2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Performance analysis of asynchronous model predictive control laws

Ravi Gondhalekar

***Abstract*— Performance analysis of *model predictive control* (MPC) laws for constrained linear periodic systems with asynchronous timing constraints on input channels is considered. The problem boils down to performance analysis of autonomous periodic *piecewise affine* (PWA) systems of time-dependent state dimension. A periodic reverse reachability algorithm that constructs the explicit periodic *piecewise quadratic* (PWQ) running cost function over the region(s) of attraction of the origin is described. Available explicitly, the running cost function enables exact and rapid performance evaluation of the system as a whole, and facilitates the strategic optimization of an asynchronous controller's initial condition, while circumnavigating the need to perform a large number of simulations.**

***Index Terms*— Model predictive control; Asynchronous control; Periodic control; Periodic system; Performance analysis**

I. Introduction

Control law synthesis for discrete-time systems with asynchronous inputs has been investigated for a while. Control of systems with multi-rate inputs, where each input channel[1] may be updated at a unique frequency, was studied in [1,2,3,4]. Multiplexed control, where input channels are updated in ordered sequence, was considered in [5,6]. Synthesis of general asynchronous control laws for constrained linear periodic (includes *linear time-invariant* (LTI)) systems was treated using a state-feedback MPC framework in [7], were the employed models of linear periodic systems with asynchronous inputs generally have time-dependent state and input dimensions. The resulting control laws are periodic and PWA, resulting in autonomous periodic PWA closed-loop system models with time-dependent state dimension.

The rigorous performance analysis of asynchronous control laws is crucial and the focus of this paper. For example, when selecting an actuator for use in a given system, there may be a trade-off between cost and actuation speed. Alternatively, in [5,6] a multi-input plant with synchronous inputs is intentionally treated as a multiplexed system for the purpose of reducing controller complexity. With m input channels there are $(m-1)!$ unique possibilities for updating each input channel individually. In both of these examples a rigorous means of evaluating the effects of the input asynchronicity is indispensable for making informed design decisions.

A further design decision of potential interest in asynchronous control concerns optimal controller initialization. The modeling technique of [7] employs an augmented state that contains the actual system state as well as information of past control actions. When initializing an asynchronous controller the past control actions are non-existent. Thus the initial state of the system model is subject to design, and a strategy for optimal controller initialization may be useful.

A simple approach for performance evaluation and controller initialization involves performing simulations from a large number of initial states. However, this is at best approximate, generally provides no error bounds, and is extremely time-consuming to do accurately. For the nominal case the need for simulations is avoided in this paper by determining an explicit characterization of the running cost function over the entire region(s) of attraction of the origin, where the running cost is the cost of an infinite-horizon trajectory starting from a particular initial state. Available explicitly as a periodic PWQ function of the model state, the function can be integrated exactly over the entire region of attraction of the origin to evaluate the average performance of the system as a whole, and can furthermore be employed for rigorous and strategic optimization of a controller's initial condition.

This paper makes two contributions. The first is the periodic reverse reachability algorithm of Section V for constructing the explicit periodic PWQ running cost function of autonomous periodic PWA systems. This algorithm is an extension to periodic systems of the time-invariant reverse reachability algorithm presented in [8]. The second, less tangible, contribution is the proposition that rigorous methods of studying asynchronous control law performance are important. Control and measurement systems are increasingly implemented using computers. They are also becoming more complex, with different components operating at different speeds. Thus the analysis (and design) of systems subject to asynchronous timing-constraints is of increasing importance.

Notation: The set of reals is denoted by $\mathbb{R}$ ($\mathbb{R}_0$: non-negative), the set of non-negative integers by $\mathbb{N}$ ($\mathbb{N}_+ := \mathbb{N}\backslash\{0\}$), the set of consecutive non-negative integers $\{j, \ldots, k\}$ by $\mathbb{N}_j^k$, the spectral radius of matrix A by $\rho(A)$, element k of vector a by $a_{[k]}$. Let $\mathrm{mod} : \mathbb{N} \times \mathbb{N}_+ \to \mathbb{N}$, $\mathrm{mod}(i,j) := \min_{k\in\mathbb{N}}\{i - kj | i - kj \geq 0\}$.

II. Background: (Asynchronous) MPC of constrained linear periodic systems

Consider constrained discrete-time linear periodic system (1) subject to constraints (2), with step $i \in \mathbb{N}$, period length $p \in \mathbb{N}_+$, inter-period step index $j := \mathrm{mod}(i,p) \in \mathbb{N}_0^{p-1}$, state $x_i \in \mathbb{R}^{n_j}$, input $u_i \in \mathbb{R}^{m_j}$, and state and input dimensions $n_j \in \mathbb{N}_+$, $m_j \in \mathbb{N}$ $\forall j \in \mathbb{N}_0^{p-1}$. System (1) is autonomous at step i if $m_{\mathrm{mod}(i,p)} = 0$.

$$x_{i+1} = A_j x_i + B_j u_i \tag{1}$$

$$E_j x_i + G_j u_i \leq W_j \tag{2}$$

Assumption 1: System (1) is stabilizable.

R. Gondhalekar is at the Dept. of Mechanical Engineering, Osaka University, Japan. ravi.gondhalekar@wakate.frc.eng.osaka-u.ac.jp

[1]An input channel is one dimension $u_{[k]}$ of multi-dimensional input u.

