channels, which is different from the framework in [4], [10] that has only one communication channel. This point would become clearer in the development. We investigate the stabilizing condition for the multirate NCS. By using the lifting technique and channel resource allocation, we show that a multirate NCS could be stabilized by state feedback under an appropriate resource allocation if and only if the overall network capacity is larger than the topological entropy of the plant. We further apply this result to multirate quantized control systems and obtain a sufficient condition for stabilization which shows a trade-off between the densities of time quantization and spatial quantization.

The remainder of this paper is organized as follows. The problem is formulated in section II. The main result is stated and proved in section III. Section IV applies the result to the trade-off between the densities of time quantization and spatial quantization. Section V gives an illustrative example. Finally, some conclusion remarks follow in section VI.

II. Problem Formulation

The setup of a multirate NCS studied in this paper is shown in Fig. 1. We use solid lines for continuous-time signals and dotted lines for discrete-time signals. The plant $\boldsymbol{G}$ is a continuous-time LTI system. The states are available for feedback with sampling interval T. Different hold intervals $K_1T, K_2T, \ldots, K_mT$ are allowed for different input channels, where $K_1, K_2, \ldots, K_m$ are relative prime integers. Assume all the hold and sampling circuits are synchronized at time 0. F is a static state feedback gain. The control signals generated by F would be transmitted through a multirate network before reaching the plant. In many practical applications, the actuators are located separately from each other and from the controller. Hence, we adopt a parallel transmission strategy, i.e., each element $v_i(k)$ of the control signal is separately sent through an independent channel of the network to the actuator.

Fig. 2 shows one of the channels of the network, which can be considered as the cascade of a downsampling system and an ideal transmission system with a unity transfer function together with an additive norm bounded uncertainty. The uncertainty $\boldsymbol{\Delta}_i$ can be nonlinear, time-varying, or a dynamic system. The only requirement is that its $\mathcal{H}_\infty$ norm is bounded

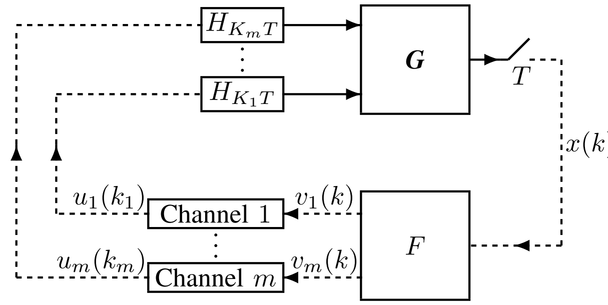

Fig. 1. Multirate networked control system

by δ_i, i.e.,

$$\|\Delta_i\|_\infty = \sup_{\tilde{v}_i(k_i)\in\ell_2} \frac{\|e_i(k_i)\|_2}{\|\tilde{v}_i(k_i)\|_2} \leq \delta_i.$$

By modeling the channels this way, we can address many different kinds of uncertainties in the network. Moreover, different sampling rates could be adopted in different channels. The advantage of multirate sampling stands out not only in theoretical studies but also in practical applications. For example, in complex, multivariable control systems, sampling all physical signals uniformly at one single rate is often unrealistic, then one is forced to use multirate sampling. Also, multirate sampling can often reduce the required storage space or computational complexity for signal processing.

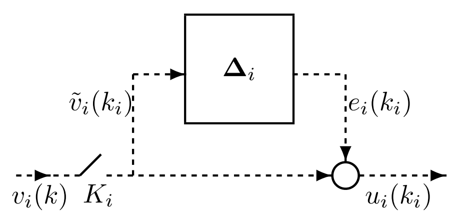

Fig. 2. A network channel

Intuitively, if the sampling rates of the network are too slow or there exists too much uncertainty such that little information of the control signals could be transmitted, then the multirate NCS can hardly be stabilized. Only when enough information is transmitted per time unit can stabilization become possible. Our objective is to find the minimum amount of information transmitted through the network per time unit so as to make stabilization of the multirate NCS possible.

