2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Finite Time Stabilization of Perturbed Double Integrator with Jumps in Velocity

Harshal B. Oza, Yury V. Orlov and Sarah K. Spurgeon

***Abstract*— In this paper, finite time stabilization of a perturbed double integrator is considered, incorporating jumps in the velocity at the unstable equilibrium. Rigid body inelastic impacts are considered. A robust control synthesis is presented in the presence of uniformly bounded persistent disturbances. The second order sliding mode (twisting) controller is utilized. Firstly, a non-smooth state transformation is employed to transform the original system into a jump-free system. The transformed system is shown to be a switched homogeneous system with negative homogeneity degree whose solutions are well-defined. Secondly, a non-smooth Lyapunov function is identified to establish uniform asymptotic stability of the transformed system. The global finite time stability then follows from the homogeneity principle of switched systems. Thus, using a single Lyapunov function, the global finite time stability of the origin of the system with velocity jumps is established without having to analyze the Lyapunov function at the jump instants. A finite upper bound on the settling time is also computed.**

I. INTRODUCTION

The study of discontinuous systems has received considerable interest amongst control theorists and practitioners. Discontinuous systems are studied in very different research fields such as economics, electrical circuit theory, mechanical engineering, biosciences, systems and control theory. Many different frameworks therefore exist to describe various classes of discontinuous systems, for example, differential inclusions [1], measure differential inclusions [2] and complementarity systems [3] to name a few. Discontinuities appear either due to the very nature of the system dynamics or due to discontinuous feedback control. A survey article on discontinuous dynamical systems can be found in [4], which discusses various kinds of discontinuities, their respective solutions concepts and stability tools available in the literature. The monograph [3] details methods for specifying solutions, the Lyapunov stability framework and control synthesis for non-smooth mechanical systems with friction and collision. The rigorous theoretical developments in the theory of non-smooth mechanics have been accompanied by applications such as biped robots [5], [6], [7] thereby emphasizing the importance of control theoretic aspects.

The main focus of this paper is on mechanical systems with resets in velocity. A linear double integrator is considered with jumps in its velocity when it hits a constraint surface. The velocity undergoes an instantaneous jump when the inelastic collision occurs. It is assumed in this paper that the restitution in velocity, representing loss of energy which occurs at the time of impact, is fully known. It is clear from the existing Lyapunov stability frameworks [3], [8] that a jump in the Lyapunov function occurs whenever the velocity undergoes a jump. Therefore the Lyapunov stability needs to be specifically defined for all possible jumps in the Lyapunov function. An alternative approach is followed in this paper. First, a non-smooth state transformation [3] is utilized to render a jump-free system with its solutions clearly defined in the sense of Filippov [1]. As an immediate consequence, the resulting transformed system turns out to be a valid candidate for the stability analysis via smooth and non-smooth Lyapunov functions. A non-smooth Lyapunov function is identified for proving global uniform asymptotic stability. In turn, the quasi-homogeneity principle [9] is shown to be applicable to the transformed system which, while being locally homogeneous with negative homogeneity degree, proves to be finite time stable.

The work presented contains several contributions. Firstly, although results exist for asymptotic stabilization of continuous and discrete dynamics [8], finite time stabilization in the presence of velocity jumps is a novel concept. A finite upper bound on the settling time is also computed. Secondly, the presented method, substantiated by a single non-strict Lyapunov function, gives proof of global finite time stability of the transformed and the original impact system without having to analyze the jumps of the Lyapunov function. Finally, the 'twisting' controller [10] is shown to stabilize the double integrator with impacts in finite time.

II. PROBLEM STATEMENT

Consider the following open loop system [3]:

$$\begin{aligned} &\dot{x}_1 = x_2 \\ &\dot{x}_2 = u(x_1,x_2) + \omega(t) \\ &x_1 \geq 0 \\ &x_2(t_k^+) = -\bar{e}\, x_2(t_k^-) \qquad \text{if} \qquad x_2(t_k^-) < 0,\, x_1(t_k) = 0, \end{aligned} \tag{1}$$

where x_1, x_2 are the position and the velocity respectively, u is the control input, $\omega(t)$ is disturbance, t_k is the k^{th} jump time instant where the velocity undergoes a reset or jump, $\bar{e}$ represents the loss of energy and $x_2(t_k^+)$ and $x_2(t_k^-)$ represent right and left limits respectively of x_2 at the jump time t_k. The third inequality represents the dynamics with unilateral constraints on position x_1 [3]. It is assumed that the jump event occurs instantaneously within an infinitesimally

This work was supported by EPSRC via research grant EP/G053979/1. The second author wishes to thank Consejo Nacional de Ciencia y Tecnología de México for financial support of his sabbatical stay at the University of Kent.

H. B. Oza, Yury V. Orlov and S. K. Spurgeon are with School of Engineering and Digital Arts, University of Kent, Canterbury, Kent CT2 7NT, UK hbo2@kent.ac.uk,S.K.Spurgeon@kent.ac.uk

Yury V. Orlov is also with SCICESE Research Center, Ensenada, Mexico yorlov@cicese.mx

small time and hence mathematically can be represented by *Newton's restitution rule* [3], [6] given by the fourth equality of (1). The twisting control law [10] in (x_1, x_2) coordinates is given as follows:

$$u(x_1, x_2) = -\mu_1 \operatorname{sign}(x_2) - \mu_2 \operatorname{sign}(x_1) \tag{2}$$

where, $\mu_2 > \mu_1 > 0$. It should be noted that the above control law undergoes a jump whenever the state x_2 undergoes a jump. As in [9], the disturbance ω is assumed to admit a uniform upper bound

$$ess \sup_{t \geq 0} |\omega(t)| \leq M \tag{3}$$

on its magnitude such that

$$0 < M < \mu_1 < \mu_2 - M. \tag{4}$$

It is noteworthy that the solutions of the closed-loop system (1),(2), which involve switched terms along with impact, can be defined using existing methods (see [2], [3], [11] and [12] for a differential inclusion solution with both friction and collisions terms on the right hand side).

