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Control of Some Partially Observed Linear Stochastic Systems with Fractional Brownian Motions

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Abstract— **A control problem is formulated for a linear stochastic system with noisy, partial observations of the state and a cost that is a quadratic functional of the state and the control. Both the system noise and the observation noise can be fractional Brownian motions. An optimal control is explicitly described and this control is compared to the well known optimal control where the state and the observation noises are Brownian motions.**

Key Words: linear quadratic Gaussian control, fractional Brownian motion, partially observed linear stochastic systems, linear regulator.

I. Introduction

The linear-quadratic Gaussian (LQG) control problem for the control of a completely observed linear stochastic system with a Brownian motion (white Gaussian noise) and a quadratic cost functional of the state and the control (e.g. [6]) is the most well known and basic solvable stochastic control problem for stochastic systems with continuous sample paths. Similarly the linear-quadratic Gaussian control problem for the control of a partially observed linear stochastic (state) equation with a Brownian motion, linear observations with an additive Brownian motion (white Gaussian noise) and a quadratic cost functional of the state and the control is the most well known, solvable partially observed control problem for stochastic systems with continuous sample paths. The noise or perturbations of a system are typically modeled by a Brownian motion because such a process is Gauss-Markov and has independent increments. However empirical data from many physical phenomena suggest that Brownian motion is often inappropriate to use in the mathematical models of these phenomena. A family of processes that has empirical evidence of wide physical applicability is the collection of fractional Brownian motions. This collection of fractional Brownian motions is a family of Gaussian processes that was defined by Kolmogorov [10] in his study of turbulence. While this family of processes includes Brownian motion, it also includes other processes that describe behavior that is bursty or has a long range dependence. All of the processes in this family except Brownian motion are neither Markov nor semimartingales. The first empirical evidence of the usefulness of these latter processes was provided by Hurst [7] in his statistical analysis of rainfall in the Nile Basin. Subsequently empirical justifications for modeling with fractional Brownian motions have been noted for a wide variety of physical phenomena, such as economic data, flicker noise in electronic devices, turbulence, internet traffic, biology, and medicine.

Since fractional Brownian motions (FBMs) have a wide variety of potential applications, it is natural to consider the control of a linear stochastic system with an FBM and a quadratic cost functional. It is natural to call such problems, linear-quadratic fractional Gaussian (LQFG) control. Some initial work has been done on these problems. Kleptsyna, Le Breton and Viot [8] consider a scalar linear stochastic system with the index (Hurst parameter) for the FBMs restricted to $(1/2, 1)$ instead of the full family with index set $(0, 1)$. Duncan and Pasik-Duncan [3], [4] consider a multidimensional linear stochastic system with an FBM having an arbitrary Hurst parameter. Duncan, Maslowski, and Pasik-Duncan [2] consider a linear-quadratic control

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problem for systems in an infinite dimensional Hilbert space with fractional Brownian motions with the Hurst parameter in $(\frac{1}{2}, 1)$. These latter systems can model controlled linear stochastic partial differential equations in particular where the noise or the control can be restricted to the boundary of the domain or to discrete points in the domain. All of these aforementioned results on control assume that complete observations of the state are available to the controller. Kleptsyna, Le Breton and Viot [9] solved a control problem for a scalar partially observed linear stochastic system with a fractional Brownian motion having the Hurst parameter $H \in (\frac{1}{2}, 1)$.

In this paper a linear-quadratic control problem is formulated and solved where the linear stochastic system is driven by a fractional Brownian motion and only noisy partial observations are available for the controller. This problem is solved by giving an explicit description of an optimal control. The approach to solving the control problem initially finds an optimal control that is not adapted to the observations or the state and then the constraint of the control being adapted to the observations is introduced.

A brief outline of the paper is given now. In Section II some information on the family of fractional Brownian motions is given that includes some elementary stochastic calculus for these processes. Furthermore the controlled linear system, the partial observations, the quadratic cost functional, and the family of admissible controls are given. In Section III the main result for an optimal control is given for the case of a fractional Brownian motion and the optimal cost is also given. A generalization to other noise processes is noted. In Section IV some concluding remarks are made.

II. Preliminaries

The collection of fractional Brownian motions (FBMs) is a family of Gaussian processes that is indexed by the Hurst parameter $H \in (0,1)$. Initially the definition for an FBM is given.