Assumption 2: $W_j > 0 \; \forall j \in \mathbb{N}_0^{p-1}$ (origin in interior of (2))

The reason for considering linear periodic system (1) with time-dependent dimensions is that LTI and linear periodic systems (with time-invariant dimensions) subject to asynchronous inputs may conveniently be modeled as such (see Sec. 5 of [7]). "Asynchronous inputs" implies that the control input u must satisfy timing constraints $u_{[k]i} = u_{[k]i-1}$ if $k \in \mathcal{I}_j$, where $\mathcal{I}_j \subset \{\mathbb{N}_1^{m_j} \cup \emptyset\}$ denotes the set of indices of input channels that cannot be updated at step j. For the purpose of control law synthesis, where one desires a procedure to determine new values only of control input channels that *can* be updated, this is modeled here as in [7], by splitting input vector u_i into two parts, where the part $\bar{u}_i$ is a vector containing the values of all input channels that *can* updated at step i, and the part $\hat{u}_i$ is a vector containing values of all input channels that *cannot* be updated at step i, but must remain unchanged from the previous step $i-1$. In a physical system, applying new values for $\hat{u}_i$ at step i either has no effect, or must be avoided. In the model, input channels $\hat{u}_i$ that cannot be updated simply do not exist as inputs. Instead, the values are appended to the model's state at a previous step, for later use. The resulting model is a linear periodic system of the form of (1) with input $\bar{u}_i$ and state $\bar{x}_i := [x_i^\top, \hat{u}_i^\top]^\top$, where the dimensions of both $\bar{u}_i$ and $\bar{x}_i$ are generally time-dependent. For example, consider a discrete-time LTI system (i.e. system (1) with $p = 1$) with two input channels (i.e. $m_0 = 2$) updated according to Fig. 1. This is equivalent to a linear periodic system with $p = 4$, with equal dynamics and constraints at each inter-period step, subject to asynchronous timing constraints with $\mathcal{I}_0 = \emptyset$, $\mathcal{I}_1 = \{1\}$, $\mathcal{I}_2 = \{2\}$, $\mathcal{I}_3 = \{1, 2\}$. The model of this system is a linear periodic system with $p = 4$, $\{m_0, \ldots, m_3\} = \{2, 1, 1, 0\}$, $\{n_0, \ldots, n_3\} = \{2, 3, 3, 4\}$:

$$\begin{aligned}
&\bar{u}_0 = u_0 \;, \; \bar{u}_1 = u_{[2]1} \quad, \; \bar{u}_2 = u_{[1]2} \quad, \; \bar{u}_3 = \emptyset \qquad \ldots \\
&\bar{x}_0 = x_0 \;, \; \bar{x}_1 = \begin{bmatrix} x_1 \\ u_{[1]0} \end{bmatrix}, \; \bar{x}_2 = \begin{bmatrix} x_2 \\ u_{[2]1} \end{bmatrix}, \; \bar{x}_3 = \begin{bmatrix} x_3 \\ u_{[1]2} \\ u_{[2]1} \end{bmatrix} \ldots
\end{aligned}$$

A (given) periodic state-feedback control law is denoted by $\kappa_j : \mathbb{R}^{n_j} \to \mathbb{R}^{m_j}$. For initial state x_i at time i the measure of control performance is given by the quadratic cost function $\Gamma : \mathbb{R}^{n_{\mathrm{mod}(i,p)}} \to \mathbb{R}_0$,

$$\begin{aligned}
\Gamma(x_i) := \textstyle\sum_{k=i}^{\infty} \big[& x_k^\top Q_{\mathrm{mod}(k,p)} x_k + \kappa^\top(x_k) R_{\mathrm{mod}(k,p)} \kappa(x_k) \\
& + 2\kappa^\top(x_k) S_{\mathrm{mod}(k,p)} x_k \big]
\end{aligned} \tag{3}$$

with $\left[\begin{smallmatrix} Q_j & S_j^\top \\ S_j & R_j \end{smallmatrix}\right] \succ 0 \; \forall j \in \mathbb{N}_0^{p-1}$, where lower cost is better.

Control law synthesis for system (1) subject to (2) and (3) was tackled in [7], and the reader is referred there for procedures to synthesize control laws with properties required to apply the methods of this paper (Assumption 4). Control law synthesis is not a contribution of this paper, thus not repeated

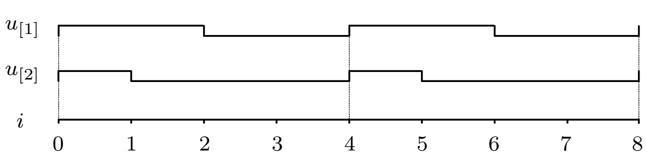

Fig. 1. Asynchronous input timing over two periods: $p = 4$.

here. However, MPC problems that are least-restrictive and strongly feasible, and periodic MPC control laws that are stabilizing and/or optimal, were designed in Theorems 16, 19 and 20 of [7], respectively. Regardless of the MPC problem design details, the resulting MPC problems are periodic QP problems (i.e. one QP problem for each $j \in \mathbb{N}_0^{p-1}$), resulting in a periodic PWA control law (see [9] for LTI case)

$$\kappa_j(x) := K_j^{[k]} x + a_j^{[k]} \quad \text{if} \quad x \in \mathbb{X}_j^{[k]} \tag{4}$$

with regions $\mathbb{X}_j^{[k]} \subseteq \mathbb{R}^{n_j} \; \forall k \in \mathbb{N}_1^{\sigma_j} \; \forall j \in \mathbb{N}_0^{p-1}$, where $\sigma_j \in \mathbb{N}_+ \; \forall j \in \mathbb{N}_0^{p-1}$ denotes the number of regions in the PWA partition of κ_j. The regions' interiors do not overlap, and the control law is continuous across region boundaries. Due to Assumption 2, at each step j the origin is contained within the interior of exactly one region, for simplicity the region with index one: $0 \in \mathrm{int}(\mathbb{X}_j^{[1]}) \; \forall j \in \mathbb{N}_0^{p-1}$. If the origin is in a region with index other than one then a simple re-numbering of the partition is required. As system (1) is linear it holds that $a_j^{[1]} = 0 \; \forall j \in \mathbb{N}_0^{p-1}$, i.e. κ_j is linear in a neighborhood of the origin. In the unconstrained case κ_j is globally linear, i.e. $\mathbb{X}_j^{[1]} = \mathbb{R}^{n_j}$, $\sigma_j = 1 \; \forall j \in \mathbb{N}_0^{p-1}$.