The aim of this paper is to (i) prove finite time stability and (ii) to establish a finite upper bound on the settling time $\mathscr{T}$ of the closed-loop system (1), (2).

The existing approaches [9], [10], [13] do not apply to the case of jumps in the velocity dynamics. The motivation to achieve finite time stability is twofold. Firstly, the effect of the jump in velocity when $x_1 = 0$ is a destabilizing one for the double integrator. This is contrary, for example, to the self-stabilizing nature of a bouncing ball where impact with the ground stabilizes the motion [3], with loss of energy at the time of impact. Hence proving finite time stability is a theoretical challenge due to the complexity of the definition of the solutions of the closed-loop system (1), (2) [12]. Although limited, system (1) is not too restrictive as such planar systems may occur when non-linear constraints of the form $F(x) = 0$ are transformed into $\tilde{x}_1 = 0$ (see [14]). Secondly, stabilization results will motivate a similar development for nonlinear systems with resets in velocity such as biped robots. It is noted that the latest advances in the literature of numerical schemes [15] aid the implementation of discontinuous control laws.

III. GLOBAL FINITE TIME STABILITY

The method of non-smooth transformation [3] is employed to transform the impact system (1) into a jump-free system. Let the non-smooth transformation be defined as follows:

$$x_1 = |s|, x_2 = R\, v \operatorname{sign}(s), R = 1 - k \operatorname{sign}(s\, v), k = \tfrac{1-\bar{e}}{1+\bar{e}} \tag{5}$$

The variable structure-wise transformed system

$$\begin{aligned} \dot{s} &= R\, v \\ \dot{v} &= R^{-1} \operatorname{sign}(s)\, \left(u(|s|, R\, v \operatorname{sign}(s)) + \omega(t)\right) \end{aligned} \tag{6}$$

is then derived by employing (5) (see [3, Sec. 1.4.2] for detailed explanation of this derivation). By combining (2) and (5), the controller (2) can be represented in the transformed coordinates as follows:

$$u(|s|, R\, v \operatorname{sign}(s)) = -\mu_1 \operatorname{sign}(sv) - \mu_2 \tag{7}$$

Substituting (7) into (6), the closed-loop system in the new coordinate frame can be obtained as follows:

$$\begin{aligned} \dot{s} &= R\, v \\ \dot{v} &= -\mu_1\, R^{-1} \operatorname{sign}(v) - \mu_2\, R^{-1} \operatorname{sign}(s) + R^{-1} \operatorname{sign}(s) \omega(t) \end{aligned} \tag{8}$$

Remark 1: The origin $s = v = 0$ of the system (8) corresponds to the origin $x_1 = x_2 = 0$ of the system (1),(2). Since the transformation (5) is not invertible, one starts from the closed-loop system (8) and that the original dynamics can be recovered via (5). The solutions of (8) are well defined in the sense of Filipov [1]. Furthermore, such a formulation admits both friction and jump phenomena, while guaranteeing existence of the solutions. Finally, the formulation (8) captures the infinite rebounds [8] (the so-called Zeno behavior) once the system stabilizes on the origin and in turn on the constraint surface.

Lemma 1: Let the dynamical system be given by (8). Also assume $\bar{e} \in (0, 1)$, then the following is true:

$$\operatorname{sign}(sv) \operatorname{sign}(R - R^{-1}) = -1 \tag{9}$$

Proof: The parameter R is defined in (5). For the case when $\operatorname{sign}(sv) = -1$, R can be computed as $R = 1 - k\operatorname{sign}(sv) = 1 + k = \frac{2}{1+\bar{e}}$. Hence $R - R^{-1} = \frac{(\bar{e}+3)(1-\bar{e})}{2(1+\bar{e})}$. Noting that $\bar{e} \in (0, 1)$, it is indeed clear that $\operatorname{sign}(R - R^{-1}) = 1$. Hence the result $\operatorname{sign}(sv) \operatorname{sign}(R - R^{-1}) = -1$. For the case when $\operatorname{sign}(sv) = 1$, R can be computed as $R = 1 - k\operatorname{sign}(sv) = 1 - k = \frac{2\bar{e}}{1+\bar{e}}$. Hence $R - R^{-1} = \frac{(3\bar{e}+1)(\bar{e}-1)}{2\bar{e}(1+\bar{e})}$. Noting that $\bar{e} \in (0, 1)$, it is indeed clear that $\operatorname{sign}(R - R^{-1}) = -1$. □

Remark 2: The transformed system (8) is not a standard double integrator system and is not a candidate of existing methods [10] because R causes discontinuity in the right hand side of the first equation of (8). Hence, proving finite time stability is not as trivial as merely combining the existing controller and existing non-smooth transformation technique. Furthermore, finite time stability of (8) has never been studied.