Definition 1: Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and let $H \in (0,1)$ be fixed. On this probability space a real-valued standard fractional Brownian motion, $(B(t), t \geq 0)$, with Hurst parameter H is a Gaussian process with continuous sample paths such that

$$\mathbb{E}[B(t)] = 0$$
$$\mathbb{E}[B(s)B(t)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

for all $s,t \in \mathbb{R}_+$.

An $\mathbb{R}^n$-valued standard fractional Brownian motion, $(B(t), t \geq 0)$, with Hurst parameter H is an n-vector of independent real-valued standard fractional Brownian motions with the same Hurst parameter H. If B is an FBM with $H = 1/2$ then B is a Brownian motion.

Fix $H \in (\frac{1}{2}, 1)$ and let $f : \to \mathbb{R}^n$ be a Borel measurable function such that

$$< f, f >_H = \int_0^T \int_0^T f(s) f(r) \phi_H(s-r) dr ds < \infty \qquad \text{(II.1)}$$

where $\phi_H(s) = H(2H-1)|s|^{2H-2}$. Then $\int_0^T f dB$ is a zero mean Gaussian random variable with second moment

$$\mathbb{E}|\int_0^T f dB|^2 = < f, f >_H \qquad \text{(II.2)}$$

Consider the control system given by the following controlled linear stochastic differential equation with a fractional Brownian motion

$$\begin{aligned} dX(t) &= (AX(t) + CU(t))dt + dB(t) \qquad \text{(II.3)} \\ X(0) &= X_0 \end{aligned}$$

where X_0 is an $N(0,\Sigma)$ $\mathbb{R}^n$-valued random variable, $X(t) \in \mathbb{R}^n, U(t) \in \mathbb{R}^m, A \in L(\mathbb{R}^n, \mathbb{R}^n), C \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $(B(t), t \geq 0)$ is an $\mathbb{R}^n$-valued standard fractional Brownian motion whose components $(B_1, ..., B_n)$ are independent real-valued standard fractional Brownian motions each with the Hurst parameter $\bar{H} \in (\frac{1}{2}, 1)$. The (normal) $N(0,\Sigma)$-random vector X_0 and the process B are independent and are defined on the complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

The observation process $(Y(t), t \in [0,T])$ satisfies the following stochastic equation

$$dY(t) = FX(t)dt + dV(t) \quad \text{(II.4)}$$

$$Y(0) = 0 \quad \text{(II.5)}$$

where $F \in L(\mathbb{R}^n, \mathbb{R}^p)$ and $(V(t), t \geq 0)$ is an $\mathbb{R}^P$-valued standard fractional Brownian motion whose components $(V_1, ..., V_p)$ are independent real-valued fractional Brownian motions each with the Hurst parameter $\widehat{H} \in (\frac{1}{2}, 1)$ and are also defined on $(\Omega, \mathscr{F}, \mathbb{P})$. It is assumed that $X_0, B,$ and V are independent. The quadratic cost functional J is

$$J(U) = \frac{1}{2}E[\int_0^T < QX(s), X(s) > + < RU(s), U(s) > ds] + \frac{1}{2}E < MX(T), X(T) > \quad \text{(II.6)}$$

where $Q > 0, R > 0$, and $M \geq 0$ are symmetric linear transformations, $T > 0$ is fixed, and $<,>$ is the standard inner product on the appropriate Euclidean space. Let $(\mathscr{G}(t), t \in [0,T])$ be the filtration for the observations $(Y(t), t \in [0,T])$ so that $\mathscr{G}(t)$ is the $\mathbb{P}$ completion of $\sigma(Y(s), s \in [0,t])$. This filtration for the observations $(\mathscr{G}(t), t \in [0,T])$ can be defined for the control $U \equiv 0$. The equality of this observation filtration and the observation filtration for the optimal control can be verified because the equation is linear ([5], [11]). The family of admissible controls $\mathscr{U}$ is defined as

$\mathscr{U} = \{U : U$ is an $\mathbb{R}^m$-valued process adapted to $(\mathscr{G}(t), t \in [0,T])$ such that $U \in L^2([0,T])$ a.s$\}$

III. Main Result

The following theorem provides a solution to the control problem (II.3), (II.4), and (II.6). It generalizes the result in [9] by solving the problem for multidimensional linear systems, by using more direct methods to find an optimal control and the associated optimal cost and by being applicable to other continuous processes.