Let $\boldsymbol{G}$ have a state space realization:

$$\dot{x}(t) = A_c x(t) + B_c u(t), x(0) = x_0,$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$. Define $\mathscr{H}$ and $\mathscr{S}$ as the hold and sampling operators:

$$\mathscr{H} = \begin{bmatrix} H_{K_1T} & & \\ & \ddots & \\ & & H_{K_mT} \end{bmatrix}, \quad \mathscr{S} = \begin{bmatrix} S_T & & \\ & \ddots & \\ & & S_T \end{bmatrix}.$$

Then $\mathscr{S}\boldsymbol{G}\mathscr{H}$ is a multirate sampled-data system. For simplicity, we denote it by $\boldsymbol{G}_d$. Now let

$$N = \text{LCM}\{K_1, K_2, \ldots, K_m\},$$

where LCM means the least common multiple. It is easy to see that NT is the least common period for all input channels, in other words, it is the shortest time interval for which the hold schedule repeats itself.

Lifting [5] is a common and efficient method to deal with multirate sampled-data systems. Let ℓ be the space of sequences, perhaps vector valued, defined on the time set $\{0, 1, 2, \ldots\}$. The lifting operator over ℓ is given by

$$\begin{aligned} L_p : &\{u(0), u(1), u(2), \ldots\} \mapsto \\ &\left\{ \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(p-1) \end{bmatrix}, \begin{bmatrix} u(p) \\ u(p+1) \\ \vdots \\ u(2p-1) \end{bmatrix}, \ldots \right\}. \end{aligned}$$

It is well known that the lifting operator is invertible and norm preserving [5].

Let $N_i = \frac{N}{K_i}, i = 1, 2, \ldots, m$. We lift the multirate system $\boldsymbol{G}_d$ to an equivalent LTI system

$$\underline{\boldsymbol{G}_d} = \begin{bmatrix} L_N & & \\ & \ddots & \\ & & L_N \end{bmatrix} \boldsymbol{G}_d \begin{bmatrix} L_{N_1}^{-1} & & \\ & \ddots & \\ & & L_{N_m}^{-1} \end{bmatrix}.$$

Let $A = e^{A_c T}, B = \int_0^T e^{A_c(T-\tau)} B_c \mathrm{d}\tau$, and denote B_j as the jth column of B, then a state space realization of $\underline{\boldsymbol{G}_d}$ with state $\xi(k) = x(kNT)$ is given by [15]

$$\xi(k+1) = A_e \xi(k) + B_e u_e(k), \xi(0) = x_0,$$

where

$$\begin{aligned} A_e &= A^N, \quad B_e = [B_{e1} \quad B_{e2} \quad \ldots \quad B_{em}], \\ B_{ej} &= \Big[\textstyle\sum_{q=1}^{K_j} A^{(N-q)} B_j \quad \sum_{q=1}^{K_j} A^{(N-K_j-q)} B_j \\ &\quad \ldots \quad \textstyle\sum_{q=1}^{K_j} A^{(K_j-q)} B_j \Big], \end{aligned}$$

$$u_e(k) = \begin{bmatrix} u_1(kNT) \\ u_1(kNT + K_1 T) \\ \vdots \\ u_1(kNT + (N - K_1)T) \\ u_2(kNT) \\ u_2(kNT + K_2 T) \\ \vdots \\ u_2(kNT + (N - K_2)T) \\ \vdots \\ u_m(kNT + (N - K_m)T) \end{bmatrix}.$$

Examining the detailed structure of $u_e(k)$ implies that it is obtained by lifting the inputs of each channel first and then grouping them all together. Clarifying this would make it easier to understand the later design of the state feedback gain F.

The control signal is generated with time period T by the feedback law $v(k) = Fx(kT)$. Let F_i be the ith row of F. To see the behavior of the controller in the time period NT, we apply the lifting technique to get

$$v_e(k) = F_e \xi(k) = \begin{bmatrix} F_1 \\ F_1 A \\ \vdots \\ F_1 A^{N-1} \\ F_2 \\ F_2 A \\ \vdots \\ F_2 A^{N-1} \\ \vdots \\ F_m A^{N-1} \end{bmatrix} \xi(k). \tag{1}$$

Clearly, $v_e(k)$ is the lifted controller output. We will come back to the structure of F_e as shown in (1) when we design the controller in section III.

As introduced before, different components of the control signal are transmitted through independent communication channels each of which may have a different sampling rate and uncertainty bound. Applying the lifting technique to the transmission process yields

$$u_e = (\boldsymbol{I} + \underline{\boldsymbol{\Delta}}) Z v_e,$$

where $\boldsymbol{I}$ is the identity system, $\underline{\boldsymbol{\Delta}} = \text{diag}\{\underline{\boldsymbol{\Delta}_1}, \underline{\boldsymbol{\Delta}_2}, \ldots, \underline{\boldsymbol{\Delta}_m}\}$ denotes the lifted uncertainty, $Z = \text{diag}\{Z_1, Z_2, \ldots, Z_m\}$ describes the downsampling scheme of the ith channel with Z_i having dimension $N_i \times N$. Let Z_i^{jk} be the (j, k)th element of Z_i, then

$$Z_i^{jk} = \begin{cases} 1 & \text{when } k = (j-1)K_i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since lifting is norm preserving, we have $\|\underline{\boldsymbol{\Delta}_i}\|_\infty \leq \delta_i$.

