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Optimality Conditions for A Trend-Following Strategy

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***Abstract*— Based on trend-following trading strategies that are widely used in the investment world, this work provides a set of sufficient conditions that determines the optimality of the traditional trend-following strategies when the trends are completely observable. A dynamic programming approach is used to verify the optimality under these conditions. The value functions are characterized by the associated HJB equations, and are shown to be either linear functions or infinity depending on the parameter values. The results reveal two counter-intuitive facts: (a) trend following may not lead to optimal reward in some cases even when/if the investor knows exactly when a trend change occurs; (b) stock volatility is not relevant in trend following when trends are observable.**

***Index Terms*— regime-switching process, quasi-variational inequality, trend-following strategy.**

I. Introduction

Active market participants can be classified in accordance with their trading strategies: Those who trade contra-trend and those who follow the trend. In this paper, we focus on the trend-following (TF) trading strategies. The basic premise underlying the trend-following rules is that the market can be regarded either as a bull market or a bear market at a given time. Trend-following strategies are concerned with trading rules that trade with the market, i.e., go long in a bull market or go short in a bear market. One way to capture the market trends is to use the geometric Brownian motions with regime switching. A standard geometric Brownian motion (GBM) model involves two parameters, the expected rate of return and the volatility, both assumed to be deterministic constants. In a model with regime switching, these key parameters are allowed to be market trend (or regime) dependent. The regime-switching model was first introduced by Hamilton [10] to describe a regime-switching time series. Subsequently, it is extensively studied in connection with option pricing; see Di Masi et al. [7], Bollen [1], Buffington and Elliott [2], Yao et al. [19], and references there in.

In the financial engineering literature, stock trading rules have been studied under various diffusion models for many years. For example, Øksendal [18, Examples 10.2.2-4] considered optimal exit strategy for stocks whose price dynamics were modeled by a geometric Brownian motion. Stock selling rules under regime-switching models have gained increasing attention. For example, Zhang [23] considered a selling rule determined by two threshold levels, a target price and a stop-loss limit. Under the regime-switching model, optimal threshold levels were obtained by solving a set of two-point boundary value problems. In Guo and Zhang [9], the results of Øksendal [18] were extended to incorporate a model with regime switching. In addition to these analytical results, various mathematical tools have been developed to compute these threshold levels. A stochastic approximation technique was used in Yin et al. [20], and a linear programming approach was developed in Helmes [11]; and the fast Fourier transform was used in Liu et al. [15]. Furthermore, consideration of capital gain taxes and transaction costs in connection with selling can be found in Cadenillas and Pliska [3], Constantinides [4], and Dammon and Spatt [5] among others. Recently, there has been growing effort concerning trading rules that involved both buying and selling. For instance, Zhang and Zhang [22] developed optimal trading strategies in a mean-reverting market, which validated a well-known contra-trend trading method. In particular, they established two threshold prices (buy and sell) that maximized overall discounted return if one traded at those prices. These results are extended to allow short sales in Kong and Zhang [13]. In addition to the results obtained in [22] along this line of research, an investment capacity expansion/reduction problem was considered in Merhi and Zervos [17]. Under a geometric Brownian motion market model, the authors used the dynamic programming approach and obtained an explicit solution to the singular control problem. A more general diffusion market model was treated by Løkka and Zervos [16] in connection with an optimal investment capacity adjustment problem. More recently, Johnson and Zervos [12] studied an optimal timing of investment problem under a general diffusion market model. The objective was to maximize the expected cash flow by choosing when to enter an investment and when to exit the investment. An explicit analytic solution was obtained in [12].

In this paper, we consider a regime-switching model for the stock price dynamics. The price of the stock follows a geometric Brownian motion whose drift switches between two different regimes representing the up trend (bull market) and down trend (bear market), respectively. We model the switching as a Markov chain. In addition, we assume trading one share with a fixed percentage slippage cost. As in Zhang and Zhang [22] we introduce optimal value functions that correspond to starting net position being either flat or long. We focus on a fundamental issue in trend-following trading. Under the framework of a regime-switching market, we pose the following question: If the investor has the full knowledge of market trends, i.e., she or he knows exactly when the market turns from bull to bear (or bear to bull), will she or

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he always be profitable? We address this best-case scenario. In particular, we aim at classifying parameter regions so that the optimal trading strategy varies on each of these regions. We use a dynamic programming approach, and derive a system of two variational inequalities, which can be casted into the form of HJB equations. We find solutions to these equations and construct the corresponding trading rules. We also provide verification theorems to justify the optimality of these trading rules. The results reveal two counter intuitive facts:

(a) trend following may not lead to optimal reward in some cases even the investor knows exactly when a trend change occurs;
(b) stock volatility is not relevant in trend following when trends are observable.

We point out that analytic or closed-form solutions in stochastic control problems are rarely obtainable. An analytic solution is desirable in practice because it provides a clear picture on dependence of random variables. It can also be useful for the related computational methods because it usually reveals the basic structure of the underlying problem. This paper reports a set of closed-form solutions to the optimal control problem under consideration. It adds to the list of 'solvable' stochastic control problems in the literature. In this paper, we present the main ideas and results. The detailed development and verbatim proofs can be found in our recent paper [14].

In Section II, problem setup is constructed. In Section III, classification of parameter regions are provided so that the optimal trading rules have the same structure on each of these regions. In Section IV, the associated HJB equations and their solutions are studied. Closed-form solutions are obtained. In Section V, verification theorems with sufficient conditions are given. Finally, Section VI concludes the paper with further remarks.