Theorem 1: For the control problem described by (II.3), (II.4), and (II.6), there is an optimal control $(U^*(t), t \in [0,T])$ in $\mathscr{U}$ that is given by

$$U^*(t) = -R^{-1}C^T(P(t)\widehat{X}(t|t) + \widehat{\phi}(t|t)) \quad \text{(III.1)}$$

where $(P(t), t \in [0,T])$ is the unique, symmetric, positive definite solution of the Riccati equation

$$\frac{dP}{dt} = -PA - A^TP + PCR^{-1}C^TP - Q \quad \text{(III.2)}$$
$$P(T) = M$$

and

$$\widehat{X}(t|t) = \mathbb{E}[X(t)|\mathscr{G}(t)] \quad \text{(III.3)}$$

$$\phi(t) = \int_t^T \Phi_P(s,t)P(s)dB(s) \quad \text{(III.4)}$$

$$\widehat{\phi}(t|t) = \mathbb{E}[\phi(t)|\mathscr{G}(t)] \quad \text{(III.5)}$$

and Φ_P satisfies the following equation

$$\frac{d\Phi_P(s,t)}{dt} = -(A^T - P(t)CR^{-1}C^T)\Phi_P(s,t) \quad \text{(III.6)}$$

$$\Phi_(s,s) = I \quad \text{(III.7)}$$

The optimal cost is

$$J(U^*) = \frac{1}{2}\mathbb{E}\int_0^T |R^{-1}C^T(P(t)\widetilde{X}(t|t) + \widetilde{\phi}(t|t)|^2 dt + \int_0^T \int_0^s tr(P(s)\Phi_P^T(s,r))\phi_{\bar{H}}(s-r)drds + \frac{1}{2}tr(P(0)\Sigma) - \frac{1}{2}\mathbb{E}\int_0^T |R^{-\frac{1}{2}}C^T\phi|^2 dt \quad \text{(III.8)}$$

where $\phi_{\bar{H}}(s) = \bar{H}(2\bar{H}-1)|s|^{2\bar{H}-2}$, $\widetilde{X}(t|t) = X(t) - \widehat{X}(t|t)$, and $\widetilde{\phi}(t|t) = \phi(t) - \widehat{\phi}(t|t)$.

Proof: (Sketch). The following presentation describes the major ideas of the proof. Let $(B(t), t \geq 0)$ be the $\mathbb{R}^n$-valued standard fractional Brownian motion in (II.3). For each $n \in \mathbb{N}$, let $T_n = \{t_j^{(n)}, j \in \{0, ..., n\}\}$ be a partition of $[0,T]$ such that $0 = t_0^{(n)} < t_1^{(n)}, ... < t_n^{(n)} = T$. Assume that $T_{n+1} \supset T_n$ for each $n \in \mathbb{N}$ and that the sequence $(T_n, n \in \mathbb{N})$ becomes dense in $[0,T]$. For each $n \in \mathbb{N}$, let $(B_n(t), t \in [0,T])$ be the piecewise linear process obtained from $(B(t), t \in [0,T])$ and T_n as follows

$$B_n(t) = [B(t_j^{(n)}) + \frac{B(t_{j+1}^{(n)}) - B(t_j^{(n)})}{t_{j+1}^{(n)} - t_j^{(n)}}(t - t_j^{(n)})]\mathbb{1}_{[t_j^{(n)}, t_{j+1}^{(n)})}(t)$$

Initially a "nonadapted" control problem for the controlled system (II.3) with $(B(t), t \in [0,T])$ replaced by $(B_n(t), t \in [0,T], n \in \mathbb{N})$ is solved for almost all sample paths of B_n and the solution of (II.3) with B_n is considered as a translation of the deterministic linear system without B_n.

Let $(B_n(t), t \in [0,T], n \in \mathbb{N})$ be the sequence of processes above that converges uniformly almost surely to $(B(t), t \in [0,T])$. Fix $n \in \mathbb{N}$ and consider a sample path of B_n. For this sample path the dependence on $\omega \in \Omega$ is suppressed for notational convenience.