Now we define the concept of channel capacity to measure the amount of information transmitted through the channels per time unit. The individual channel capacity is given by

$$\mathfrak{C}_i = \frac{1}{T_i} \ln \delta_i^{-1}, i = 1, 2, \ldots, m,$$

where $T_i = K_i T$. This capacity depends linearly on the sampling frequency $\frac{1}{T_i}$ and the logarithm of the inverse uncertainty bound δ_i^{-1}. We can consider δ_i^{-1} as the worst case signal-to-error ratio since

$$\|\Delta_i\|_\infty^{-1} = \inf_{\tilde{v}_i(k_i) \in \ell_2} \frac{\|\tilde{v}_i(k_i)\|_2}{\|e_i(k_i)\|_2} \geq \delta_i^{-1}.$$

Clearly, larger δ_i indicates that less accurate information could be transmitted through the channel. Therefore, the capacity $\mathfrak{C}_i$ measures properly how much information per time unit can be transmitted through the i-th channel. To measure the amount of information transmitted through the whole network pet time unit, we define the overall network capacity by summing up all the capacities $\mathfrak{C}_i$, i.e., $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$.

So far, we have obtained quite much knowledge on the structure of the multirate NCS. The lifted closed-loop system would follow directly from the equivalent LTI systems associated with the plant, controller and network. In view of [15], the closed-loop multirate NCS is stable if and only if the lifted closed-loop system is stable. Therefore, our problem becomes to find the stabilizing conditions for the lifted system.

This would result in an $\mathcal{H}_\infty$ robust control problem. Due to the existence of more than one uncertainties in the loop, the robust stability and stabilization problem is called a structured problem. Let $T(z)$ be the complementary sensitivity function of the lifted feedback system:

$$T(z) = ZF_e(zI - A_e - B_eZF_e)^{-1}B_e.$$

If the uncertainty bounds $\delta_1, \delta_2, \dots, \delta_m$ and the state feedback gain F_e are given, the uncertain system is stabilized for all possible uncertainty satisfying the bounds if and only if [20]

$$\inf_{D\in\mathcal{D}} \left\|D^{-1}T(z)D\Psi\right\|_\infty < 1, \tag{2}$$

where $\mathcal{D}$ is the set of all diagonal matrices with the structure

$$\mathrm{diag}\{d_1 I_{N_1}, d_2 I_{N_2}, \dots, d_m I_{N_m}\}$$

and

$$\Psi = \mathrm{diag}\{\delta_1 I_{N_1}, \delta_2 I_{N_2}, \dots, \delta_m I_{N_m}\}.$$

Note that if we specify the factor causing the uncertainty, e.g., quantization, the inequality (2) is sufficient for stabilization apparently. However, the necessity may not be true. We will come across this situation when we study the multirate quantized control systems in section IV.

The minimization problem in (2) is convex and can be solved easily. However, the design problem, i.e., to find a stabilizing F_e such that (2) holds, is very difficult. We can formulate the design problem as the following minimization problem:

$$\inf_{F_e : A_e + B_e Z F_e \text{ is stable}} \left[\inf_{D\in\mathcal{D}} \left\|D^{-1}T(z)D\Psi\right\|_\infty\right]. \tag{3}$$

The objective function in (3) is convex over D and also convex over F_e. However, unfortunately, it is not jointly convex.

To handle this difficulty, channel resource allocation can play a crucial role. In the NCSs, quite often the channel capacity is determined by the available resource. If we allocate more resource to one channel, e.g., use better and more expensive hardware or allocate more communication bandwidth, then we are able to increase its capacity. For the current problem, we might have a constraint on the overall network capacity but we do have the freedom to allocate the individual channel capacities. Notice that allocating the channel capacity actually involves two aspects. One is allocating the sampling rates and the other is allocating the uncertainty bounds. By looking into the structure of Ψ, we find these two aspects are simultaneously contained in Ψ. Therefore, the constraint on the overall capacity could be given in terms of $\delta = \det\Psi = \Pi_{i=1}^m \delta_i^{N_i}$. Applying the channel resource allocation yields a further nested minimization problem:

$$\inf_{\det\Psi=\delta}\left\{\inf_{F_e : A_e + B_e Z F_e \text{ is stable}} \left[\inf_{D\in\mathcal{D}} \left\|D^{-1}T(z)D\Psi\right\|_\infty\right]\right\}.$$

At first sight, this problem looks even harder than problem (3), however, surprisingly, it can be analytically solved, which will be elaborated in the next section.