Let the two values of R be defined as follows:

$$R = \begin{cases} R_1 = \frac{2}{1+\bar{e}}, & \text{if } \operatorname{sign}(sv) = -1; \\ R_2 = \frac{2\bar{e}}{1+\bar{e}}, & \text{if } \operatorname{sign}(sv) = 1. \end{cases} \tag{10}$$

Then, it is trivial to note that given $\bar{e} \in (0, 1)$, the following is true from the computations in Lemma 1:

$$\begin{aligned} R_1 > R_2 > 0\, , R_1^{-1} < R_2^{-1}, |R_1 - R_1^{-1}| < |R_2 - R_2^{-1}| \\ |R_1 - R_1^{-1}| = \tfrac{3+\bar{e}}{2}|k|\, , |R_2 - R_2^{-1}| = \tfrac{3\bar{e}+1}{2\bar{e}}|k| \end{aligned} \tag{11}$$

Theorem 1: Given $M = 0$, the impact system (1), (2) and its transformed version (6), (7) are globally finite time stable.

Proof: Lyapunov stability analysis can be performed in the transformed coordinates since both the set of expressions (1), (2) and (6), (7) represent the same system. Let a Lyapunov function candidate be given as follows:

$$V(s, v) = \mu_2\, |s| + \tfrac{1}{2}\, v^2 \tag{12}$$

The following temporal derivative of (12) is computed along the system trajectories in (6), (7) with $M = 0$:

$$\begin{aligned}\dot{V} &= \mu_2 \text{sign}(s) R v + v(-\mu_1 R^{-1} \text{sign}(v) - \mu_2 R^{-1} \text{sign}(s)) \\ &\leq \mu_2 \text{sign}(sv)\,|v|\,R - \mu_2 R^{-1}\,|v| \text{sign}(sv) - \mu_1 R^{-1}\,|v| \\ \dot{V} &\leq \mu_2 |v|\,|R - R^{-1}| \text{sign}(sv) \text{sign}(R - R^{-1}) - \mu_1 R^{-1} |v|\end{aligned} \tag{13}$$

where substitution $v = \text{sign}(v)|v|$ has been utilised. From Lemma 1, Eq. (13) can be simplified as,

$$\dot{V} \leq -\mu_2 |v|\,|R - R^{-1}| - \mu_1 R^{-1}\,|v| \tag{14}$$

It can be verified that $R^{-1} > 0$ for either sign of sign$(s\,v)$ since $\bar{e} \in (0,1)$. Since, the equilibrium point $s = v = 0$ is the only trajectory of (8) on the invariance manifold $v = 0$ where $\dot{V}(s,v) = 0$, the differential inclusion (6), (7) is globally uniformly asymptotically stable by applying the invariance principle [16], [17]. Moreover, the system described in (6), (7) is globally homogeneous of the negative degree $q = -1$ with respect to dilation $r = (2,1)$ and is globally uniformly finite time stable according to [9, Theorem 3.1]. □

The closed loop system (8) is a globally homogeneous system if $\omega(t) = 0\ \forall t \geq 0$. Given $M \neq 0$, the discontinuous control law (7) can reject any disturbance ω with a uniform upper bound (3). The following result can be stated.

Theorem 2: The closed-loop impact system (1), (2) and its transformed version (6), (7) are globally finite time stable, regardless of whichever disturbance ω, satisfying condition (3) with $M < \mu_1 < \mu_2 - M$, affects the system.

Proof: The proof is achieved in several steps.

1. Global Asymptotic Stability Under the conditions of the theorem, the time derivative of the Lyapunov function (12), computed along the trajectories of (6), (7) is estimated as follows:

$$\begin{aligned}\dot{V} &= \mu_2 |v|\,|R - R^{-1}| \text{sign}(sv) \text{sign}(R - R^{-1}) \\ &\quad - \mu_1 R^{-1}\,|v| + R^{-1}\,|v| \text{sign}(sv)\,\omega \\ &\leq -\mu_2 |v|\,|R - R^{-1}| - (\mu_1 - M) R^{-1} |v|\end{aligned} \tag{15}$$

The first term in the last inequality follows from Lemma 1. Since $M < \mu_1$ by a condition of the theorem, the global asymptotic stability of (6), (7) is then established by applying the invariance principle [16], [17].

2. Semiglobal Strong Lyapunov Functions.

The goal of this step is to show the existence of a parameterized family of local Lyapunov functions $V_{\tilde{R}}(s,v), \tilde{R} > 0$ such that each $V_{\tilde{R}}(s,v)$ is well-posed on the corresponding compact set

$$D_{\tilde{R}} = \{(s,v) \in \mathbf{R}^2 : V(s,v) \leq \tilde{R}\}. \tag{16}$$

In other words, $V_{\tilde{R}}(s,v)$ is to be positive definite on $D_{\tilde{R}}$ and its derivative, computed along the trajectories of the uncertain system (6), (7) with initial conditions within $D_{\tilde{R}}$, is to be negative definite in the sense that,

$$\dot{V}_{\tilde{R}}(s,v) \leq -W_{\tilde{R}}(s,v) \tag{17}$$

for all $(s,v) \in D_{\tilde{R}}$ and some $W_{\tilde{R}}(s,v)$, positive definite on $D_{\tilde{R}}$. A parameterized family of Lyapunov functions $V_{\tilde{R}}(s,v), \tilde{R} > 0$, with the properties defined above are constructed by combining the Lyapunov function V of (12), whose time derivative along the system motion is only negative semi-definite, with the indefinite function $U(s,v) = s\,v$:

$$V_{\tilde{R}}(s,v) = V(s,v) + \kappa_{\tilde{R}} U(s,v) = \mu_2\,|s| + \tfrac{1}{2} v^2 + \kappa_{\tilde{R}}\, s\, v \tag{18}$$

where the weight parameter $\kappa_{\tilde{R}}$ is chosen small enough namely,

$$\kappa_{\tilde{R}} < min\left\{1,\ \frac{2\mu_2{}^2}{\tilde{R}},\ \frac{\mu_2|R_1 - R_1^{-1}| + R_1^{-1}(\mu_1 - M)}{R_1\sqrt{2\tilde{R}}}\right\} \tag{19}$$

where R_1 is defined in (10). It can be noted from (16) that the following inequalities hold true:

$$|s| \leq \tfrac{\tilde{R}}{\mu_2}, \quad |v| \leq \sqrt{2\tilde{R}}. \tag{20}$$