II. Problem Setup

Let X_t denote the price of the asset under consideration at time t. We consider the case when X_t is a regime-switching geometric Brownian motion governed by

$$dX_t = X_t(\mu(\zeta_t)dt + \sigma(\zeta_t)dW_t), \tag{1}$$

where $\zeta_t \in \{1, 2\}$ is a two-state Markov chain, $\mu(1) = \mu_1$, $\mu(2) = \mu_2$ are the expected return rates, $\sigma(1) = \sigma_1$ and $\sigma(2) = \sigma_2$ the volatilities, and W_t is a standard Brownian motion. In this paper, $\zeta_t = 1$ indicates a bull market and $\zeta_t = 2$ a bear market, i.e., $\mu_1 > 0$ and $\mu_2 < 0$. Assume that ζ_t is observable and its generator is given by $Q = \begin{pmatrix} -\eta_1 & \eta_1 \\ \eta_2 & -\eta_2 \end{pmatrix}$, for some $\eta_1 > 0$ and $\eta_2 > 0$. Assume also that $\{\zeta_t\}$ and $\{W_t\}$ are independent.

Remark 2.1: The observability of ζ_t is imposed mainly to simplify the matter to the extent that we can extract useful information without undue technical difficulties. It allows us to formulate/visualize the issue in more depth and is helpful in providing optimality conditions that are otherwise hard to see. In addition, under the best case scenario, we can identify market conditions potentially to avoid trades which might be unprofitable even under the best market information. Finally, the corresponding value functions will provide an upper bound for trading performance which can be used as a general guide to rule out unrealistic expectations.

In this paper, we allow to buy or sell at most one share at a time. Moreover, we consider the case that the net position at any time can be either flat (no stock holding) or long (with one share of stock holding). Let $0 \le b_1 \le s_1 \le b_2 \le s_2 \le \cdots$ be a sequence of stopping times. A buying decision is made at b_n and a selling decision is made at s_n, for $n = 1, 2, \ldots$. Let k_t denote the net position with $k_t = \begin{cases} 0, & \text{flat,} \\ 1, & \text{long one share.} \end{cases}$ If the initial net position is long ($k_{0^-} = 1$), then one should sell the stock before acquiring any share. Similarly, if the initial net position is flat ($k_{0^-} = 0$), then one should first buy a share before a subsequent selling. We define the sequence of stopping times for each initial position k as follows:

$$\begin{array}{ll} \Xi_0 = (b_1, s_1, b_2, s_2, \ldots) & \text{if } k = 0, \\ \Xi_1 = (s_0, b_1, s_1, b_2, s_2, \ldots) & \text{if } k = 1. \end{array}$$

In this paper, we impose slippage cost on each transaction. Slippage cost usually refers to the spread between expected price and the actual price paid. Slippage affects all trading activities especially those with frequent transactions and those with larger orders. In this paper, we assume that a fixed percentage of slippage cost δ is incurred with each transaction. The value for δ depends on the liquidity of the underlying stock. Its normal range is from 0.01% to 1%. Let $r > 0$ be the discount rate. Given the initial states $X_0 = x$, $\zeta_0 = \zeta$, and initial net position $k = 0, 1$, the reward functions of decision sequences, Ξ_k, are given as follows:

$$\begin{aligned} & J_k(x, \zeta, \Xi_k) \\ & = \begin{cases} \mathsf{E} \displaystyle\sum_{i=1}^{\infty} [e^{-rs_i} X_{s_i}(1-\delta) - e^{-rb_i} X_{b_i}(1+\delta)], \\ \hfill \text{if } \ k = 0, \\ \mathsf{E}\{e^{-rs_0} X_{s_0}(1-\delta) \\ \qquad + \displaystyle\sum_{i=1}^{\infty} (e^{-rs_i} X_{s_i}(1-\delta) - e^{-rb_i} X_{b_i}(1+\delta))\}, \\ \hfill \text{if } \ k = 1. \end{cases} \end{aligned}$$

Given initial position k, let $V_k(x, \zeta)$ denote the value functions with the initial states $X_0 = x$ and $\zeta_0 = \zeta$. That is,

$$V_k(x, \zeta) = \sup_{\Xi_k} J_k(x, \zeta, \Xi_k). \tag{2}$$

Let $0 \le b_1^* \le s_1^* \le b_2^* \le s_2^* \le \cdots$ denote the corresponding jump times of ζ_t, i.e., $b_1^* = \inf\{t \ge 0 : \zeta_t = 1\}$, $s_i^* = \inf\{t \ge b_i^* : \zeta_t = 2\}$, and $b_{i+1}^* = \inf\{t \ge s_i^* : \zeta_t = 1\}$ for $i = 1, 2, \ldots$. In the rest of this paper, we focus on the trend-following rule: Buy at b_n^* and sell at s_n^*. In the next section, we find regions for (η_1, η_2) so that the trend-following strategy is optimal.

III. Classification of (η_1, η_2)-Regions and Assumptions

First we note that if $r \geq \mu_1$ then "no trading is optimal." In fact, it is easily seen that for any given Ξ_0,

$$\mathsf{E}e^{-rs_i}X_{s_i} - \mathsf{E}e^{-rb_i}X_{b_i} = \mathsf{E}\int_{b_i}^{s_i} e^{-rt}X_t(-r+\mu(\zeta_t))dt.$$

Note that $(-r+\mu(\zeta_t)) \leq 0$ under $r \geq \mu_1 > 0$ and $\mu_2 < 0$. This implies that

$$\mathsf{E}[e^{-rs_i}X_{s_i}(1-\delta) - e^{-rb_i}X_{b_i}(1+\delta)] \leq 0.$$

It follows that $J_0(x,\zeta,\Xi_0) \leq 0$. Therefore, $V_0(x,\zeta) = 0$. Similarly, $V_1(x,\zeta) = (1-\delta)x$, i.e., one has to sell the share right away at $t = 0$.