Assumption 3: $\max_{z \in \mathbb{Z}_j} |z| < \infty$, $\mathbb{Z}_j := \{z \in \mathbb{R}^{n_j + m_j} | [E_j, G_j] z \leq W_j\} \; \forall j \in \mathbb{N}_0^{p-1}$ (constraints (2) are bounded)

Under Assumption 3 each region of PWA control law (4) is a polytope: $\mathbb{X}_j^{[k]} = \{x \in \mathbb{R}^{n_j} | G_j^{[k]} x \leq W_j^{[k]}\}$, $\max_{x \in \mathbb{X}_j^{[k]}} |x| < \infty \; \forall k \in \mathbb{N}_1^{\sigma_j} \; \forall j \in \mathbb{N}_0^{p-1}$.

The methods of this paper employ the explicit characterization of periodic PWA control law (4), as determined by *multi-parametric quadratic programming* (mpQP) [9]. The mpQP solution may be computed using the *multi-parametric toolbox* (MPT) [10]. It is assumed henceforth that periodic PWA control law (4) has been obtained in explicit form.

We define the following $\forall k \in \mathbb{N}_1^{\sigma_j} \; \forall j \in \mathbb{N}_0^{p-1}$:

$$\begin{aligned}
A_j^{[k]} &:= A_j + B_j K_j^{[k]} \;\in\; \mathbb{R}^{n_{\mathrm{mod}(j+1,p)} \times n_j} \\
b_j^{[k]} &:= B_j a_j^{[k]} \;\in\; \mathbb{R}^{n_{\mathrm{mod}(j+1,p)}} \\
H_j^{[k]} &:= Q_j + K_j^{[k]\top} R_j K_j^{[k]} + 2K_j^{[k]\top} S_j \;\in\; \mathbb{R}^{n_j \times n_j} \\
L_j^{[k]} &:= 2a_j^{[k]\top} \big(R_j K_j^{[k]} + S_j\big) \;\in\; \mathbb{R}^{1 \times n_j} \\
C_j^{[k]} &:= a_j^{[k]\top} R_j a_j^{[k]} \;\in\; \mathbb{R} \;.
\end{aligned}$$

The closed-loop system then evolves according to periodic PWA dynamics (5) with $j = \mathrm{mod}(i, p)$, and incurs periodic PWQ single-step cost (6) with $J_j : \mathbb{R}^{n_j} \to \mathbb{R}_0 \; \forall j \in \mathbb{N}_0^{p-1}$.

$$x_{i+1} = A_j^{[k]} x_i + b_j^{[k]} \quad \text{if} \quad x_i \in \mathbb{X}_j^{[k]} \tag{5}$$

$$J_j(x) := x^\top H_j^{[k]} x + L_j^{[k]} x + C_j^{[k]} \quad \text{if} \quad x \in \mathbb{X}_j^{[k]} \tag{6}$$

Assumption 4: $\rho(\hat{A}) < 1$, $\hat{A} := \prod_{j=0}^{p-1} A_j^{[1]} = A_{p-1}^{[1]} \cdots A_0^{[1]}$

Assumption 4 states that the closed-loop dynamics governing the origin are stable. Note that a sensible control law synthesis procedure (e.g. those described in [7]) yields a stabilizing control law, even if the control law is sub-optimal (i.e. does not *minimize* (3)), and even if closed-loop stability is not guaranteed *a priori*, but must be determined after the control law is characterized. If Assumption 4 does not hold the methods of this paper are rendered inapplicable and irrelevant.

III. Problem description

We now consider periodic PWA system (5) and define the regions $\mathbb{A}_j \subseteq \mathbb{R}^{n_j}\ \forall j \in \mathbb{N}_0^{p-1}$ of attraction of the origin:

$$\mathbb{A}_j := \Big\{x_j \in \mathbb{R}^{n_j} | x_{i+1} = A^{[k]}_{\mathrm{mod}(i,p)} x_i + b^{[k]}_{\mathrm{mod}(i,p)} \ \text{ if } \ x_i \in \mathbb{X}^{[k]}_{\mathrm{mod}(i,p)} \ \ \forall i \in \mathbb{N}_j^{\infty}, \ \ \lim_{i\to\infty} |x_i| = 0\Big\} \ .$$

We further define the *running cost* $V_j : \mathbb{A}_j \to \mathbb{R}_0\ \forall j \in \mathbb{N}_0^{p-1}$ of a trajectory subject to dynamics (5) and periodic PWQ stage cost (6):

$$V_j(x) := \textstyle\sum_{i=j}^{\infty} J_{\mathrm{mod}(i,p)}(x_i) \ \ , \ x_j = x \ . \tag{7}$$

Note that $a_j^{[1]} = 0$ implies $C_j^{[1]} = 0$ (also $L_j^{[1]} = 0$) and therefore $V_j(x) < \infty\ \forall x \in \mathbb{A}_j\ \forall j \in \mathbb{N}_0^{p-1}$.

The main aim and contribution of this paper is a method to derive an explicit characterization of V_j. Available explicitly V_j can subsequently be employed for analysis purposes. It turns out (unsurprisingly) that V_j is a periodic PWQ function of x, where the partition of V_j is generally different from the partition of periodic PWA dynamics (5) and periodic PWQ stage cost (6). The algorithm to construct the periodic PWQ running cost function V_j is described later in Section V.