Before proceeding, let us define the topological entropy of a continuous-time LTI system. Recall that the Mahler measure [13] of a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$M(T) = \prod_{i=1}^n \max\{1, |\lambda_i|\},$$

and the topological entropy [3] of T is given by

$$h(T) = \ln M(T) = \sum_{|\lambda_i|>1} \ln|\lambda_i|,$$

where λ_i are the eigenvalues of T. Here, we take the natural logarithm to be consistent with the channel capacity notion defined before. In fact, the base of the logarithm does not affect our main result except for multiplication by a constant. Based on the topological entropy of a linear map, we define the topological entropy of a continuous-time system

$$\dot{x}(t) = A_c x(t)$$

as $H_c(A_c) = h(e^{A_c}) = \sum_{\Re(\lambda_i)>0} \lambda_i$, where λ_i are the eigenvalues of A_c.

III. Main Result

A mild assumption is needed to establish our main result: for the NCS in Fig. 1, assume that NT is nonpathological with respect to A_c, i.e., $\lambda_i - \lambda_j \neq \frac{2k\pi\sqrt{-1}}{NT}, k = 1, 2, \dots,$ for any two eigenvalues λ_i and λ_j of A_c [5]. With this assumption satisfied, we have the following theorem.

Theorem 1: The multirate NCS in Fig. 1 is stabilizable by state feedback under an appropriate resource allocation if and only if the overall network capacity is larger than the topological entropy of the plant, i.e., $\mathfrak{C} > H_c(A_c)$.

Proof: To simplify the proof, assume that all the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A_c lie on the open right half complex plane. This assumption can be removed following the same argument as in [9]. Under this assumption, all the eigenvalues of A_e lie outside the unit circle. In view of [18], (A_e, B_e) is stabilizable if (A_c, B_c) is stabilizable when NT is nonpathological with respect to A_c.

We first prove the necessity part. Assume that there exists a stabilizing state feedback gain F_e and a $D \in \mathcal{D}$ such that (2) holds, then it has been verified in [9] that

$$\delta^{-1} > M(A_e).$$

Since

$$\delta^{-1} = \Pi_{i=1}^m (\delta_i^{-1})^{N_i}, \quad M(A_e) = M(A^N) = e^{NT\sum\lambda_i},$$

after some calculations, we have

$$\delta^{-1} > M(A_e) \Leftrightarrow \mathfrak{C} > H_c(A_c).$$

To show the sufficiency part, for any given $\mathfrak{C} > H_c(A_c)$, we find a $D \in \mathcal{D}$, a stabilizing state feedback gain F_e and a factorization

$$\delta = \Pi_{i=1}^m \delta_i^{N_i}$$

such that (2) holds. Without loss of generality, we assume that (A_c, B_c) has the following Wonham decomposition [21]:

$$A_c = \begin{bmatrix} A_{c_1} & * & \cdots & * \\ 0 & A_{c_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_{c_m} \end{bmatrix},$$

$$B_c = \begin{bmatrix} B_{c_1} & * & \cdots & * \\ 0 & B_{c_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & B_{c_m} \end{bmatrix},$$

where each pair $(A_{c_i}, B_{c_i}), i = 1, 2, \ldots, m$ is stabilizable with state dimension n_i. Clearly, we have $\sum_{i=1}^m n_i = n$. Then the associated equivalent LTI system (A_e, B_e) has the following structure:

$$A_e = \begin{bmatrix} A_{e_1} & * & \cdots & * \\ 0 & A_{e_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_{e_m} \end{bmatrix},$$

$$B_e = \begin{bmatrix} \tilde{B}_{e_1} & * & \cdots & * \\ 0 & \tilde{B}_{e_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \tilde{B}_{e_m} \end{bmatrix}.$$

Since (A_{c_i}, B_{c_i}) is stabilizable, it follows that $(A_{e_i}, \tilde{B}_{e_i})$ is stabilizable for all $i = 1, 2, \ldots, m$.