Note also that if $r + \eta_1 - \mu_1 \leq 0$, then, in view of (12) developed in Section 4, when $\zeta_0 = 1$, we have $\mathsf{E}e^{-rs_1^*}X_{s_1^*} = x\eta_1\int_0^\infty e^{-(r+\eta_1-\mu_1)u}du = \infty$. In this case, it is easy to see that buying at $(b_1^* = 0)$ and selling at (s_1^*) is optimal because the corresponding payoff $J = \infty$. Similarly, if $\zeta_0 = 2$, it can be seen that buying at (b_2^*) and selling at (s_2^*) gives $J = \infty$.

Assumptions. In this paper, we assume $\mu_1 > r > 0$, $\mu_2 < 0$, and $r + \eta_1 - \mu_1 > 0$.

Next we determine necessary conditions that guarantee the optimality of trend-following-trading rules. We consider the equation $(r+\eta_1-\mu_1)(r+\eta_2-\mu_2) - \eta_1\eta_2 = 0$. Such equation is used in Guo and Zhang [9] to determine the region on which the optimal return is infinite. Let $\beta_1 > \beta_2$ be its roots. Then, in view of Zhang [9, Lemma 1], we have $\lim_{s_1\to\infty}\mathsf{E}e^{-rs_1}X_{s_1} = \infty$, when $\beta_2 < r < \beta_1$ which is equivalent to

$$(r+\eta_1-\mu_1)(r+\eta_2-\mu_2) - \eta_1\eta_2 < 0. \quad (3)$$

In this case, the buy $(b_1 = 0)$ and hold $(s_1 = \infty)$ strategy is optimal and the corresponding payoff $J = \infty$.

Under our trading rule, i.e., buy at b_n^* and sell at s_n^*, in order to generate nonnegative returns, we expect

$$\mathsf{E}[e^{-rs_i^*}X_{s_i^*}(1-\delta) - e^{-rb_i^*}X_{b_i^*}(1+\delta)] \geq 0,\ i = 1,2,\ldots.$$

In particular, if $i = 1$ and $b_1 = 0$, we can show, by writing $X_{s_1^*}$ in terms of ζ_t and W_t (detailed development is given later in this paper in Lemma 5.1), that $\mathsf{E}e^{-rs_1^*}X_{s_1^*} = \frac{\eta_1 x}{r+\eta_1-\mu_1}$. Note that $r+\eta_1-\mu_1 > 0$ when $(r+\eta_1-\mu_1)(r+\eta_2-\mu_2) - \eta_1\eta_2 > 0$. It suffices to require that $\frac{\eta_1}{r+\eta_1-\mu_1} > \frac{1+\delta}{1-\delta}$.

For notational simplicity, define $F_1 = \frac{\eta_1}{r+\eta_1-\mu_1}$ and $F_2 = \frac{\eta_2}{r+\eta_2-\mu_2}$. Using this notation, we construct the following parameter regions.

$$\begin{aligned}
\mathrm{I} &= \left\{(\eta_1,\eta_2) > 0:\ F_1F_2 < 1,\ F_1 > \frac{1+\delta}{1-\delta}\right\},\\
\mathrm{II} &= \left\{(\eta_1,\eta_2) > 0:\ F_1F_2 \leq 1,\ F_1 \leq \frac{1+\delta}{1-\delta}\right\},\\
\mathrm{III} &= \left\{(\eta_1,\eta_2) > 0:\ F_1F_2 \geq 1,\ F_1 > \frac{1+\delta}{1-\delta}\right\},\\
\mathrm{IV} &= \left\{(\eta_1,\eta_2) > 0:\ F_1F_2 > 1,\ F_1 \leq \frac{1+\delta}{1-\delta}\right\}.
\end{aligned}$$

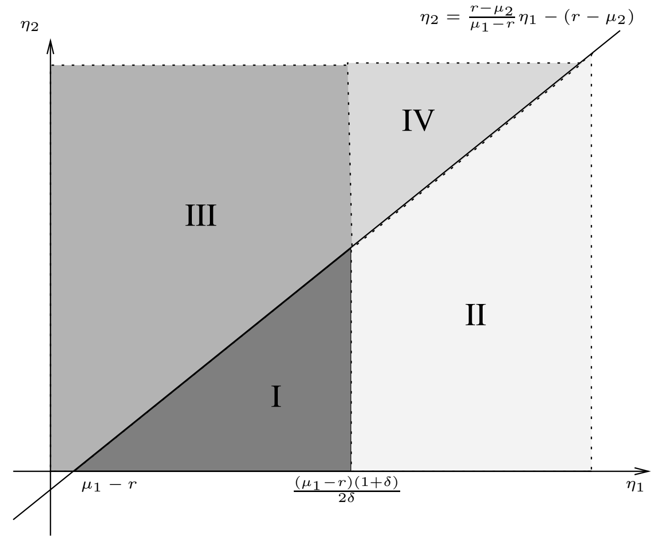

Fig. 1. (η_1, η_2) regions

It is easy to see that these four regions consist of a partition of $\{(\eta_1,\eta_2):\ \eta_1 > 0, \eta_2 > 0\}$, as shown on figure 1.

