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Interconnection and Composition of Dirac Structures for Lagrange–Dirac Systems

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***Abstract*— There is much known on the port-Hamiltonian theory of interconnection of Dirac structures through shared variables. This interconnection is known as *Composition of Dirac structures*. In this paper, we will show an alternative interconnection of Dirac structures called *Bowtie interconnection* in the context of Lagrange-Dirac dynamical systems. In particular, we try to illustrate the following two things: Firstly, how composition of Dirac structures may be used in the Lagrangian theory of LC-circuits. Secondly, how composition of Dirac structures may be linked with bowtie interconnection.**

I. Introduction

Dirac structures were first introduced in [?] and [?] to generalize pre-symplectic and almost Poisson structures as relations between tangent and cotangent bundles. It was soon realized that many systems (including circuits) could be formulated with Dirac structures and a well chosen Hamiltonian [?], referred to as an *implicit Hamiltonian system*. Additionally it was realized that power conserving interconnection through the "shared variables" of two Dirac structures could produce a new Dirac structure where the shared variables are absent. This process is known as *composition of Dirac structures* [?] and it has been shown that the composition of Dirac structures is a powerful tool in the context of implicit port-Hamiltonian systems.

In parallel, a Lagrangian analogue of was developed in [?], [?]. The implicit port-Lagrangian system was also developed in [?], [?], [?]. Subsequently, the bowtie–product has been introduced in [?] as a means of using D_{int} and applied to an LC-circuit in [?], where $D_{\text{int}} = \Delta_{\text{int}} \oplus \Delta_{\text{int}}^{\circ}$ denotes the power conserving interconnection induced from a constraint Δ_{int} (see [?]). However, the Lagrange-Dirac and Hamilton-Dirac theories for the interconnection of Dirac structures appear vastly different. In this paper, we attempt to shine light on a link between the interconnection produced by *composition of Dirac structures* and that produced by *bowtie interconnection* as well as demonstrate how composition of Dirac structures may be used in the Lagrange-Dirac theory with an illustrative example of an LC–circuits.

II. Background

We use the notation following [?] as well as [?], [?].

A. The Pontryagin Bundle

Consider a manifold M. We denote the tangent bundle of M by TM with the fibration $\tau_M : TM \to M; (q,v) \to q$, where (q,v) are the local coordinates for TM. The sections of TM are the set of smooth vector field, $\mathfrak{X}(M)$. Additionally we denote the cotangent bundle by T^*M with the fibration $\pi_M : T^*M \to M; (q,p) \mapsto q$, where (q,p) are the local coordinates of T^*M. Sections of the cotangent bundle are the set of one-forms, $\Lambda^1(M)$. Let $\pi_1 : P_1 \to B, \pi_2 : P_2 \to B$ be two vector bundles over the same base-manifold B. The *Pontryagin bundle* over M is the Whitney sum bundle over M:

$$\begin{aligned}\mathbb{T}M &:= TM \oplus T^*M \\ &= \{(v,\alpha) \in TM \times T^*M | \tau_M(v) = \pi_M(\alpha)\}\end{aligned}$$

Note that the sections of $\mathbb{T}M$ are simply elements of the cartesian product $\mathfrak{X}(M) \times \Lambda^1(M)$. Locally the coordinates of $\mathbb{T}M$ may be expressed by (m,v,p) for $m \in M$, $v \in T_mM$ and $p \in T_m^*M$.

B. Dirac Structures

There exists a canonical inner-product:

$$\ll, \gg : \mathbb{T}M \oplus \mathbb{T}M \to \mathbb{R},$$

which is defined as:

$$\ll (v,\alpha), (w,\beta) \gg = \langle \beta, v \rangle + \langle \alpha, w \rangle.$$

In the above, $\langle , \rangle$ denotes the canonical pairing between TM and T^*M. A *Dirac structure* is a maximal isotropic sub-bundle of $\mathbb{T}M$ with respect to $\ll, \gg$[1]. For the case in which M has finite dimensions, one may think of a Dirac structure D_m as a "smooth" $\dim(M)$ dimensional subspace on the $2\dim(M)$ dimensional fibers of $\mathbb{T}M$ at each $m \in M$ such that for any $(X,\alpha), (Y,\beta) \in D_m$

$$\ll (X,\alpha), (Y,\beta) \gg = 0.$$

[1] We simply mention a 'Dirac structure' though this is an *almost* Dirac structure [?].