Choose

$$D = \begin{bmatrix} I_{N_1} & 0 & \cdots & 0 \\ 0 & \epsilon I_{N_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \epsilon^{m-1} I_{N_m} \end{bmatrix}$$

with a small real number ϵ. Define

$$S = \begin{bmatrix} I_{n_1} & 0 & \cdots & 0 \\ 0 & \epsilon I_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \epsilon^{m-1} I_{n_m} \end{bmatrix}.$$

Let

$$F_{e_D} = D^{-1} F_e, \quad B_{e_D} = B_e D,$$

then

$$\begin{aligned} (4)\ D^{-1}T(z)D\Psi &= ZF_{e_D}(zI - A_e - B_{e_D} Z F_{e_D})^{-1} B_{e_D} \Psi \\ &= ZF_{e_D} S (zI - S^{-1} A_e S - S^{-1} B_{e_D} Z F_{e_D} S)^{-1} \\ &\quad \cdot S^{-1} B_{e_D} \Psi, \end{aligned}$$

where

$$S^{-1} A_e S = \begin{bmatrix} A_{e_1} & o(\epsilon) & \cdots & o(\epsilon) \\ 0 & A_{e_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & o(\epsilon) \\ 0 & \cdots & 0 & A_{e_m} \end{bmatrix},$$

$$S^{-1} B_{e_D} = \begin{bmatrix} \tilde{B}_{e_1} & o(\epsilon) & \cdots & o(\epsilon) \\ 0 & \tilde{B}_{e_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & o(\epsilon) \\ 0 & \cdots & 0 & \tilde{B}_{e_m} \end{bmatrix},$$

and $\frac{o(\epsilon)}{\epsilon}$ approaches to a finite constant as $\epsilon \to 0$.

Since $\mathfrak{C} > H_c(A_c)$, i.e., $\delta < M(A_e)^{-1}$, we can always possibly choose δ_i such that $\delta_i^{N_i} < M(A_{e_i})^{-1}, i = 1, 2, \ldots, m$ and $\delta = \Pi_{i=1}^m \delta_i^{N_i}$. This in fact realizes the allocation of the individual input channel capacity $\mathfrak{C}_i$ such that $\mathfrak{C}_i > H_c(A_{c_i})$ and $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$.

With this allocation of capacity, we consider each single-input NCS corresponding to (A_{c_i}, B_{c_i}). Discretizing (A_{c_i}, B_{c_i}) with time period $K_i T$ yields a discretized system (A_{d_i}, B_{d_i}):

$$A_{d_i} = A_{s_i}^{K_i}, \quad B_{d_i} = \sum_{q=1}^{K_i} A_{s_i}^{K_i - q} B_{s_i},$$

where $A_{s_i} = e^{A_{c_i} T}, B_{s_i} = \int_0^T e^{A_{c_i}(T-\tau)} B_{c_i} \mathrm{d}\tau$. Since $\delta_i^{N_i} < M(A_{e_i})^{-1}$, it follows directly that $\delta_i < M(A_{d_i})^{-1}$. According to [8], a state feedback gain F_{s_i} could be designed such that the single-input NCS associated with (A_{c_i}, B_{c_i}) is stable for all uncertainties satisfying the norm bound δ_i. Moreover, we have

$$\|F_{s_i}(zI - A_{d_i} - B_{d_i} F_{s_i})^{-1} B_{d_i}\|_\infty \delta_i < 1.$$

Applying the lifting technique in accordance with the time period NT yields the lifted feedback gain

$$F_{e_i} = \begin{bmatrix} F_{s_i} \\ F_{s_i} A_{s_i} \\ \vdots \\ F_{s_i} A_{s_i}^{N-1} \end{bmatrix}$$

and the lifted complementary sensitivity function

$$T_i(z) = Z_i F_{e_i} (zI - A_{e_i} - \tilde{B}_{e_i} Z_i F_{e_i})^{-1} \tilde{B}_{e_i}.$$

Since the lifting operator preserves norms, we have

$$\|T_i(z)\delta_i\|_\infty < 1.$$

Let $F = \mathrm{diag}\{F_{s_1}, F_{s_2}, \ldots, F_{s_m}\}$. In view of the structure of F_e in (1), we get $F_{e_D} S = D^{-1} F_e S = \mathrm{diag}\{F_{e_1}, F_{e_2}, \ldots, F_{e_m}\} + o(\epsilon)$. Now go back to (4), we have

$$D^{-1}T(z)D\Psi = \begin{bmatrix} T_1(z)\delta_1 & 0 & \cdots & 0 \\ 0 & T_2(z)\delta_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & T_m(z)\delta_m \end{bmatrix} + o(\epsilon; z).$$

Since $\|T_i(z)\delta_i\|_\infty < 1$ and $o(\epsilon; z) \to 0$ as $\epsilon \to 0$ for each $|z| \geq 1$, it follows that $\|D^{-1}T(z)D\Psi\|_\infty < 1$ for sufficiently small ϵ. This completes the proof.