In the subsequent sections, we will show

- On Region I: Trend following gives the optimal strategies with finite optimal payoff;
- On Region II: No trade is optimal if there is no initial position; otherwise, hold the position till the first time entering a bear market;
- On Region III: Trend following is optimal with infinite optimal payoff. In this case, the buy and hold strategy is also optimal.
- Finally, on Region IV: The buy and hold strategy is optimal. Trend following on the other hand is not optimal.

In the next few sections, we first focus on the optimality of trend-following strategies on Region I. Then we discuss the results on other regions.

IV. HJB Equations

In this section, we study the corresponding HJB equations. Let $\mathcal{A}$ be the generator of (X_t, ζ_t) given by

$$\mathcal{A}f(x,\zeta) = \frac{x^2\sigma^2(\zeta)}{2}\frac{\partial^2}{\partial x^2}f(x,\zeta) + x\mu(\zeta)\frac{\partial}{\partial x}f(x,\zeta) + Qf(x,\cdot)(\zeta)$$

where

$$Qf(x,\cdot)(\zeta) = \begin{cases} \eta_1(f(x,2) - f(x,1)), & \text{if } \zeta = 1,\\ \eta_2(f(x,1) - f(x,2)), & \text{if } \zeta = 2.\end{cases}$$

Formally, the associated system of HJB equations should have the form:

$$\begin{cases} \min\{rv_0(x,\zeta) - \mathcal{A}v_0(x,\zeta), \\ \qquad v_0(x,\zeta) - v_1(x,\zeta) + x(1+\delta)\} = 0, \\ \min\{rv_1(x,\zeta) - \mathcal{A}v_1(x,\zeta), \\ \qquad v_1(x,\zeta) - v_0(x,\zeta) - x(1-\delta)\} = 0. \end{cases} \quad (4)$$

Our trend-following rule says that one should buy at the switching times of bear-to-bull and sell at the switching times of bull-to-bear. In terms of the market trend ζ_t and net position k_t, we have the following strategies. When $\zeta_t = 1$,

$$\begin{array}{ll} \text{if } \; k_t = 0, & \text{buy one share,} \\ \text{if } \; k_t = 1, & \text{hold the share.} \end{array}$$

When $\zeta_t = 2$,

$$\begin{array}{ll} \text{if } \; k_t = 0, & \text{stay flat,} \\ \text{if } \; k_t = 1, & \text{sell one share.} \end{array}$$

Therefore, $v_k(x, \zeta)$ has to satisfy the following conditions to qualify for being solutions to the HJB equations (4):

$$\left\{ \begin{array}{l} rv_1(x,1) - \mathcal{A}v_1(x,1) = 0, \\ rv_0(x,2) - \mathcal{A}v_0(x,2) = 0, \\ v_0(x,1) - v_1(x,1) + x(1+\delta) = 0, \\ v_1(x,2) - v_0(x,2) - x(1-\delta) = 0, \\ rv_0(x,1) - \mathcal{A}v_0(x,1) > 0, \\ rv_1(x,2) - \mathcal{A}v_1(x,2) > 0, \\ v_1(x,1) - v_0(x,1) - x(1-\delta) > 0, \\ v_0(x,2) - v_1(x,2) + x(1+\delta) > 0. \end{array} \right. \tag{5}$$

From the first two equations of (5), we have

$$\left\{ \begin{array}{l} rv_1(x,1) = \dfrac{x^2\sigma_1^2}{2}\dfrac{\partial^2}{\partial x^2}v_1(x,1) + x\mu_1\dfrac{\partial}{\partial x}v_1(x,1) \\ \qquad\qquad +\eta_1(v_1(x,2) - v_1(x,1)), \\ rv_0(x,2) = \dfrac{x^2\sigma_2^2}{2}\dfrac{\partial^2}{\partial x^2}v_0(x,2) + x\mu_2\dfrac{\partial}{\partial x}v_0(x,2) \\ \qquad\qquad +\eta_2(v_0(x,1) - v_0(x,2)). \end{array} \right. \tag{6}$$

Consider the case when the value functions are linear with respect to the initial state x, i.e.

$$\left\{ \begin{array}{lcl} v_1(x,1) & = & A_1 x, \\ v_0(x,2) & = & A_2 x, \end{array} \right. \tag{7}$$

for some $A_1, A_2 \geq 0$. Then by the third and fourth equations of (5), we obtain

$$\left\{ \begin{array}{lcl} v_0(x,1) & = & [A_1 - (1+\delta)]x, \\ v_1(x,2) & = & [A_2 + (1-\delta)]x. \end{array} \right. \tag{8}$$

Substitute (7) and (8) into (6) to get

$$\left\{ \begin{array}{lcl} A_1 & = & \dfrac{\eta_1[(r+\eta_2-\mu_2)(1-\delta) - \eta_2(1+\delta)]}{(r+\eta_1-\mu_1)(r+\eta_2-\mu_2) - \eta_1\eta_2}, \\ \\ A_2 & = & \dfrac{\eta_2[\eta_1(1-\delta) - (r+\eta_1-\mu_1)(1+\delta)]}{(r+\eta_1-\mu_1)(r+\eta_2-\mu_2) - \eta_1\eta_2}. \end{array} \right.$$