In the absence of constraints (2) system (5) is a periodic (globally) linear system, and stage cost (6) is periodic and (globally) purely quadratic: $J_j(x) = x^\top H_j^{[1]} x\ \forall x \in \mathbb{R}^{n_j}$. In this case V_j is a periodic (globally) purely quadratic function

$$V_j(x) = x^\top \mathcal{H}_j^{[1]} x \quad \forall x \in \mathbb{R}^{n_j} \ \ \forall j \in \mathbb{N}_0^{p-1}$$

where $\{\mathcal{H}_0^{[1]}, \ldots, \mathcal{H}_{p-1}^{[1]}\}$, $\mathcal{H}_j^{[1]} \succeq 0\ \forall j \in \mathbb{N}_0^{p-1}$ is the solution to periodic Lyapunov Eq. (10) (see [11]). In the unconstrained case the methods of this paper are not required.

Remark 5: In the case that a given constrained control law is known to be *optimal* w.r.t (3) the methods of this paper are not strictly required, but are applicable. If a given control law κ_j is known to be optimal then one can formulate an MPC problem according to Theorem 20 of [7], and obtain an explicit PWQ characterization of V_j via mpQP [9].

The motivation for this paper is analysis of constrained *sub*-optimal periodic (asynchronous) control laws, i.e. control laws that do not minimize (3) and for which the explicit PWQ characterization of V_j cannot be determined via a control problem formulation and its corresponding mpQP solution.

IV. Pre-processing the closed-loop system

In order to be able to apply the algorithm described in Section V closed-loop PWA system (5) must satisfy one more property, one that is not generally a natural consequence of the MPC framework described in Section II and [7]. The set $\{\mathbb{X}_0^{[1]}, \ldots, \mathbb{X}_{p-1}^{[1]}\}$ must constitute a periodic positively invariant set (see [7]), i.e. must satisfy (recall $b_j^{[1]} = 0$):

$$A_j^{[1]} x \in \mathbb{X}^{[1]}_{\mathrm{mod}(j+1,p)} \quad \forall x \in \mathbb{X}_j^{[1]} \quad \forall j \in \mathbb{N}_0^{p-1} \ . \tag{8}$$

If a given control law κ_j does not satisfy (8) then the partition of closed-loop PWA system (5) must be processed before applying the algorithm of Section V. For brevity the pre-processing procedure is only described in outline here.

One must first determine *some* periodic positively invariant set containing the origin within its interior, for the periodic linear dynamics governing the origin, i.e. determine $\{\bar{\mathbb{X}}_0, \ldots, \bar{\mathbb{X}}_{p-1}\}$ such that $\bar{\mathbb{X}}_j \subseteq \mathbb{X}_j^{[1]}$, $0 \in \mathrm{int}(\bar{\mathbb{X}}_j)\ \forall j \in \mathbb{N}_0^{p-1}$ and $A_j^{[1]} x \in \bar{\mathbb{X}}_{\mathrm{mod}(j+1,p)}\ \forall x \in \bar{\mathbb{X}}_j\ \forall j \in \mathbb{N}_0^{p-1}$. As regions $\mathbb{X}_j^{[k]}$ are polytopes this is straightforward to do [7].

The origin regions $\mathbb{X}_j^{[1]}$ are then re-partitioned such that $\bar{\mathbb{X}}_j$ are the new origin regions, and the sets $\mathbb{X}_j^{[1]} \backslash \bar{\mathbb{X}}_j$ are split into new regions. Each new region is assigned the same dynamics and stage cost of the original region $\mathbb{X}_j^{[1]}$, so this re-partitioning procedure has no effect on the closed-loop dynamics, the stage cost, and on running cost V_j. Note, however, that the re-partitioning procedure does affect the *partition* of the explicit periodic PWQ characterization of V_j.

We henceforth make Assumption 6.

Assumption 6: Eq. (8) holds.

V. Periodic reverse reachability algorithm

In this section we describe a periodic reverse reachability algorithm that constructs the explicit periodic PWQ running cost function V_j. The algorithm proceeds as follows. First, given the properties of system (5) described above, the running cost V_j within the periodic positively invariant set $\{\mathbb{X}_0^{[1]}, \ldots, \mathbb{X}_{p-1}^{[1]}\}$ is given by periodic purely quadratic function

$$V_j(x) = x^\top \mathcal{H}_j^{[1]} x \quad \forall x \in \mathbb{X}_j^{[1]} \ \ \forall j \in \mathbb{N}_0^{p-1} \tag{9}$$

where $\{\mathcal{H}_0^{[1]}, \ldots, \mathcal{H}_{p-1}^{[1]}\}$, $\mathcal{H}_j^{[1]} \succeq 0\ \forall j \in \mathbb{N}_0^{p-1}$ is the solution to periodic Lyapunov Eq. (10) (see [11]):

$$A_j^{[1]\top} \mathcal{H}^{[1]}_{\mathrm{mod}(j+1,p)} A_j^{[1]} - \mathcal{H}_j^{[1]} + H_j^{[1]} = 0 \ . \tag{10}$$