III. Lagrange–Dirac Systems

Let us review Lagrange-Dirac systems as in [**?**]. Let Q be a configuration manifold and let $L : TQ \to \mathbb{R}$ be a Lagrangian, possibly, degenerate. A partial vector field is a map $X : \mathbb{T}Q \to TT^*Q$ such that $\tau_{T^*Q} \circ X(q, v, p) = (q, p)$. Define the generalized energy $E : \mathbb{T}Q \to \mathbb{R}$ by

$$E(q, v, p) = \langle p, v\rangle - L(q, v)$$

and recall that the stationary condition of $E(q, v, p)$ associated to the second component v implies the generalized Legendre transformation (see [**?**], [**?**]). Additionally let $\Delta \subset TQ$ be a velocity constraint. Let $\Delta_{T^*Q} = T\pi_Q^{-1}(\Delta)$. We define an *induced Dirac structure on* T^*Q by

$$\begin{aligned} D_\Delta = \{(v, \alpha) \in \mathbb{T}T^*Q : v \in \Delta_{T^*Q}, \\ \Omega^\flat_{T^*Q}(v) - \alpha \in \Delta^\circ_{T^*Q}\}. \end{aligned}$$

Locally we have the canonical coordinates (q, p) on T^*Q and denote the coordinates on $T_{(q,p)}T^*Q$ by $(\dot{q}, \dot{p})$ and the coordinates on $T^*_{(q,p)}T^*Q$ by (α, w). In local coordinates, we have, for each $(q, p) \in T^*Q$,

$$D_\Delta(q, p) = \{(\dot{q}, \dot{p}, \alpha, w) : \; \dot{q} = w \in \Delta_q, \; \dot{p} + \alpha \in \Delta^\circ_q\}. \tag{1}$$

Note that locally we may write:

$$\mathbf{d}E(q, v, p) = -\frac{\partial L}{\partial q} dq + \left(p - \frac{\partial L}{\partial v}\right) dv + v dp$$

and the restriction to $P = \mathbb{F}L(\Delta) \subset T^*Q$ leads to

$$\mathbf{d}E(q, v, p)|_{TP} = -\frac{\partial L}{\partial q} dq + v dp,$$

for all v $\in \Delta \in T^*P$. Given a partial vector-field $X : \mathbb{T}Q \to TT^*Q$, the Lagrange-Dirac system is defined as

$$(X(q, v, p), \mathbf{d}E(q, v, p)|_{TP}) \in D_\Delta(q, p). \tag{2}$$

The local expression of the Lagrange-Dirac system is given by

$$\dot{q} = v \in \Delta_q, \;\; \dot{p} - \frac{\partial L}{\partial q} \in \Delta^\circ_q, \;\; p = \frac{\partial L}{\partial v} \in P \tag{3}$$

Equations (**??**) are called the *implicit Lagrange–d'Alembert equations*; let us denote it by ILDA equations for brevity. It was shown in [**?**] that they are equivalent to the *Lagrange–d'Alembert –Pontryagin principle*

$$\delta \int_a^b \langle p, \dot{q}\rangle - E(q, v, p) dt = 0$$

for $v(t) \in \Delta_{q(t)}$ and variations $\delta q(t) \in \Delta_{q(t)}$ and $\delta v(t), \delta p(t)$ arbitrary, where $t \in [a, b] \subset \mathbb{R}$ denote the time.

IV. Lagrange-Dirac Systems for L-C circuits

Consider an LC–circuit consisting of n_L inductors and n_C capacitors. The configuration manifold is $Q = \mathbb{R}^{n_L} \times \mathbb{R}^{n_C}$ referring to the charge associated to all the branches (or components). This means $v \in TQ$ denotes the current through the components[2].