It is not difficult to see that these two constants are positive on Region I. In fact, for $(\eta_1, \eta_2) \in \mathrm{I}$, we have

$$(r+\eta_1-\mu_1)(r+\eta_2-\mu_2) - \eta_1\eta_2 > 0, \tag{9}$$

$$\eta_1(1-\delta) - (r+\eta_1-\mu_1)(1+\delta) > 0. \tag{10}$$

Note that $r + \eta_2 - \mu_2 > 0$ as $\mu_2 < 0$, so (9) implies that $r + \eta_1 - \mu_1 > 0$. Consequently,

$$\begin{array}{l} (r+\eta_1-\mu_1)[(r+\eta_2-\mu_2)(1-\delta) - \eta_2(1+\delta)] \\ \quad = (r+\eta_1-\mu_1)(r+\eta_2-\mu_2)(1-\delta) \\ \qquad -(r+\eta_1-\mu_1)\eta_2(1+\delta) \\ \quad > \eta_1\eta_2(1-\delta) - (r+\eta_1-\mu_1)\eta_2(1+\delta) \text{ (By (9))} \\ \quad = \eta_2[\eta_1(1-\delta) - (r+\eta_1-\mu_2)(1+\delta)] \\ \quad > 0 \qquad \text{(By (10)).} \end{array}$$

As a result, we have the following inequality

$$(r+\eta_2-\mu_2)(1-\delta) - \eta_2(1+\delta) > 0, \tag{11}$$

which implies $A_1 > 0$. Similarly, under (9) and (10), it can be shown that $A_1 > 1+\delta$. Therefore, $v_k(x, \zeta) \geq 0$ on Region I.

Next we show that the value functions $v_k(x, \zeta)$ defined in (7) and (8) satisfy the system of HJB equations. It suffices to check the last four inequalities of (5). Indeed,

$$\left\{ \begin{array}{l} rv_0(x,1) - \mathcal{A}v_0(x,1) = \\ \quad [\eta_1(1-\delta) - (r+\eta_1-\mu_1)(1+\delta)]x > 0, \\ rv_1(x,2) - \mathcal{A}v_1(x,2) = \\ \quad [(r+\eta_2-\mu_2)(1-\delta) - \eta_2(1+\delta)]x > 0, \\ v_1(x,1) - v_0(x,1) - x(1-\delta) = 2\delta x > 0, \\ v_0(x,2) - v_1(x,2) + x(1+\delta) = 2\delta x > 0, \end{array} \right.$$

by (9) and (10).

V. A Verification Theorem

We give a verification theorem to show that the solution $v_k(x, \zeta)$ of the HJB equations (4) are equal to the value functions $V_k(x, \zeta)$, and sequences of optimal stopping times can be constructed accordingly. We need the following lemma in the proof of the verification theorem. Recall the jump times of ζ_t defined as $b_1^* = \inf\{t \geq 0 : \zeta_t = 1\}$, $s_i^* = \inf\{t \geq b_i^* : \zeta_t = 2\}$, and $b_{i+1}^* = \inf\{t \geq s_i^* : \zeta_t = 1\}$ for $i = 1, 2, \ldots$. Recall that $F_1 = \frac{\eta_1}{r+\eta_1-\mu_1}$ and $F_2 = \frac{\eta_2}{r+\eta_2-\mu_2}$.

Lemma 5.1: For each $n = 1, 2, \ldots$, we have

$$\left\{ \begin{array}{l} \mathsf{E}e^{-rs_n^*}X_{s_n^*} = \left\{ \begin{array}{ll} (F_1F_2)^{n-1}F_1 x & \text{if } \zeta_0 = 1, \\ (F_1F_2)^n x & \text{if } \zeta_0 = 2, \end{array} \right. \\ \mathsf{E}e^{-rb_n^*}X_{b_n^*} = \left\{ \begin{array}{ll} (F_1F_2)^{n-1} x & \text{if } \zeta_0 = 1, \\ (F_1F_2)^{n-1}F_2 x & \text{if } \zeta_0 = 2. \end{array} \right. \end{array} \right.$$

Proof. Note that

$$X_{s_n^*} = X_{b_n^*} \exp\Big(\int_{b_n^*}^{s_n^*}\Big(\mu_1 - \frac{\sigma_1^2}{2}\Big)dt + \int_{b_n^*}^{s_n^*}\sigma_1 dW_t\Big),$$

$$X_{b_{n+1}^*} = X_{s_n^*} \exp\Big(\int_{s_n^*}^{b_{n+1}^*}\Big(\mu_2 - \frac{\sigma_2^2}{2}\Big)dt + \int_{s_n^*}^{b_{n+1}^*}\sigma_2 dW_t\Big).$$

We first consider the case when $\zeta_0 = 1$. In this case, $b_1^* = 0$. Recall that s_1^* is an exponential random variable with parameter η_1. Moreover, let

$$M_u = \exp\Big(\int_0^u -\frac{\sigma_1^2}{2}\, dt + \int_0^u \sigma_1 dw_t\Big).$$

Then, M_u is a martingale (see Elliott [Thm 13.27] [6]) and it is independent of s_1^*. It follows that, by conditioning on s_1^*,

$$\mathsf{E}e^{-rs_1^*}X_{s_1^*} = x\eta_1\int_0^\infty e^{-(r+\eta_1-\mu_1)u}du. \tag{12}$$

Recall the assumption that $r + \eta_1 - \mu_1 > 0$. We obtain $\mathsf{E}e^{-rs_1^*}X_{s_1^*} = \frac{\eta_1 x}{r+\eta_1-\mu_1} = F_1 x$. Moreover, having right continuous sample paths and being Feller continuous, (X_t, ζ_t) possesses the strong Markov property by virtue of Yin and Zhu [21, p.51, Corollary 2.19]. Thus, we have

$$\mathsf{E}e^{-rb_2^*}X_{b_2^*} \;=\; \frac{\eta_1\eta_2 x}{(r+\eta_2-\mu_2)(r+\eta_1-\mu_1)} = F_1F_2x.$$