Note that the regions $\mathbb{X}_j^{[1]}$ contain the origin within their interiors. Thus all trajectories that converge to the origin must enter one of the sets $\mathbb{X}_j^{[1]}$, for some $j \in \mathbb{N}_0^{p-1}$, after a finite number of steps. We now define the N-step regions $\mathbb{A}_j^{[N]}$ of attraction of the origin as the set of states that first enter any one of the origin regions $\mathbb{X}_j^{[1]}$ after exactly $N \in \mathbb{N}$ steps:

$$\begin{aligned} \mathbb{A}_j^{[0]} &:= \mathbb{X}_j^{[1]} \\ \mathbb{A}_j^{[N]} &:= \Big\{x \in \mathbb{R}^{n_j} | x_{j+N} \in \mathbb{X}^{[1]}_{\mathrm{mod}(j+N,p)} \ , \\ &\qquad x_i \notin \mathbb{X}^{[1]}_{\mathrm{mod}(i,p)} \ \ \forall i \in \mathbb{N}_j^{j+N-1} \ , \\ &\qquad x_{i+1} = A^{[k]}_{\mathrm{mod}(i,p)} x_i + b^{[k]} \ \text{ if} \\ &\qquad x_i \in \mathbb{X}_j^{[k]} \ \ \forall i \in \mathbb{N}_j^{\infty} \ , \ \ x_j = x\Big\} \ \ \forall N \in \mathbb{N}_+ \ . \end{aligned} \tag{11}$$

Consider a region $\mathbb{X}_j^{[k]}$ and any polytopic target region $\mathbb{Y} := \{x \in \mathbb{R}^{n_{\mathrm{mod}(j+1,p)}} | Yx \leq y\}$. Then

$$\begin{aligned} \bar{\mathbb{Y}} &:= \Big\{x \in \mathbb{X}_j^{[k]} | A_j^{[k]} x + b_j^{[k]} \in \mathbb{Y}\Big\} \\ &= \left\{x \in \mathbb{R}^{n_j} | \begin{bmatrix} G_j^{[k]} \\ Y A_j^{[k]} \end{bmatrix} x \leq \begin{bmatrix} W_j^{[k]} \\ y - Y b_j^{[k]} \end{bmatrix}\right\} \ . \end{aligned} \tag{12}$$

If $\bar{\mathbb{Y}} = \emptyset$ then $\mathbb{Y}$ is not reachable from $\mathbb{X}_j^{[k]}$. Suppose

$$V_{\mathrm{mod}(j+1,p)}(x) \;=\; x^\top \mathcal{H}_{\mathrm{y}} x + \mathcal{L}_{\mathrm{y}} x + \mathcal{C}_{\mathrm{y}} \qquad \forall x \in \mathbb{Y} \;.$$

Then $\forall x \in \bar{\mathbb{Y}}$:

$$\begin{aligned} V_j(x) \;&=\; J_j(x) + V_{\mathrm{mod}(j+1,p)}\big(A_j^{[k]} x + b_j^{[k]}\big) \\ &=\; x^\top \bar{\mathcal{H}}_{\mathrm{y}} x + \bar{\mathcal{L}}_{\mathrm{y}} x + \bar{\mathcal{C}}_{\mathrm{y}} \qquad\qquad (13) \\ \bar{\mathcal{H}}_{\mathrm{y}} \;&:=\; H_j^{[k]} + A_j^{[k]\top} \mathcal{H}_{\mathrm{y}} A_j^{[k]} \;\; \in \mathbb{R}^{n_j \times n_j} \\ \bar{\mathcal{L}}_{\mathrm{y}} \;&:=\; L_j^{[k]} + 2 b_j^{[k]\top} \mathcal{H}_{\mathrm{y}} A_j^{[k]} + \mathcal{L}_{\mathrm{y}} A_j^{[k]} \;\; \in \mathbb{R}^{1 \times n_j} \\ \bar{\mathcal{C}}_{\mathrm{y}} \;&:=\; C_j^{[k]} + b_j^{[k]\top} \mathcal{H}_{\mathrm{y}} b_j^{[k]} + \mathcal{L}_{\mathrm{y}} b_j^{[k]} + \mathcal{C}_{\mathrm{y}} \;\; \in \mathbb{R} \;. \end{aligned}$$

The reverse reachability algorithm recursively determines the N-step regions $\mathbb{A}_j^{[N]}$ of attraction of the origin, and associated periodic PWQ running cost function(s), for increasing values of N, starting at $N = 1$. For each N the reverse reachability algorithm progresses by employing Eqs. (12) and (13), first for each inter-period step $j \in \mathbb{N}_0^{p-1}$, and then for each combination of starting region $\mathbb{X}_j^{[k]}$ $\forall k \in \mathbb{N}_1^{\sigma_j}$ and target region within $\mathbb{A}_{\mathrm{mod}(j+1,p)}^{[N-1]}$. By Eq. (11), for $N = 1$ there exists only one target region for each $j \in \mathbb{N}_0^{p-1}$. Regions $\mathbb{A}_j^{[N]}$ for $N \geq 1$ generally consist of multiple regions, each a polytope.

The algorithm continues to a value $\hat{N}$ that achieves termination condition $\mathbb{A}_j^{[\hat{N}]} = \emptyset$ $\forall j \in \mathbb{N}_0^{p-1}$. Subsequently

$$\mathbb{A}_j \;=\; \textstyle\bigcup_{N=0}^{\infty} \mathbb{A}_j^{[N]} \;\;=\;\; \bigcup_{N=0}^{\hat{N}} \mathbb{A}_j^{[N]} \quad \forall j \in \mathbb{N}_0^{p-1} \;.$$

Note that the algorithm may not terminate. However, due to Assumption 3 the regions $\mathbb{A}_j$ of attraction can be approximated to arbitrary accuracy in a finite number of steps.

An exact algorithm definition is not provided, for brevity.

VI. Application example: Exact performance analysis of constrained control laws

We consider the discrete-time LTI system

$$x_{i+1} \;=\; \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} x_i + \begin{bmatrix} 0.02 & 0.02 \\ 0.2 & 0.2 \end{bmatrix} u_i \quad , \qquad (14)$$

i.e. system (1) with $p = 1$ and $n_0 = m_0 = 2$. The constraints are $\|u_i\|_\infty \leq 1$, $|[1,1]u_i| \leq 1$, $\|x_i\|_\infty \leq 2$. The control objective is the minimization of (3) with $Q_0 = I$, $R_0 = I$ and $S_0 = 0$. All computations were performed on a 3.33 GHz x86-64 processor running Matlab and the MPT [10]. Stated run-times are 20 run averages.