■

Finally, we solve the problem and obtain a sufficient and necessary condition for the stabilization of the multirate NCS with channel resource allocation. The minimum network capacity required for stabilization is equal to the topological entropy of the plant. Once again, we witness the power and efficiency of the channel-controller co-design. One has to simultaneously design the channels and the controller to accomplish stabilization in the case when the network capacity is minimal.

IV. Stabilization of Multirate Quantized Control Systems

In this section, we apply the result in section III to the case of multirate quantized control systems. A sufficient condition for stabilization is obtained which shows a trade-off between the densities of time quantization and spatial quantization.

The problem setup is the same as shown in Fig. 1 except that now the network channels are specifically composed of quantizers. The time quantization is just downsampling. For the spatial quantization, we adopt logarithmic quantizers advocated in [7]. As shown in Fig. 3, the logarithmic quantizer is given by the following nonlinear mapping:

$$u_i = Q_{\delta_i}(v_i) := \begin{cases} \rho_i^l v_{i_0}, & \text{if } \frac{\rho_i^l v_{i_0}}{1+\delta_i} < v_i \leq \frac{\rho_i^l v_{i_0}}{1-\delta_i}, \\ 0, & \text{if } v_i = 0, \\ -Q_{\delta_i}(-v_i), & \text{if } v_i < 0, \end{cases}$$

where $v_{i_0} > 0$, $0 < \rho_i < 1$, $\delta_i = \frac{1-\rho_i}{1+\rho_i}$, and $l = 0, \pm 1, \pm 2, \cdots$. The uncertainty resulted from such a quantizer satisfies the norm bound

$$\frac{\|v_i(k) - u_i(k)\|_2}{\|v_i(k)\|_2} \leq \delta_i, \quad i = 1, 2, \ldots, m.$$

Clearly, larger δ_i means larger quantization errors and consequently, corresponds to a coarser spatial quantization.

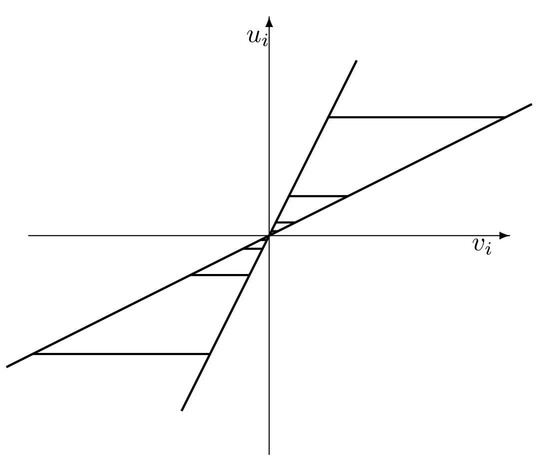

Fig. 3. A logarithmic quantizer

Apply the channel capacity notion to this case, we have

$$\mathfrak{C}_i = \frac{1}{T_i} \ln \delta_i^{-1} = \frac{1}{T_i} \ln \frac{1+\rho_i}{1-\rho_i}, \qquad \mathfrak{C} = \sum_{i=1}^{m} \mathfrak{C}_i.$$

Here, $\frac{1}{T_i}$ is apparently time quantization density and $\ln \delta_i^{-1}$ can be considered as a measure of the spatial quantization density. Therefore, the capacity $\mathfrak{C}_i$ reflects both quantization in time and space which makes it an appropriate measure of the information constraint of the channel.

Although the quantizer adopted is nonlinear, the uncertainty resulted is static without any dynamics. As we mentioned before, because of this specification on the uncertainty, the inequality (2) may not be necessary for the stabilization of closed-loop system in this case. Nevertheless, we can apply the sufficiency part of Theorem 1 to obtain a sufficient condition for the stabilization of the multirate quantized control systems with resource allocation.

Theorem 2: The multirate quantized control system is stabilizable by state feedback under an appropriate resource allocation if the overall network capacity is larger than the topological entropy of the plant, i.e., $\mathfrak{C} > H_c(A_c)$.