Continuing this way, we have

$$\begin{cases} \mathsf{E}e^{-rs_n^*}X_{s_n^*} = (F_1F_2)^{n-1}F_1x, \\ \mathsf{E}e^{-rb_n^*}X_{b_n^*} = (F_1F_2)^{n-1}x. \end{cases}$$

Similarly, if $\zeta_0 = 2$, we can show

$$\begin{cases} \mathsf{E}e^{-rs_n^*}X_{s_n^*} = (F_1F_2)^{n}x \\ \mathsf{E}e^{-rb_n^*}X_{b_n^*} = (F_1F_2)^{n-1}F_2x. \end{cases}$$

The proof is complete. □

Theorem 5.2:. Let $(\eta_1, \eta_2) \in \mathrm{I}$ and

$$\begin{cases} v_1(x,1) &=& A_1x, \\ v_0(x,2) &=& A_2x, \\ v_0(x,1) &=& [A_1-(1+\delta)]x, \\ v_1(x,2) &=& [A_2+(1-\delta)]x, \end{cases} \tag{13}$$

with

$$\begin{cases} A_1 &=& \dfrac{\eta_1[(r+\eta_2-\mu_2)(1-\delta)-\eta_2(1+\delta)]}{(r+\eta_1-\mu_1)(r+\eta_2-\mu_2)-\eta_1\eta_2}, \\ \\ A_2 &=& \dfrac{\eta_2[\eta_1(1-\delta)-(r+\eta_1-\mu_1)(1+\delta)]}{(r+\eta_1-\mu_1)(r+\eta_2-\mu_2)-\eta_1\eta_2}. \end{cases}$$

Then, $v_k(x,\zeta) = V_k(x,\zeta)$, for $k = 0,1$ and $\zeta = 1,2$.

In addition, when $k = 0$, let $\eta_0^* = (b_1^*, s_1^*, b_2^*, s_2^*, \ldots)$, where the stopping times $b_1^* = \inf\{t \geq 0 : \zeta_t = 1\}$, $s_i^* = \inf\{t \geq b_i^* : \zeta_t = 2\}$, and $b_{i+1}^* = \inf\{t \geq s_i^* : \zeta_t = 1\}$ for $i = 1,2,\ldots$. When $k = 1$, let $\eta_1^* = (s_0^*, \eta_0^*)$ with $s_0^* = \inf\{t \geq 0 : \zeta_t = 2\}$. Then η_0^* and η_1^* are optimal.

Proof. The proof is divided into two steps. In the first step, we show that $v_k(x,\zeta) \geq J_k(x, \Xi_{k,\zeta})$ for all $\Xi_{k,\zeta}$. Then in the second step, we show that $v_k(x,\zeta) = J_k(x, \Xi_{k,\zeta}^*)$. Therefore, $v_k(x,\zeta) = V_k(x,\zeta)$ and $\Xi_{k,\zeta}^*$ is optimal.

It is clear that $v_k(x,\zeta)$ satisfy the HJB equations (4). Using $rv_k(x,\zeta) - \mathcal{A}v_k(x,\zeta) \geq 0$ and Dynkin's formula, we have, for any stopping times $0 \leq \theta_1 \leq \theta_2$, a.s.,

$$\mathsf{E}e^{-r\theta_1}v_k(X_{\theta_1}, \zeta_{\theta_1}) \geq \mathsf{E}e^{-r\theta_2}v_k(X_{\theta_2}, \zeta_{\theta_2}),$$

for $k = 0,1$.

Recall that $v_0 \geq v_1 - x(1+\delta)$ and $v_1 \geq v_0 + x(1-\delta)$. Given $\Xi_{k,\zeta} = (b_1, s_1, b_2, s_2, \ldots)$, we have

$$\begin{aligned} v_0(x,\zeta_0) \quad \geq \quad & \mathsf{E}e^{-rb_2}v_0(X_{b_2}, \zeta_{b_2}) + \mathsf{E}e^{-rs_1}X_{s_1}(1-\delta) \\ & -\mathsf{E}e^{-rb_1}X_{b_1}(1+\delta). \end{aligned}$$

Continuing this way and using the fact that $v_k \geq 0$, we get

$$v_0(x,\zeta) \geq \mathsf{E}\sum_{i=1}^{N}\{e^{-rs_i}X_{s_i}(1-\delta) - e^{-rb_i}X_{b_i}(1+\delta)\}.$$

Sending $N \to \infty$, we have $v_0(x,\zeta) \geq J_0(x,\Xi_0)$ for all Ξ_0. This implies that $v_0(x,\zeta) \geq V_0(x,\zeta)$. Similarly, we can show that $v_1(x,\zeta) \geq V_1(x,\zeta)$.