In Sections VI-A through VI-D four different control laws are contrasted. The performance differences are small due to the simplistic nature of this numerical example. However, the point is to motivate the use of exact performance analysis over simulations, and to give the reader a feel for the type of analyses and results that the presented methods permit.

A. Optimal synchronous constrained LQR

The optimal synchronous constrained LQR control law is given in (15) with accompanying partition in Fig. 2 (left). The explicit PWQ running cost function V_0 was computed in 93 milli-seconds and 14 iterations using the reverse reachability algorithm of Section V. The resulting partition is plotted in Fig. 2 (right) and consists of 27 regions. Note that in this (optimal) case V_0 could have been obtained by solving an MPC problem according to Theorem 20 of [7] with prediction horizon greater than 13 using mpQP (see Remark 5). Available explicitly, the PWQ running cost function can be integrated exactly, semi-analytically (see [8]), in order to compute the average running cost $\bar{V}$, of (16), of the system as a whole. Note that the integral and volume function of (16) are unsigned. Given the explicit PWQ running cost function V_0, computing the average running cost $\bar{V}$ of (16) required 76 milli-seconds. This includes the computation time required for vertex enumeration and Delaunay triangulation, which are the two biggest drivers of computational complexity.

$$u_i \;=\; \begin{cases} \begin{bmatrix} -0.568 & -0.944 \\ -0.568 & -0.944 \end{bmatrix} x_i & \text{if} \quad x_i \in \mathbb{X}_1 \\ \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^\top & \text{if} \quad x_i \in \mathbb{X}_2 \\ -\begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^\top & \text{if} \quad x_i \in \mathbb{X}_3 \end{cases} \qquad (15)$$

$$\bar{V} \;:=\; \tfrac{1}{\mathrm{Vol}(\mathbb{A}_0)} \int_{\mathbb{A}_0} V_0(x)\,\mathrm{d}x \;\;=\;\; 18.204 \qquad (16)$$

Suppose we approximate the average $\bar{V}$ by performing simulations from a finite number of starting states, and taking the average of the trajectories' individual running cost. The starting states are the nodes of an even grid with $M \in \mathbb{N}_2^{300}$ nodes per dimension. Each trajectory's exact running cost is obtained by terminating the simulation only when the trajectory enters the maximum positively invariant set where the exact running cost is given by (9). Trajectories that exit the dynamics' partition are ignored. The approximate average

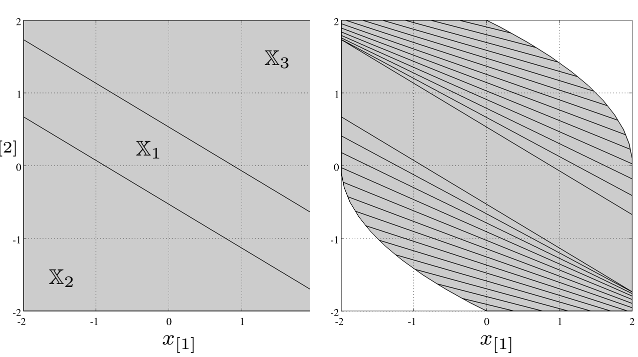

Fig. 2. Synchronous constrained case: PWA optimal LQR control law partition (left), and resulting PWQ running cost function partition (right). Number of regions in PWQ partition: $\sigma_1 = 27$.

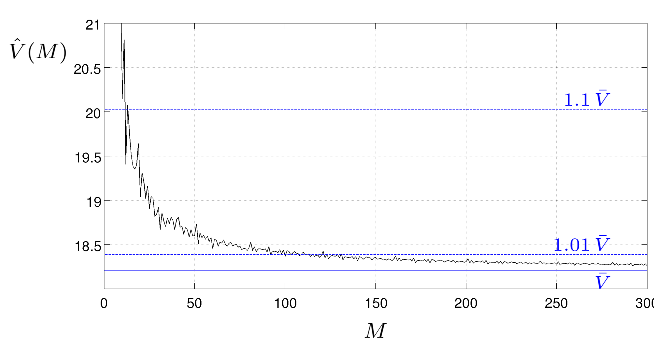

Fig. 3. Approximate average running cost $\hat{V}(M)$ computed via M^2 simulations starting from an even grid with M nodes per dimension.

running cost is denoted by $\hat{V}(M)$ and plotted in Fig. 3. It holds that $\hat{V}(M) \leq 1.01 \cdot \bar{V}$ (i.e. 1% accuracy) for all $M \geq 132$. Computing $\hat{V}(132)$ required 9.95 seconds. Note that $132^2 = 17424$ is a little more simulations than most authors perform to demonstrate the efficacy of their control laws. Thus even this trivial, low-dimensional, synchronous, time-invariant example demonstrates that the reverse reachability algorithm of Section V may be useful for quickly and accurately determining the performance of a system as a whole.

B. Sub-optimal synchronous constrained LQR

Next we consider the sub-optimal synchronous constrained LQR control law obtained by employing MPC with a unit length prediction horizon according to Theorem 16 of [7]. Using the reverse reachability algorithm, and integrating the resulting PWQ running cost function, the average $\tilde{V}$ of the running cost was determined: $\tilde{V} = \bar{V}$. In this case, due to the simple dynamics, the control performance is not reduced as a result of reducing the prediction horizon length. The PWQ running cost function partition is the same as for the optimal case plotted in Fig. 2 (right).