Hence, the Lyapunov function (18) is positive definite on compact set (16) for all $(s,v) \in D_{\tilde{R}} \backslash \{0,0\}$ and $\kappa_{\tilde{R}} > 0$ satisfying (19) as shown below:

$$\begin{aligned}V_{\tilde{R}}(s,v) &= \mu_2\,|s| + \tfrac{1}{2}\,v^2 + \kappa_{\tilde{R}}\, s\, v \geq \mu_2 |s| + \tfrac{1}{2}\,v^2 - \tfrac{1}{2}\kappa_{\tilde{R}}(s^2 + v^2) \\ &\geq \left(\mu_2 - \tfrac{\kappa_{\tilde{R}}\tilde{R}}{2\mu_2}\right)|s| + \tfrac{1}{2}(1 - \kappa_{\tilde{R}})\, v^2 \quad > 0\end{aligned} \tag{21}$$

The time derivative of the indefinite function $U(s,v)$ along the trajectories of the uncertain system (6), (7) is obtained as follows:

$$\begin{aligned}\dot{U}(s,v) &= R\,v^2 + s\begin{pmatrix} -\mu_1 R^{-1} \text{sign}(v) - \mu_2 R^{-1}\ \text{sign}(s) \\ +R^{-1} \text{sign}(s)\omega \end{pmatrix} \\ &= R\,v^2 - \mu_1 R^{-1} |s| \text{sign}(sv) - \mu_2 R^{-1}\,|s| + R^{-1} |s|\,\omega \\ &\leq R\,v^2 - R^{-1} |s|\,(\mu_2 - \mu_1 - M)\end{aligned} \tag{22}$$

Then, by combining (15) and (22) the time derivative of (18) can be obtained as follows:

$$\begin{aligned}\dot{V}_{\tilde{R}} &\leq -\mu_2 |v|\,|R - R^{-1}| - (\mu_1 - M) R^{-1}\,|v| + \kappa_{\tilde{R}} R\, v^2 \\ &\quad - \kappa_{\tilde{R}} R^{-1} |s|\,(\mu_2 - \mu_1 - M)\end{aligned} \tag{23}$$

The parameter R in (23) keeps switching between the two values as shown in Lemma 1. In turn, the rate of decay of the Lyapunov function (21) switches depending on R. Considering the slowest decay, a conservative estimate of the upper bound (23) can be readily obtained using Lemma 1 and Eq.(11) as follows:

$$\begin{aligned}\dot{V}_{\tilde{R}} &\leq -\mu_2 |v|\,|R_1 - R_1^{-1}| - (\mu_1 - M) R_1^{-1} |v| + \kappa_{\tilde{R}} R_1\, v^2 \\ &\quad - \kappa_{\tilde{R}} R_1^{-1} |s|\,(\mu_2 - \mu_1 - M)\end{aligned} \tag{24}$$

Noting that, due to (15), all possible solutions of the uncertain system (6), (7), initialized at $t_0 \in \mathbf{R}$ within the compact set (16), are a priori estimated by

$$\sup_{t \in [t_0, \infty)} V(s,v) \leq \tilde{R}, \tag{25}$$

and that (20) holds true within the compact set (16), (24) can be re-written as follows:

$$\begin{aligned}\dot{V}_{\tilde{R}} &\leq -\left[\ \mu_2 |R_1 - R_1^{-1}| + (\mu_1 - M) R_1^{-1} - \kappa_{\tilde{R}} R_1 \sqrt{2\tilde{R}}\ \right] |v| \\ &\quad - \kappa_{\tilde{R}} R_1^{-1}(\mu_2 - \mu_1 - M)|s| \leq -c_{\tilde{R}}\,[|s| + |v|]\end{aligned} \tag{26}$$

where

$$c_{\tilde{R}} = \min \left\{ \begin{array}{l} \kappa_{\tilde{R}} R_1^{-1}(\mu_2 - \mu_1 - M), \\ \mu_2 |R_1 - R_1^{-1}| + (\mu_1 - M) R_1^{-1} - \kappa_{\tilde{R}} R_1 \sqrt{2\tilde{R}} \end{array} \right\} \tag{27}$$

It follows from (19) that $c_{\tilde{R}} > 0$. Hence (26) results in

$$\dot{V}_{\tilde{R}} \leq -K_{\tilde{R}} V_{\tilde{R}}(s,v) \tag{28}$$

where

$$K_{\tilde{R}} = c_{\tilde{R}} \left[\max \left\{ \frac{2\mu_2^2 + \kappa_{\tilde{R}} \tilde{R}}{2\mu_2}, \sqrt{\frac{\tilde{R}}{2}} (1 + \kappa_{\tilde{R}}) \right\} \right]^{-1} > 0$$

and the upper estimate

$$V_{\tilde{R}} \leq \frac{2\mu_2^2 + \kappa_{\tilde{R}} \tilde{R}}{2\mu_2} |s| + \sqrt{\frac{\tilde{R}}{2}} (1 + \kappa_{\tilde{R}}) |v|$$

of the Lyapunov function (21) on compact set (16) has been used. Hence the desired uniform negative definiteness (17) is obtained with $W_{\tilde{R}}(s,v) = K_{\tilde{R}} V_{\tilde{R}}(s,v)$.