Theorem 2 shows a tradeoff between the densities of time quantization and spatial quantization. If the time quantization is finer, i.e., sampling faster, then the spatial quantization can be coarser, vice versa. In [7], this tradeoff has been studied for single-input systems with the assumption that the sampling and hold scheme use the same time period. It has been concluded that for a given sampling interval T, the feedback system can be stabilized if and only if

$$\rho > \frac{e^{T \sum_{\Re(\lambda_i)>0} \lambda_i} - 1}{e^{T \sum_{\Re(\lambda_i)>0} \lambda_i} + 1}.$$

Comparatively, our study is more general, allowing for multirate sampling and hold scheme. Simple derivation yields

$$\rho > \frac{e^{T \sum_{\Re(\lambda_i)>0} \lambda_i} - 1}{e^{T \sum_{\Re(\lambda_i)>0} \lambda_i} + 1} \Leftrightarrow \frac{1+\rho}{1-\rho} > e^{T \sum_{\Re(\lambda_i)>0} \lambda_i}$$
$$\Leftrightarrow \mathfrak{C} > H_c(A_c).$$

Therefore, Theorem 2 extends the result in [7].

V. An Illustrative Example

In this section, we give an example to illustrate stabilization of a multirate quantized control system. Given an unstable continuous-time system $[A_c|B_c]$ with

$$A_c = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Clearly, it is stabilizable. However, $[A_c|\alpha_1 B_{c_1} + \alpha_2 B_{c_2}]$ is not stabilizable for any $\alpha_1, \alpha_2 \in \mathbb{R}$, since the matrix $\begin{bmatrix} \lambda I - A_c & \alpha_1 B_{c_1} + \alpha_2 B_{c_2} \end{bmatrix}$ loses row rank when $\lambda = 1$. This means that it is impossible to convert $[A_c|B_c]$ to a stabilizable single-input system by a linear combination of the two inputs. The topological entropy of the plant is

$$H_c(A_c) = 2 + 1 + 1 = 4.$$

Let the overall capacity be given by $\mathfrak{C} = 4 + 2 \times 10^{-2}$. We allocate the capacity among the two input channels as $\mathfrak{C}_1 = 3 + 10^{-2}, \mathfrak{C}_2 = 1 + 10^{-2}$. Let $T = 0.1(\text{sec})$ and $K_1 = 3, K_2 = 2$, then the two input channels are characterized by

$$\begin{aligned} T_1 &= 0.3, & \delta_1 &= e^{-\mathfrak{C}_1 T_1} = 0.405, \\ T_2 &= 0.2, & \delta_2 &= e^{-\mathfrak{C}_2 T_2} = 0.817. \end{aligned}$$

We proceed to design the state feedback gain. Discretize the following two continuous-time single-input systems

$$\left[\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \text{ and } \left[\begin{array}{c|c} 1 & 1 \end{array}\right]$$

with time period T_1 and T_2 respectively. Solving the $\mathcal{H}_\infty$ control problem for the two discretized systems yields the optimal feedback gains $F_{s_1} = \begin{bmatrix} -7.758 & 3.23 \end{bmatrix}, F_{s_2} = -3.155$. Let

$$F = \begin{bmatrix} -7.758 & 3.23 & 0 \\ 0 & 0 & -3.155 \end{bmatrix}.$$

With the above co-design of input channels and state feedback gain F, the continuous-time evolution of the plant states starting from an initial condition stimulated by an impulse is shown in Fig. 4. The state converges to zero asymptotically. Fig. 5 shows the quantized control signal in this case.

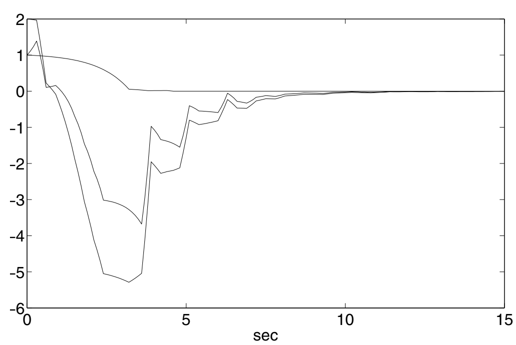

Fig. 4. Closed-loop plant state evolution.