Let $(\phi_n(t), t \in [0,T])$ be the solution of the linear differential equation

$$\begin{aligned} \frac{d\phi_n}{dt} &= -[(A^T - P(t)CR^{-1}C^T)\phi_n(t) \quad \text{(III.9)} \\ &+ P(t)\frac{dB_n}{dt}] \\ \phi_n(T) &= 0 \\ \phi_n(t) &= \int_t^T \Phi_P(s,t)P(s)dB_n(s) \quad \text{(III.10)} \end{aligned}$$

This differential equation is defined for almost all t and its solution is well defined from the results for linear ordinary differential equations.

By a completion of squares method for deterministic affine systems (e.g. [12]) it can be shown that

$$\begin{aligned} & J_n^0(U) - \frac{1}{2} < P(0)X_0, X_0 > - < \phi_n(0), X_0 > \\ = & \frac{1}{2}[\int_0^T (< RU, U > + < PCR^{-1}C^T PX_n, X_n > \\ + & 2 < C^T PX_n, U > + 2 < PCR^{-1}C^T \phi_n, X_n > \\ + & 2 < \phi_n, CU >)dt + \int_0^T 2 < \phi_n, dB_n >] \end{aligned}$$

$$\begin{aligned} = & \frac{1}{2}\int_0^T [(|R^{-1/2}[RU + C^T PX_n + C^T \phi_n]|^2 \\ & -|R^{-1/2}C^T \phi_n|^2)dt + 2 < \phi_n, dB_n >] \quad \text{(III.11)} \end{aligned}$$

$$\begin{aligned} = & \frac{1}{2}[\int_0^T (< R^{-1/2}(RU + C^T PX_n + C^T \phi_n), R^{-1/2}(RU \\ & + C^T PX_n + C^T \phi_n) > \\ & - < R^{-1/2}C^T \phi_n, R^{-1/2}C^T \phi_n >)dt + 2 < \phi_n, dB_n >] \end{aligned}$$

where

$$\begin{aligned} J_n^0(U) &= \frac{1}{2}\int_0^T < QX_n(s), X_n(s) > + < RU(s), U(s) > ds \\ &+ \frac{1}{2} < MX_n(T), X_n(T) > \end{aligned}$$

Since the arbitrary control U only appears in the first term of (III.11) and this term is quadratic, an optimal control U_n^* is

$$U_n^*(t) = -R^{-1}(C^T P(t)X_n(t) + C^T \phi_n(t)) \quad \text{(III.12)}$$

This optimal control is satisfied for almost all $\omega \in \Omega$.

Letting $n \to \infty$ in (III.11) it can be shown that $(\widehat{U}(t), t \in [0,T])$ is an optimal, nonadapted control for (II.3), (II.6) where

$$\widehat{U}(t) = R^{-1}C^T(P(t)X(t) + \phi(t)) \quad \text{(III.13)}$$

Now it is shown that $(U^*(t), t \in [0,T])$ is an optimal control in $\mathscr{U}$ where

$$\begin{aligned} U^*(t) &= -R^{-1}C^T(P(t)\widehat{X}(t|t) + \widehat{\phi}(t|t)) \quad \text{(III.14)} \\ \widehat{X}(t|t) &= \mathbb{E}[X(t)|\mathscr{G}(t)] \quad \text{(III.15)} \\ \widehat{\phi}(t|t) &= \mathbb{E}[\phi(t)|\mathscr{G}(t)] \quad \text{(III.16)} \end{aligned}$$

It follows that

$$\begin{aligned} & \lim_{t\to\infty} \mathbb{E}J_n^0(U) \quad \text{(III.17)} \\ = & \frac{1}{2}tr(P(0)\Sigma) + \frac{1}{2}\mathbb{E}\int_0^T [(|R^{-1/2}[RU + C^T PX + C^T \phi]|^2 \\ & -|R^{-1/2}C^T \phi|^2)dt] + \lim_{n\to\infty} \mathbb{E}\int_0^T < \phi_n, dB_n > \end{aligned}$$

It follows from the relation between stochastic integrals of Ito-type and Statonovich-type for a fractional Brownian motion [1] that

$$\begin{aligned} & \lim_{n\to\infty} \mathbb{E}\int_0^T \phi_n dB_n \quad \text{(III.18)} \\ = & \int_0^T \int_0^s tr(P(s)\Phi_P^T(s,r))\phi_{\bar{H}}(s-r)drds \end{aligned}$$

where

$$\phi_{\bar{H}}(s) = \bar{H}(2\bar{H}-1)|s|^{2\bar{H}-2} \quad \text{(III.19)}$$