3. Global Uniform Asymptotic Stability Since the inequality (28) holds on the solutions of the uncertain system (6), (7), initialized within the compact set (16), the function $V_{\tilde{R}}(s,v)$ decays exponentially

$$V_{\tilde{R}}(s(t),v(t)) \leq V_{\tilde{R}}(s(t_0),v(t_0)) e^{-K_{\tilde{R}}(t-t_0)} \tag{29}$$

on these solutions with decay rate $K_{\tilde{R}}$ which depends on the gain parameters μ_1, μ_2, bound M on disturbance ω and the system property R_1. On the compact set (16), the following inequality holds (see (21)):

$$L_{\tilde{R}} V(s,v) \leq V_{\tilde{R}}(s,v) \leq M_{\tilde{R}} V(s,v) \tag{30}$$

for all $(s,v) \in D_{\tilde{R}}$ and positive constants $L_{\tilde{R}}, M_{\tilde{R}}$, satisfying

$$L_{\tilde{R}} < \min \left\{ \frac{2\mu_2^2 - \tilde{R}\kappa_{\tilde{R}}}{2\mu_2^2}, 1 - \kappa_{\tilde{R}} \right\}, M_{\tilde{R}} > \max \left\{ \frac{2\mu_2^2 + \tilde{R}\kappa_{\tilde{R}}}{2\mu_2^2}, 1 + \kappa_{\tilde{R}} \right\} \tag{31}$$

The above inequalities (29) and (30) ensure that the function $V(s,v)$ decays exponentially

$$\begin{aligned} V(s(t),v(t)) &\leq L_{\tilde{R}}^{-1} M_{\tilde{R}} V(s(t_0),v(t_0)) e^{-K_{\tilde{R}}(t-t_0)} \\ &\leq L_{\tilde{R}}^{-1} M_{\tilde{R}} \tilde{R} e^{-K_{\tilde{R}}(t-t_0)} \end{aligned} \tag{32}$$

on the solutions of (6), (7) uniformly in ω and the initial data, located within an arbitrarily large set (16). This proves that the uncertain system (6), (7) is globally uniformly asymptotically stable around the origin $(s,v) = (0,0)$.

4. Global Uniform Finite Time Stability.

Due to (4), the piece-wise continuous [1], [9] uncertainty $R_1^{-1}\omega(t) sign(s)$ in the right hand side of the system (6), (7) is locally uniformly bounded by $R_1^{-1} M$ whereas the remaining part of the feedback is globally homogeneous with homogeneity degree $q = -1$ with respect to dilation $r = (2,1)$. Noting that $q + r_2 \leq 0$, the globally uniformly asymptotically stable system (6), (7) and in turn the original impact system (1), (2) are globally finite time stable according to [9, Theorem 3.2]. □

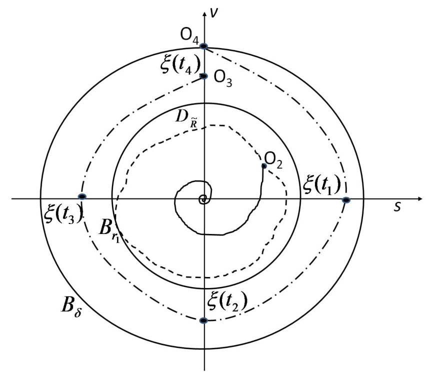

Fig. 1. Finite settling time behavior of the transformed system (6),(7)

IV. SETTLING TIME ESTIMATE

A finite upper bound on the settling time of the closed-loop system (6), (7) is computed in this section. When the trajectories are initialized on the positive vertical semi-axis $e_1^+ = \{x \in \mathbf{R}^2 : x_1 = 0, x_2 > 0\}$ at O_4, the factor by which it gets closer to the origin after one revolution can be computed. The value of the intercept (point O_3) on the positive vertical semi-axis after one revolution should be greater than the radius r_1 of the ball B_{r_1} containing the level set $D_{\tilde{R}}$ defined in (16) (see appendix for the definition of r_1 to render the relation $D_{\tilde{R}} \subset B_{r_1}$ to hold true). The choice of δ, such that $\delta\Psi > r_1$ where Ψ is the factor by which the trajectory gets closer to the origin after one revolution at point O_3, will ensure that the settling time estimate will be more conservative than the one computed with the initialization on the level set $D_{\tilde{R}}$ (point O_2). The motivation for such a choice of initialization of the trajectories on the ball B_δ stems from the fact that the trajectory, containing O_2 on the level set $D_{\tilde{R}}$, starting from any arbitrary point below O_4 (see Figure 1) on the e_1^+ axis cannot intersect the trajectory starting from the point O_4. The basis for this is the fact that the solutions of (8) are unique everywhere. In fact, the solution does not remain on the axes $s = 0, v = 0$ for finite time and always crosses the axes except at the origin. Hence, different trajectories have no intersections because otherwise they would coincide outside the origin with each other due to the uniqueness of the solution. The approach utilized in the following is a two step process. Firstly, a comparison system [18], the trajectory of which encompasses the actual system, is defined as shown in Figure 1 (also see work on the majorant curves [10] for the twisting controller). Secondly, the comparison system is then initialized on the positive vertical semi-axis e_1^+ with the coordinates $(0,\delta)$. Then the finite settling time is computed for the comparison system subject to the condition $\delta\Psi > r_1$. Let the right hand side of (8) be written as follows:

$$\begin{aligned} &\phi_1 = R\, v \\ &\phi_2 = -\mu_1 R^{-1} \mathrm{sign}(v) - \mu_2 R^{-1} \mathrm{sign}(s) + R^{-1} \mathrm{sign}(s) \omega(t) \end{aligned} \tag{33}$$