As yet the *periodic* property of the periodic reverse reachability algorithm of Section V has not been exploited. The results of Sections VI-A and VI-B could have been obtained using the original, time-invariant version of the reverse reachability algorithm presented in [8].

C. Optimal asynchronous constrained LQR

We next consider system (14) with asynchronous inputs updated according to Fig. 1. This is modeled by linear periodic system (1) with $p = 4$, $\{m_0, \ldots, m_3\} = \{2, 1, 1, 0\}$, $\{n_0, \ldots, n_3\} = \{2, 3, 3, 4\}$ as described in Section II and [7]. The optimal asynchronous constrained LQR control law is obtained via a periodic MPC formulation according to Theorem 20 of [7] using a prediction horizon greater than 15. The periodic PWQ running cost functions $V_0, \ldots, V_3$ were computed in 3.37 seconds and 16 iterations using the periodic reverse reachability algorithm of Section V. The partition of V_0 is plotted in Fig. 4 (left) and consists of 21 regions. Note that the partitions of V_1, V_2, V_3 are in $\mathbb{R}^3$, $\mathbb{R}^3$, $\mathbb{R}^4$, respectively. The average $\hat{V}$ of V_0 over $\mathbb{A}_0$ was computed: $\hat{V} = 1.0027 \cdot \bar{V}$, i.e. the asynchronous timing constraint induces a 0.27% increase in average, optimal cost.

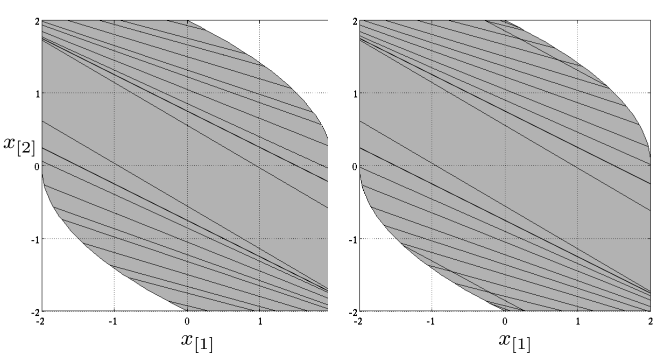

Fig. 4. Asynchronous constrained case: Optimal (left, number of regions: $\sigma_1 = 21$) and sub-optimal (right, number of regions: $\sigma_1 = 31$) PWQ running cost function partition.

D. Sub-optimal asynchronous constrained LQR

We consider again the asynchronous system (14) considered in Section VI-C. This time a sub-optimal asynchronous constrained LQR control law is obtained via a periodic MPC formulation according to Theorem 16 of [7] using a unit length prediction horizon. The periodic PWQ running cost functions $V_0, \ldots, V_3$ were computed in 3.46 seconds and 16 iterations using the periodic reverse reachability algorithm of Section V. The partition of V_0 is plotted in Fig. 4 (right) and consists of 31 regions. The average $\check{V}$ of V_0 over $\mathbb{A}_0$ was computed: $\check{V} = 1.0309 \cdot \bar{V}$, i.e. the asynchronous timing constraint, in combination with the unit prediction horizon, induces a 3.09% increase in average cost.

This result could not have been obtained exactly without employing the proposed periodic reverse reachability algorithm of Section V. As the differences in average performance of the differing control laws in this illustrative example are subtle, the errors resulting from even a vast number of simulations may have led to highly distorted conclusions.

VII. Application example: Asynchronous control law initialization

We focus next on the problem of optimally initializing an asynchronous controller. Recall from Section II that the state $\bar{x}_i$ of a periodic *model* of a system with asynchronous inputs is an augmented state, and contains the *actual* system state x_i as well as the control input values $\hat{u}_i$ of input channels that cannot be updated from the value they had at the previous step $i-1$. Thus the augmented part $\hat{u}_i$ is historic data that in the physical system either cannot, or must not, be changed at step i. However, at the initial step this historic data is non-existent and should, when possible, be chosen the best way; to minimize (3). If one has determined the running cost function V_j for the model of a system with asynchronous inputs (i.e. $V_j(\bar{x})$ not $V_j(x)$) then one has explicitly characterized the running cost for each combination of physical state x, initial condition of the historic data $\hat{u}$, and inter-period step j, for which the running cost is finite, i.e. the controller stabilizing.

It seems natural to suppose a trajectory starts at time $i = 0$. However, an asynchronous input update pattern of length p, for example that of Fig. 1 with length $p = 4$, could be drawn in p different ways, starting at $i = 0$, to yield the same update sequence. In this paper we suppose that the update pattern is fixed, and for a given actual system state x consider determining the optimal initial inter-period time-

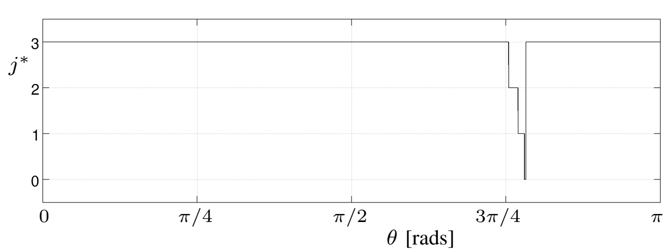

Fig. 5. Unconstrained asynchronous case: State-dependent optimal initial inter-period step $j^*(x)$, $x = [\ \cos(\theta)\ \ \sin(\theta)\ ]^\top$.

step $j^*(x) \in \mathbb{N}_0^{p-1}$ to start a trajectory at, and accompanying initial condition $\hat{u}_{j^*(x)}(x)$ for the asynchronous controller.