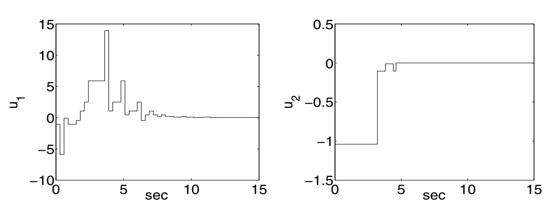

Fig. 5. Quantized control signal.

VI. Conclusion

In this paper, we study the stabilization of multirate NCSs with norm bounded uncertainties in the input channels. The key idea is to use the channel resource allocation, i.e., given the overall capacity of the transmission network, we do have the freedom to allocate the capacities among different input channels. With this idea, we successfully show that a multirate NCS could be stabilized by state feedback under an appropriate resource allocation if and only if the overall network capacity is larger than the topological entropy of the plant. We also apply the result to multirate quantized NCSs. A sufficient condition for stabilization is obtained which shows a trade-off between the densities of time quantization and spatial quantization.

References

[1] P. Antsaklis and J. Baillieul (Guest Editors), "Special Issue on Networked Control Systems", *IEEE Trans. Automat. Contr.*, vol. 49, 2004.

[2] P. Antsaklis and J. Baillieul (Guest Editors), "Special Issue on Technology of Networked Control Systems", *Proc. of IEEE*, vol. 95, 2007.

[3] R. Bowen, "Entropy for group endomorphisms and homogeneous spaces", *Trans. of the American Mathematical Society*, vol. 153, pp. 401-414, 1971.

[4] R. Brockett, "Stabilization of motor networks", in *Proc. 34st IEEE Conf. Decision Contr.*, pp. 1484-1488, 1995.

[5] T. Chen and B. Francis, *Optimal Sampled-Data Control Systems*, Springer, 1995.

[6] N. Elia, "Remote stabilization over fading channels", *Systems & Control Letters*, vol. 54, pp. 237-249, 2005.

[7] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information", *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1384-1400, 2001.

[8] M. Fu and L. Xie, "The sector bound approach to quantized feedback control", *IEEE Trans. Automat. Contr.* vol. 50, pp. 1698-1711, 2005.

[9] G. Gu and L. Qiu, "Networked stabilization of multi-input systems with channel resource allocation", *Reprints of the 17th IFAC World Congress*, 2008.

[10] D. Hristu, K. Morgansen, "Limited communication control", *Systems & Control Letters*, vol. 37, pp. 193-205, 1999.

[11] K. Li and J. Baillieul, "Robust quantization for digital finite communication bandwidth (DFCB) control", *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1573-1584, 2004.

[12] K. Li and J. Baillieul, "Robust and efficient quantization and coding for control of multidimensional linear systems under data rate constraints", *Int. J. Robust Nonlinear Control*, vol. 17, pp. 898-920, 2007.

[13] K. Mahler, "An application of Jensen's formula to polynomials", *Mathematika*. vol. 7, pp. 98-100, 1960.

[14] A. S. Matveev and A. V. Savkin, "Multirate stabilization of linear multiple sensor systems via limited capacity communication channels", *SIAM J. Control Optim.*, vol. 44, pp.584-617, 2005.

[15] D. G. Meyer, "A new class of shift-varying operators, their shift-invariant equivalents, and multirate digital systems", *IEEE Trans. Automat. Contr.*, vol. 35, pp. 429-433, 1990.

[16] L. Moreau, "Stability of multiagent systems with time-dependent communication links", *IEEE Trans. Automat. Contr.*, vol. 50, pp. 169-182, 2005.

[17] P. Ogren, E. Fiorelli, and N. E. Leonard, "Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment", *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1292-1302, 2004.

[18] L. Qiu and K. Tan, "Direct state space solution of multirate sampled-data $\mathcal{H}_2$ optimal control", *Automatica*, vol. 34, pp. 1431-1437, 1998.

[19] P. Seiler and R. Sengupta, "Analysis of communication losses in vehicle control problems", *Proc. 2001 Amer. Contr. Conf.*, vol. 2, pp. 1491-1496, 2001.

[20] J. S. Shamma, "Robust stability with time-varying structured uncertainty", *IEEE Trans. Automat. Contr.*, vol. 39, pp. 714-724, 1994.

[21] W. M. Wonham, "On pole assignment in multi-input controllable linear systems", *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 660-665, 1967.

[22] N. Xiao, L. Xie and L. Qiu, "Mean square stabilization of multi-input systems over stochastic multiplicative channels", in *Proc. 48st IEEE Conf. Decision Contr.*, pp. 6893-6898, 2009.