Let a comparison system corresponding to (8) be given as follows:

$$\begin{aligned} \dot{s} &= R\, v \\ \dot{v} &= -\left[\mu_1 - M\right] R^{-1}\mathrm{sign}(v) - \mu_2 R^{-1}\mathrm{sign}(s) \end{aligned} \tag{34}$$

In turn, the right hand side

$$\begin{aligned} \phi_1^c &= R\, v \\ \phi_2^c &= -\left[\mu_1 - M\right] R^{-1}\mathrm{sign}(v) + \mu_2 R^{-1}\mathrm{sign}(s) \end{aligned} \tag{35}$$

of the comparison system (34) relates to (33) as

$$\begin{aligned} \phi_1 &= \phi_1^c \\ \phi_2 &= \phi_2^c + \Delta\phi \end{aligned} \tag{36}$$

where $\Delta\phi = -MR^{-1}\,\mathrm{sign}(v) + R^{-1}\mathrm{sign}(s)\omega$. It is trivial to note that,

$$\Delta\phi \begin{cases} \leq 0, & \text{if } (s,v) \in (G_1 \cup G_4); \\ \geq 0, & \text{if } (s,v) \in (G_2 \cup G_3). \end{cases} \tag{37}$$

where

$$\begin{aligned} G_1 = \{(s,v) : s > 0, v > 0\}, G_2 = \{(s,v) : s > 0, v < 0\} \\ G_3 = \{(s,v) : s < 0, v < 0\}, G_4 = \{(s,v) : s < 0, v > 0\} \end{aligned} \tag{38}$$

By virtue of (36), (37), the motion of the plant (8) is dominated by that of (34) subject to the same initial condition. In other words, the solutions $(s(t), v(t))$ of the system (8) and the solutions $(s^c(t), v^c(t))$ of the comparison system (34) rotate around the origin and in each region $G_i, i = 1,2,3,4$ the plot of the trajectory $(s(t), v(t))$ is bounded by the plot of the trajectory $(s^c(t), v^c(t))$ and the switching lines $s = 0, v = 0$. Hence it suffices for the purpose of estimating the finite settling time to consider system (34) which can be represented in the matrix vector notation as follows:

$$\dot{\zeta}(t) = A\,\zeta(t) + B\,u \tag{39}$$

where

$$\zeta = \begin{bmatrix} s & v \end{bmatrix}^T, \quad A = \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$u = -(\mu_1 - M)\mathrm{sign}(v)R^{-1} - \mu_2 \mathrm{sign}(s) R^{-1} \tag{40}$$

The motion in the state space can be obtained using the convolution integral as follows:

$$\zeta(t) = e^{At}\,\zeta^0 + \int_0^t e^{A(t-\tau)} B u d\tau \tag{41}$$

$\zeta^0 = [s(t_0) \quad v(t_0)]^T$ is initial condition. Since the control switches on the the axes $\zeta_1 = s = 0, \zeta_2 = v = 0$, the integral (39) is required to be computed in each quadrant utilizing Bu as follows:

$$Bu = \begin{bmatrix} 0 \\ -(\mu_1 - M)\mathrm{sign}(v)R^{-1} - \mu_2 \mathrm{sign}(s) R^{-1} \end{bmatrix} \tag{42}$$

It is noted that using such integrals to define the solutions of the comparison system (34) is mathematically correct as the control law never generates a sliding mode on the switching lines $\zeta_1 = 0$ and $\zeta_2 = 0$. Hence the solution always crosses these switching lines except at the origin [10], [18]. The matrix exponential in (41) can be computed as follows:

$$e^{At} = I + At + \frac{At^2}{2!} + \ldots \tag{43}$$

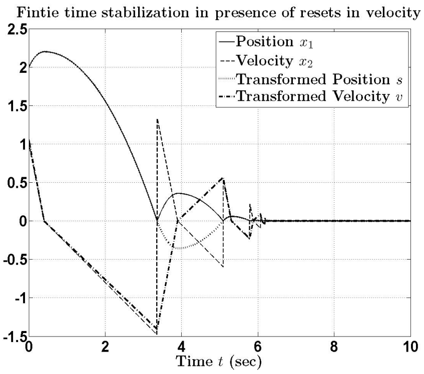

Fig. 2. Finite time stabilization of the system (1), (2) with jumps in velocity and that of the jump-free transformed system (8).

Since $A^n = 0, \forall n \geq 2$, (43) leads to the following:

$$e^{At} = I + At = \begin{bmatrix} 1 & Rt \\ 0 & 1 \end{bmatrix}, \int_0^t e^{-A\tau} d\tau = \begin{bmatrix} t & \frac{-Rt^2}{2} \\ 0 & t \end{bmatrix} \tag{44}$$