Now we establish the equalities. It is easy to see that $s_i^* < \infty$ and $b_i^* < \infty$, a.s. Recall that $v_0(x,1) = v_1(x,1) - x(1+\delta)$ and $v_1(x,2) = v_0(x,2) + x(1-\delta)$. We have

$$\begin{aligned} & v_0(x,\zeta) \\ = \quad & \mathsf{E}e^{-rb_1^*}v_0(X_{b_1^*}, \zeta_{b_1^*}) \\ = \quad & \mathsf{E}e^{-rs_1^*}v_0(X_{s_1^*}, \zeta_{s_1^*}) + \mathsf{E}[e^{-rs_1^*}X_{s_1^*}(1-\delta) \\ & -e^{-rb_1^*}X_{b_1^*}(1+\delta)]. \end{aligned}$$

Continuing this way, we obtain

$$\begin{aligned} v_0(x,\zeta) &= \mathsf{E}e^{-rs_N^*}v_0(X_{s_N^*}, \zeta_{s_N^*}) \\ &+\mathsf{E}\sum_{i=1}^{N}[e^{-rs_i^*}X_{s_i^*}(1-\delta) - e^{-rb_i^*}X_{b_i^*}(1+\delta)]. \end{aligned} \tag{14}$$

Similarly, we have

$$\begin{aligned} v_1(x,\zeta) &= \mathsf{E}e^{-rs_N^*}v_0(X_{s_N^*}, \zeta_{s_N^*}) + \mathsf{E}[e^{-rs_0^*}X_{s_0^*}(1-\delta)] \\ &+\mathsf{E}\sum_{i=1}^{N}[e^{-rs_i^*}X_{s_i^*}(1-\delta) - e^{-rb_i^*}X_{b_i^*}(1+\delta)]. \end{aligned}$$

Finally, it remains to show that $\mathsf{E}e^{-rs_N^*}v_0(X_{s_N^*}, \zeta_{s_N^*}) \to 0$ as $N \to \infty$. This follows from $v_0(X_{s_N^*}, \zeta_{s_N^*}) = A_2X_{s_N^*}$ and Lemma 5.1 together with the condition $(r+\eta_1-\mu_1)(r+\eta_2-\mu_2) > \eta_1\eta_2$. This completes the proof. □

Region II:

We show in this subsection that the trend following strategy is *not* optimal on Region II. This is mainly because η_1 is too large which lead to the short lived subsequent bull markets. Let

$$\begin{cases} v_0(x,1) = 0; \\ v_0(x,2) = 0; \\ v_1(x,1) = \dfrac{\eta_1(1-\delta)}{r+\eta_1-\mu_1}x; \\ v_1(x,2) = (1-\delta)x. \end{cases}$$

It is direct to show that these functions solve the HJB equations (4). Moreover, it can be shown similarly as in Theorem 5.2 that $v_k(x,\zeta) \geq V_k(x,\zeta)$. Furthermore, we consider the following strategies: If there is no existing position, do not trade; If the initial holding is one share, then sell it right away in a bear market and, if in a bull market, hold it till the end of the bull market and then sell it. It is easy to see that the corresponding payoff is given by $v_k(x,\zeta)$. Therefore, they are indeed the value functions and the above strategy is optimal.

Region III:

Note that on this region, $F_1F_2 \geq 1$ and $F_1(1-\delta) - (1+\delta) > 0$. It follows that, in view of Lemma 5.1,

$$\begin{aligned} & J_0(x,1,\Xi_0^*) \\ &= \sum_{i=1}^{\infty}\left[\mathsf{E}e^{-rs_i^*}X_{s_i^*}(1-\delta) - \mathsf{E}e^{-rb_i^*}X_{b_i^*}(1+\delta)\right] \\ &= \sum_{i=1}^{\infty}\left[(F_1F_2)^{i-1}F_1(1-\delta)x - (F_1F_2)^{i-1}(1+\delta)x\right] \\ &= \sum_{i=1}^{\infty}(F_1F_2)^{i-1}\left[F_1(1-\delta) - (1+\delta)\right]x = \infty. \end{aligned} \tag{15}$$

Similarly, we have

$$\begin{aligned} & J_0(x,2,\Xi_0^*) \\ &= \sum_{i=1}^{\infty}\left[(F_1F_2)^{i}(1-\delta)x - (F_1F_2)^{i-1}F_2(1+\delta)x\right] \\ &= \sum_{i=1}^{\infty}(F_1F_2)^{i-1}F_2\left[F_1(1-\delta) - (1+\delta)\right]x = \infty. \end{aligned}$$

Note also that $\Xi_1^* = (s_0^*, \Xi_0^*)$. Therefore, we have

$$J_1(x, \zeta, \Xi_1^*) \geq J_0(x, \zeta, \Xi_0^*) = \infty.$$

It follows that the trend-following strategies are optimal and $V_k(x, \zeta) = \infty$. Also, the buy and hold strategy is optimal as noted in §3 when $F_1 F_2 < 1$. In this case the value function $V = \infty$.

Region IV:

It is clear that the buy and hold is optimal on this region and the corresponding payoff $J = \infty$, which in turn implies $V = \infty$.

We next show that the trending following is not optimal. Note that on this region, $F_1 F_2 > 1$ and $F_1(1-\delta) - (1+\delta) \leq 0$. Using (15), we have

$$\begin{aligned} J_0(x, 1, \Xi_0^*) &= \sum_{i=1}^{\infty} (F_1 F_2)^{i-1} \left[F_1(1-\delta) - (1+\delta)\right] x \\ &\leq 0. \end{aligned}$$

Similarly, $J_0(x, 2, \Xi_0^*) \leq 0$. Moreover, the trend-following strategy gives

$$\begin{aligned} &J_1(x, 1, \Xi_1^*) \leq \mathsf{E} e^{-r s_0^*} X_{s_0^*}(1-\delta) = \frac{\eta_1 x(1-\delta)}{r + \eta_1 - \mu_1}, \\ &J_1(x, 2, \Xi_1^*) \leq x(1-\delta). \end{aligned}$$

Therefore, $J_k < \infty$, $k = 0, 1$. This means that the trend following strategy is not optimal.