The unconstrained case is simple. In the unconstrained case the control law is linear periodic feedback of the augmented state, i.e. $\bar{u}_i = K_j \bar{x}_i$, and the resulting running cost function is the periodic purely quadratic function (see (9))

$$V_j(x_i, \hat{u}_i) = \bar{x}_i^\top \mathcal{H}_j \bar{x}_i = \begin{bmatrix} x_i \\ \hat{u}_i \end{bmatrix}^\top \begin{bmatrix} \mathcal{Q}_j & \mathcal{S}_j^\top \\ \mathcal{S}_j & \mathcal{R}_j \end{bmatrix} \begin{bmatrix} x_i \\ \hat{u}_i \end{bmatrix}$$

for appropriate $\mathcal{Q}_j$, $\mathcal{R}_j$ and $\mathcal{S}_j$. Optimization w.r.t $\hat{u}_i$ yields:

$$\hat{u}^*(x_i) := \arg\min_{\hat{u}_i} V_j(x_i, \hat{u}_i) = -\mathcal{R}_j^{-1}\mathcal{S}_j x_i \quad (17)$$

$$V_j^*(x_i) := V_j(x_i, \hat{u}_i^*) = x_i^\top \left[\mathcal{Q}_j - \mathcal{S}_j^\top \mathcal{R}_j^{-1} \mathcal{S}_j\right] x_i \quad (18)$$

$$j^*(x) := \arg\min_{j \in \mathbb{N}_0^{p-1}} V_j^*(x) \ . \quad (19)$$

We consider system (14) with asynchronous inputs updated according to Fig. 1, but no constraints. A guess (the author's, in fact) may be that $j^*(x) = 3 \ \forall x \in \mathbb{R}^2$, because this means that both inputs are optimized at both the first and second step of the trajectory, whereas when $j^*(x) \in \{0, 1, 2\}$ only one, or no, input channel can be updated at the second step. Plotted in Fig. 5 is $j^*(x)$ for $x = [\cos(\theta) \ \sin(\theta)]^\top$ and $\theta \in [0, \pi]$. Recall that the original system (14) is LTI. The guess that $j^*(x) = 3$ is correct for most, but not all, initial states x. Furthermore, for each possible inter-period step there exists an initial state such that this step is the optimal starting step: $\forall \hat{j} \in \mathbb{N}_0^{p-1} \ \exists x \in \mathbb{R}^2$ s.t. $j^*(x) = \hat{j}$.

In the constrained case the analysis is more complicated than Eqs. (17)-(19). However, the need for simulations can again be avoided, and strategic optimization performed, by employing the explicit characterization of running cost function $V_j(\bar{x})$. Given a state x one must first determine all index pairs (j, k) such that there exists a $\hat{u}$ s.t. $\hat{x} = [x^\top \ \hat{u}^\top]^\top \in \mathbb{X}_j^{[k]}$. This can be achieved by projecting all regions $\mathbb{X}_j^{[k]}$ onto the space of x and testing set membership of x. For each located (j, k) pair the optimal controller initialization $\bar{x}^* := [x^\top \ (\hat{u}^*(x))^\top]^\top$, and resulting minimized running cost $V_j^*(x) := V_j(\bar{x}^*)$, can be found by solving a constrained QP problem (details omitted for brevity). The optimal initial inter-period time-step $j^*(x)$ is subsequently determined by optimizing $V_j^*(x)$ over all located (j, k) pairs.

We again consider system (14) with asynchronous inputs according to Fig. 1, now with constraints. The state-dependent optimal initial inter-period step $j^*(x)$ is plotted in Fig. 6 for the optimal and sub-optimal constrained control laws employed in Sections VI-C and VI-D, respectively. In a neighborhood of the origin the results are equal and correspond to the unconstrained case plotted in Fig. 5. In regions where constraints are active the results are intriguingly different. For example, in the sub-optimal case (Fig. 6, right) the optimal initial inter-period step index is 1 for most states.

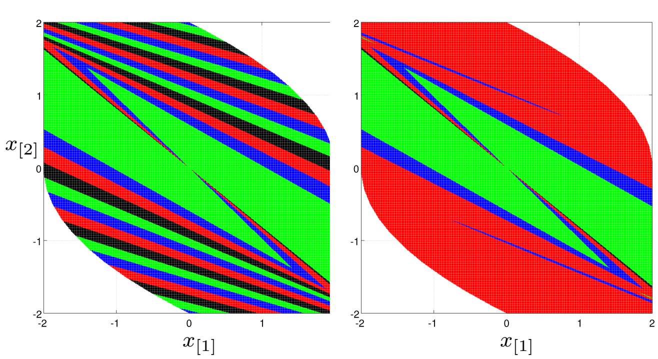

Fig. 6. Constrained asynchronous case: State-dependent optimal initial inter-period step: $j^*(x) = \{0, 1, 2, 3\} = \{$ black , red , blue , green $\}$. Optimal (left), sub-optimal (right).

The explicit running cost function has proven indispensable for exactly determining the optimal initial condition of an asynchronous control law, and its use yields conclusions far superior to conclusions achievable by performing simulations (or by guessing). Thus the periodic reverse reachability algorithm of Section V is deemed a very useful tool, should an answer to the question of optimal asynchronous controller initialization be required.

VIII. Conclusion

A periodic reverse reachability algorithm to construct the explicit PWQ running cost function of autonomous periodic PWA systems was presented. Available explicitly, the PWQ running cost function can subsequently be employed for analysis purposes. For example the performance, of the system as a whole, w.r.t asynchronous input update patterns can be determined rapidly and exactly. Furthermore, the explicit running cost function enables one to strategically determine the optimal initial condition for asynchronous control laws.

IX. Acknowledgement

The author acknowledges funding from the Program of Promotion of Environmental Improvement to Enhance Young Researchers' Independence under the Special Coordination Funds for Promoting Science and Technology, Japan Ministry of Education, Culture, Sports, Science and Technology.

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