Utilizing (41), (42) and (44), the following can be obtained:

$$t_1 = \frac{\delta R}{\mu_2(1+\eta)}, t_2 = \frac{\delta R}{\mu_2\sqrt{(1+\eta)(1-\eta)}}, t_3 = \frac{\delta R\sqrt{1-\eta}}{\mu_2(1+\eta)\sqrt{1+\eta}}, t_4 = t_1 \tag{45}$$

where $\eta = \frac{\mu_1 - M}{\mu_2}$. and t_1 is time taken by the trajectory to travel from the semi-axis $\{\zeta \in \mathbf{R}^2 : \zeta_1 = 0, \zeta_2 > 0\}$ to the semi-axis $\{\zeta \in \mathbf{R}^2 : \zeta_1 > 0, \zeta_2 = 0\}$ and so on. Furthermore, the interception of the trajectory on the positive and negative semi-axes can be obtained as follows:

$$\begin{aligned} \zeta_1(t_1) &= \frac{(\delta R)^2}{2\mu_2(1+\eta)}; & \zeta_2(t_2) &= -\frac{\delta\sqrt{1-\eta}}{\sqrt{1+\eta}}; \\ \zeta_1(t_3) &= -\frac{(R\delta)^2(1-\eta)}{2\mu_2(1+\eta)^2}; & \zeta_2(t_4) &= \frac{\delta(1-\eta)}{1+\eta}, \end{aligned} \tag{46}$$

where the intercepts $\zeta_1(t_1), \zeta_2(t_2), \zeta_1(t_3), \zeta_2(t_4)$ are depicted in the figure (1). Hence the time T_1 taken by the trajectory to travel from the point O_4 on the ball B_δ to some point O_3 on the semi-axis e_1^+ is obtained using (45) as follows:

$$T_1 = t_1 + t_2 + t_3 + t_4 = \frac{R\Delta}{\mu_2}\delta \leq \frac{R_1\Delta}{\mu_2}\delta \tag{47}$$

where $\Delta = \frac{2}{1+\eta} + \frac{1}{\sqrt{(1+\eta)(1-\eta)}} + \frac{\sqrt{1-\eta}}{(1+\eta)\sqrt{1+\eta}}$. It can be seen that the time T_1 taken by one revolution depends on the initial condition δ, gain parameters (μ_1, μ_2), system property R_1 and the bound M on the uncertainty. Hence the time T_1 and time taken by the subsequent revolutions can be computed *apriori*. Furthermore, as shown by the last equality of (46), the closed-loop trajectory decays closer to the origin by a factor Ψ of the initial condition δ where $\Psi = \frac{1-\eta}{1+\eta} < 1$. A similar computation can be repeated with the initial condition set at $\zeta_2(t_4)$ to obtain the next intersection of the trajectory with the semi-axis $e_1^+ = 0$ at the end of the second revolution as follows:

$$x(T_2) = x_2(t_4)\,\Psi = \delta\Psi^2 \tag{48}$$

where T_2 is the time at which the second revolution is completed. Noting that one revolution takes $\frac{R\Delta}{\mu_2}$ multiplied by the initial value on the vertical axis (see (47)), the total time taken for two revolutions is estimated as follows:

$$T_2 = T_1 + \frac{R\Delta}{\mu_2}\zeta_2(t_4) \leq \frac{R_1\Delta}{\mu_2}\delta\,[1+\Psi] \qquad (49)$$

where the last equality of (46) and (47) are utilized. Noting that the number of revolutions $n \to \infty$ as time $t \to \infty$, above steps can be repeated and the following generalization for the n^{th} revolution can be obtained:

$$\lim_{n\to\infty} T_n \leq \lim_{n\to\infty} \quad \frac{R_1\Delta}{\mu_2}\delta\left[1+\Psi+\Psi^2+\ldots+\Psi^{n-1}\right] \qquad (50)$$

Noting that the inequality $0 < \Psi < 1$ always holds true, the infinite series in (50) can be represented by a convergent geometric series. In turn, the upper-bound on the settling time $\mathscr{T}_s$ of the system (8) can be obtained *apriori* as follows:

$$\mathscr{T}_s = \lim_{n\to\infty} T_n \leq \lim_{n\to\infty} \quad \frac{R_1\Delta}{\mu_2}\delta\left[\frac{1-\Psi^n}{1-\Psi}\right] \leq \frac{R_1\Delta\delta}{\mu_2(1-\Psi)} < \infty \qquad (51)$$

The numerical simulation is presented in Figure 2 which gives a comparison between the system (1), (2) with $\mu_1 = 1, \mu_2 = 2, M = 0.5, \bar{e} = 0.9$ and the transformed system (8). Appropriate initial conditions $x_1(t_0) = 2, x_2(t_0) = 1$ and $s(t_0) = 2, v(t_0) = [1-k]^{-1}$ are used. It should be noted that the jump in velocity occurs when s changes sign [3]. The simulation is carried out using the event based Runge-Kutta method and it is prone to exhibit departure from the physical behaviour for both the discontinuities in the system (1), (2), namely, the 'sign' function and the jump. It should be noted that the system settles in less than $7\,sec$ which is less than the upper-bound $23.6865\,sec$ computed using (51).

V. CONCLUSIONS AND FUTURE WORK

Robust finite time stabilization is presented for the double integrator with jumps in the velocity. A non-smooth state transformation is employed to generate a jump-free system. The theoretical contribution of the presented work lies in achieving finite time stabilization of a class of impact mechanical systems without having to analyze jumps in the Lyapunov function. A finite upper-bound on the settling time is also estimated. Deriving tuning rules for the presented impact system is seen as future scope. From a practical viewpoint, the results will motivate a similar development for nonlinear impact mechanical systems such as biped robots.