Let Π be the orthogonal projection given by

$$\Pi : L^2([0,T]\times\Omega,\mathbb{R}^m) \to L^2_{\mathscr{G}}(\mathbb{R}^m) \qquad \text{(III.20)}$$

where $L^2_{\mathscr{G}}(\mathbb{R}^m)$ is the family of $\mathbb{R}^m$-valued $(\mathscr{G}(t), t\in[0,T])$ adapted (predictable) processes and let $\Pi^\perp = I - \Pi$ be the complementary orthogonal projection. By the result for nonadapted controls it follows that

$$\begin{aligned} & J(U) - \frac{1}{2}\mathbb{E} < P(0)X_0, X_0 > \qquad \text{(III.21)} \\ = \quad & \frac{1}{2}\mathbb{E}\int_0^T [|R^{-\frac{1}{2}}(RU + \Pi C^T PX + \Pi C^T\phi|^2 \\ & +|\Pi^\perp(R^{-1}C^T PX + R^{-1}C^T\phi|^2]dt \\ & +\int_0^T\int_0^s tr(P(s)\Phi_P(s,r))\phi_{\bar{H}}(s-r)drds \\ & -\frac{1}{2}\mathbb{E}\int_0^T |R^{-\frac{1}{2}}C^T\phi|^2 dt \end{aligned}$$

Clearly the RHS of (III.21) is minimized in $L^2_{\mathscr{G}}(\mathbb{R}^m)$ by choosing the control

$$\begin{aligned} U^*(t) &= R^{-1}C^T P(t)\Pi X(t) - R^{-1}C^T\Pi\phi(t) \\ &= R^{-1}C^T P(t)\widehat{X}(t|t) - R^{-1}C^T\widehat{\phi}(t|t) \quad \text{(III.22)} \end{aligned}$$

because the family of orthogonal projections is conditional expectation with respect to $(\mathscr{G}(t), t\in[0,T])$. The optimal cost follows directly from (III.21) as

$$\begin{aligned} J(U^*) = & \frac{1}{2}\mathbb{E}\int_0^T |R^{-1}C^T(P(t)\widetilde{X}(t|t) + \widetilde{\phi}(t|t)|^2 dt \quad \text{(III.23)} \\ & +\int_0^T\int_0^s tr(P(s)\Phi_P^T(s,r))\phi_H(s-r)drds \\ & +\frac{1}{2}tr(P(0)\Sigma) - \frac{1}{2}\mathbb{E}\int_0^T |R^{-\frac{1}{2}}C^T\phi|^2 dt \end{aligned}$$

■

Remark 1: The optimal control (III.21) has a similar structure as the case of complete observations for the control of a linear system with a fractional Brownian motion [3], [4], that is, the optimal control is the sum of two terms, one is the well known linear feedback control and the other is a prediction of the response of the adjoint system to the future fractional noise. On the other hand, the optimal control has a similar structure as the case for the control of partially observed linear stochastic systems with Brownian motions, that is, an estimate of the state, the conditional mean, is used to replace the state of the system in the optimal feedback control and the other term arises because the increments of the fractional Brownian motion are correlated.

IV. Concluding Remarks

The optimal control for the partially observed linear-quadratic control problem (II.3), (II.4), (II.6) has a natural intuitive structure as a sum of the linear state feedback for the case of Brownian motions and a prediction of the optimal adjoint system response to the future noise based on the noisy observations. This result shows that the requirement of Markov processes for stochastic control can be relaxed and that an explicit solution for an optimal control of a partially observed linear system can be obtained as a natural generalization of the corresponding completely observed problem. The generalization to a vector of real-valued fractional Brownian motions with different Hurst parameters in $(\frac{1}{2}, 1)$ follows directly. Using some other methods, the results can be extended to arbitrary standard fractional Brownian motions, that is, $H \in (0,1)$. Furthermore these results can be extended to other noise processes with continuous sample paths. The numerical studies of the computation of the conditional mean for the optimal control and the response of the optimal system apparently have not been investigated but are important for future study.

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