Remark 5.3: In general, a trend-following strategy requires the bull markets to last long enough to be profitable. In this paper, $1/\eta_1$ represents the expected duration of a bull market. If $\eta_1 > (\mu_1 - r)(1+\delta)/(2\delta)$ as shown in Figure 1, then $1/\eta_1$ is too small for the trend following to be profitable. This is very counter intuitive because even buy-and-hold will produce greater returns ($V_k(x, \zeta) = \infty$) on Region IV!

VI. Concluding Remarks

This paper treated a basic trend following strategy and established regions on which the traditional trend following can be profitable. It also identified regions with negative returns under the trend-following strategy, which are the regions that a trend-following investor should avoid. It would be interesting to extend the results to incorporate more realistic scenarios including the case of the market trends being not completely observable and/or with possible delays, and the case for trading a basket of stocks.

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References

[1] N.P.B. Bollen, Valuing options in regime-switching models, *Journal of Derivatives*, vol. 6, pp. 38-49, (1998).

[2] J. Buffington and R.J. Elliott, American options with regime switching, *International Journal of Theoretical and Applied Finance*, vol. 5, pp. 497-514, (2002).

[3] A. Cadenillas and S.R. Pliska, Optimal trading of a security when there are taxes and transaction costs, *Finance & Stochastics*, Vol. 3, pp. 137-165, (1999).

[4] G.M. Constantinides, Capital market equilibrium with personal tax, *Econometrica*, Vol. 51, pp. 611-636, (1983).

[5] R.M. Dammon and C.S. Spatt, The optimal trading and pricing of securities with asymmetric capital gains taxes and transaction costs, *Rev. Financial Studies*, Vol. 9, pp. 921-952, (1996).

[6] R.J. Elliott, *Stochastic Calculus and Applications*, Springer-Verlag, New York, 1982.

[7] G.B. Di Masi, Y.M. Kabanov and W.J. Runggaldier, Mean variance hedging of options on stocks with Markov volatility, *Theory of Probability and Applications*, vol. 39, pp. 173-181, (1994).

[8] X. Guo, *Inside Information and Stock Fluctuations*, Ph.D. thesis, Rutgers University, 1999.

[9] X. Guo and Q. Zhang, Optimal selling rules in a regime switching model, *IEEE Trans. Automatic Control*, Vol. 50, pp. 1450-1455, (2005).

[10] J.D. Hamilton, A new approach to the economic analysis of non-stationary time series, *Econometrica*, vol. 57, pp. 357-384, (1989).

[11] K. Helmes, Computing optimal selling rules for stocks using linear programming, Mathematics of Finance, G. Yin and Q. Zhang, (Eds), *Contemporary Mathematics*, American Mathematical Society, pp. 187-198, (2004).

[12] T.C. Johnson and M. Zervos, The optimal timing of investment decisions, *Working Paper*, (2006).

[13] H.T. Kong and Q. Zhang, An optimal trading rule of a mean-reverting asset, *Discrete and Continuous Dynamical System Series B*, Vol. 14, Number 4, pp.1403-1417, (2010).

[14] H.T. Kong, Q. Zhang, and G. Yin, A trend-following strategy: Conditions for optimality, *Automatica*, **47** (2011), 661–667.

[15] R. Liu, G. Yin, and Q. Zhang, Option pricing in a regime switching model using the fast Fourier transform, *Applied Mathematics and Stochastic Analysis*, Vol. 2006, Article ID 18109, 22 pages, 2006. doi:10.1155/JAMSA/2006/18109.

[16] A. Løkka and M. Zervos, Long-term optimal real investment strategies in the presence of adjustment costs, preprint, (2007).

[17] A. Merhi and M. Zervos, A model for reversible investment capacity expansion, *SIAM J. Control Optim.*, Vol. 46, pp. 839-876, (2007).

[18] B. Øksendal, *Stochastic Differential Equations*, 6th Ed., Springer-Verlag, New York, 2003.

[19] D.D. Yao, Q. Zhang, and X.Y. Zhou, A regime-switching model for European options, *Stochastic Processes, Optimization, and Control Theory Applications in Financial Engineering, Queueing Networks, and Manufacturing Systems*, H. Yan, G. Yin and Q. Zhang (Eds.), in International Series in Operations Research and Management Sciences, Springer-SBM, New York, 2006.

[20] G. Yin, R.H. Liu, and Q. Zhang, Recursive algorithms for stock Liquidation: A stochastic optimization approach, *SIAM J. Optim.*, Vol. 13, pp. 240-263, (2002).

[21] G. Yin and C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*, Springer, New York, 2010.

[22] H. Zhang and Q. Zhang, Trading a mean-reverting asset: Buy low and sell high, *Automatica*, Vol. 44. pp. 1511-1518, (2008)

[23] Q. Zhang, Stock trading: An optimal selling rule, *SIAM J. Contr. Optim.*, Vol. 40, pp. 64-87, (2001).

[24] Q. Zhang and G. Yin, Nearly optimal asset allocation in hybrid stock-investment models, *J. Optim. Theory Appl.*, vol. 121, pp. 419-444, (2004).

[25] X.Y. Zhou and G. Yin, Markowitz's mean-variance portfolio selection with regime switching: A continuous-time model, *SIAM Journal on Control and Optimization*, vol. 42, pp. 1466-1482, (2003).