APPENDIX I

DEFINITION OF THE RADIUS r_1 SUCH THAT $D_{\tilde{R}} \subset B_{r_1}$

The aim is to define the scalar $r_1 > 0$ such that the expression $D_{\tilde{R}} \subset B_{r_1}$ holds. The following is required:

$$\frac{\mu_2|s|}{\tilde{R}} + \frac{v^2}{2\tilde{R}} \leq 1 \Rightarrow \left(\frac{s}{r_1}\right)^2 + \left(\frac{v}{r_1}\right)^2 \leq 1 \qquad (52)$$

Impose the following inequalities:

$$\left(\frac{s}{r_1}\right)^2 \leq \frac{\mu_2|s|}{\tilde{R}}, \quad \left(\frac{v}{r_1}\right)^2 \leq \frac{v^2}{2\tilde{R}} \qquad (53)$$

Then the expression $(s,v) \in B_{r_1}$ holds true for every given point $(s,v) \in D_{\tilde{R}}$ in the (s,v) state space. Note that the following always holds true for all $(s,v) \in D_{\tilde{R}}$:

$$|s| \leq \frac{\tilde{R}}{\mu_2} \qquad (54)$$

The first inequality of (53) can be simplified as follows:

$$|s|\left(\frac{1}{r_1}\right)^2 \leq \frac{\mu_2}{\tilde{R}} \qquad (55)$$

Utilizing the relationship (54), the requirement (55) can be revised as

$$|s|\left(\frac{1}{r_1}\right)^2 \leq \frac{\tilde{R}}{\mu_2}\left(\frac{1}{r_1}\right)^2 \leq \frac{\mu_2}{\tilde{R}} \qquad (56)$$

Hence, the following upper-bound on r_1, obtained from (56), suffices to satisfy the first inequality of (53).

$$r_1 \geq \frac{\tilde{R}}{\mu_2} \qquad (57)$$

Similarly, the second inequality of (53) leads to $r_1 \geq \sqrt{2\tilde{R}}$, combining which with (57), the following estimate of the parameter r_1 is obtained:

$$r_1 = \max\left\{\frac{\tilde{R}}{\mu_2}, \sqrt{2\tilde{R}}\right\} \qquad (58)$$

REFERENCES

[1] A.F.Filippov, *Differential Equations with Discontinuous Righ thand Sides*, ser. Mathematics and its Applications. Springer, 1988, vol. 18.

[2] M. D. P. M. Marques, *Differential Inclusions in Nonsmooth Mechanical Problems: Shocks and Dry Friction*, ser. Prog. Nonlinear Differential Equations and Thier Appl. Birkhauser-Verlag Berlin, 1993, vol. 220.

[3] B. Brogliato, *Nonsmooth Impact Mechanics*, ser. Lecture Notes in Control and Information Sciences, M. Thoma, Ed. Springer Verlag London, 1996, vol. 220.

[4] J. Cortes, "Discontinuous dynamical systems," *Control Systems Magazine, IEEE*, vol. 28, no. 3, pp. 36 –73, june 2008.

[5] J.-M. Bourgeot and B. Brogliato, "Tracking control of complementarity lagrangian systems," *International Journal of Bifurcation and Chaos*, vol. 15, no. 6, pp. 1839–1866, 2005.

[6] J. Grizzle, G. Abba, and F. Plestan, "Asymptotically stable walking for biped robots: analysis via systems with impulse effects," *Automatic Control, IEEE Transactions on*, vol. 46, no. 1, pp. 51 –64, jan 2001.

[7] Y. Hurmuzlu, F. Gnot, and B. Brogliato, "Modeling, stability and control of biped robots–a general framework," *Automatica*, vol. 40, no. 10, pp. 1647 – 1664, 2004.

[8] B. Brogliato, S. I. Niculescu, and P. Orhant, "On the control of finite-dimensional mechanical systems with unilateral constraints," *IEEE Transactions on Automatic Control*, vol. 42, no. 2, pp. 200–215, 1997.

[9] Y. Orlov, "Finite time stability and robust control synthesis of uncertain switched systems," *SIAM Journal on Control and Optimization*, vol. 43, no. 4, pp. 1253–1271, 2005.

[10] A. Levant, "Sliding order and sliding accuracy in sliding mode control," *International Journal of Control*, vol. 58, no. 6, pp. 1247 – 1263, December 1993.

[11] J. J. Moreau, *Unilateral contact and dry friction in finite freedom dynamics*, ser. in Nonsmooth Mechanics and Applications, J. J. Moreau and P. D. Panagiotopoulos, Eds. Springer-Verlag, 1993, vol. 220.

[12] D. E. Stewart, "Rigid-body dynamics with friction and impact," *SIAM Review*, vol. 42, no. 1, pp. 3–39, March 2000.

[13] R. Santiesteban, L. Fridman, and J. Moreno, "Finite-time convergence analysis for "twisting" controller via a strict lyapunov function," *The 11th Workshop on Variable Structure Systems*, pp. 1–6, June 2010.

[14] N. McClamroch and D. Wang, "Feedback stabilization and tracking of constrained robots," *Automatic Control, IEEE Transactions on*, vol. 33, no. 5, pp. 419 –426, may 1988.

[15] V. Acary and B. Brogliato, "Implicit euler numerical scheme and chattering-free implementation of sliding mode systems," *Systems & Control Letters*, vol. 59, no. 5, pp. 284 – 293, 2010.

[16] J. Alvarez, Y. Orlov, and L. Acho, "An invariance principle for discontinuous dynamic systems with applications to a coulumb friction oscillator," *Journal of Dynamic Systems, Measurement and Control*, vol. 74, pp. 190 – 198, 2000.

[17] D. Shevitz and B. Paden, "Lyapunov stability theory of nonsmooth systems," *Automatic Control, IEEE Transactions on*, vol. 39, no. 9, pp. 1910 –1914, sep 1994.

[18] Y. Orlov, L. Aguilar, and J. Cadiou, "Switched chattering control vs. backlash/frcition phenomena in electrical servo motors," *International Journal of Control*, vol. 76, no. 9 & 10, pp. 959 – 967, 2003.