Each inductor with inductance l_j induces a Lagrangian $L_{l_j} : TQ_{l_j} \to \mathbb{R}$ as

$$L_{l_j}(q_{l_j}, v_{l_j}) = \frac{l_i}{2} v^2_{l_j},$$

where $(q_{l_j}, v_{l_j}) \in TQ_{l_j}$. Similarly, each capacitor c_k carries a Lagrangian $L_{c_k} : TQ_{c_k} \to \mathbb{R}$ as

$$L_{c_k}(q_{c_k}, v_{c_k}) = \frac{-1}{2c_k} q^2_{c_k}.$$

Summing up the Lagrangians from the branches of the circuit gives us the total Lagrangian as

$$L = \left(\sum_{i=1}^{n_L} \frac{l_i}{2} \dot{q}^2_{l_i}\right) - \left(\sum_{k=1}^{n_C} \frac{1}{2c_k} q^2_{c_k}\right).$$

Let $\Delta \subset TQ$ be the constraint set of currents that satisfy the Kirchhoff Circuit Law (KCL) and it follows $P = \mathbb{F}L(\Delta)$. Recall the equations of motion may be given by (**??**). For example consider a circuit consisting of a single inductor and a single capacitor. This is known as the electric harmonic oscillator, in which case $Q = \mathbb{R} \times \mathbb{R}$ and the KCL constraint is given by $\dot{q}_l = \dot{q}_c$. so that the local coordinate expression of (**??**) gives

$$\begin{aligned} \dot{q}_l = \dot{q}_c = v_l = v_c \\ \dot{p}_l = \lambda, \quad -\frac{1}{c} q_c = \lambda \\ p_l = l\dot{q}_l \quad p_c = 0, \end{aligned}$$

where λ indicates the Lagrange multiplier to enforce the KCL constraints. These equations reduce down to $\ddot{q}_c = \frac{-1}{lc} q_c$, justifying the label "harmonic oscillator".

V. Composition of Dirac Structures

The notion of "composition of Dirac structures" was introduced in [**?**] as a means for interconnection of port-Hamiltonian systems through Dirac structures. Let V_1, V_2, V_s be vector spaces. Let D_1 be a linear Dirac structure on $V_1 \oplus V_s$ and D_2 be a linear Dirac stucture on $V_s \oplus V_2$. The *composition* of D_1 and D_2 is:

$$\begin{aligned} D_1 || D_2 = \{(v_1, v_2, \alpha_1, \alpha_2) \in \mathbb{T}(V_1 \times V_2) : \\ \exists (v_s, \alpha_s) \in \mathbb{T}V_s, \text{ such that } (v_1, v_s, \alpha_1, \alpha_s) \in D_1, \\ (-v_s, v_2, \alpha_s, \alpha_s) \in D_2\}. \end{aligned}$$

It was shown in [**?**] that the set $D_1 || D_2$ is itself a Dirac structure.

[2] In this paper, following [**?**], [**?**], v stands for a choice of current through the branches of the circuit by analogy with 'velocity' in mechanics. The voltage is not addressed explicitly, however it may be analogously given as 'force' is defined by a horizontal one-form in mechanics.

A. Shared Variable Splittings

The composition developed in [?] is constructed on vector spaces. So, we shall focus on the vector space case.

The configuration manifold for circuits is given by a vector space $Q_1 \times Q_s$, where Q_s is the total charge that has flowed passed the interconnection port. Noting that the Lagrange-Dirac system is performed on the cotangent bundle, the shared variable splittings are given as follows:

$$T^*_{(q_1,q_s)}(Q_1 \times Q_s) \cong Q_1^* \times Q_s^*$$

and

$$T_{(q_1,q_s,p_1,p_s)}T^*(Q_1 \times Q_s) \cong Q_1 \times Q_s \times Q_1^* \times Q_s^*.$$

We may set V_s to be the $Q_s \times Q_s^*$ part on the fiber $T_{(q_1,q_s,p_1,p_s)}T^*(Q_1 \times Q_s)$ and set V_1 to be the $Q_1 \times Q_1^*$ part. We note that V_s and V_1 imply integrable distributions over $T^*(Q_1 \times Q_s)$. The same argument may be applied to $Q_s \times Q_2$ to get the distribution V_2.

B. Example: port-interconnection of LC-circuits

As in [?], the composition of Dirac structures is usually performed on the Hamiltonian side. Here, we will show that one can also work on the Lagrangian side.

Consider the L-C circuit depicted in Fig.??, where there are two inductors l_1 and l_2 and a capacitor c_1.

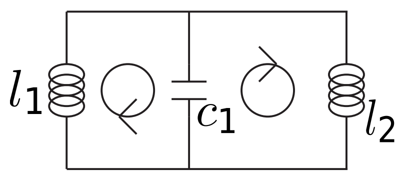

Fig. 1. A connected circuit

As in [?], [?], the L-C circuit can be formulated by using an induced Dirac structure and its associated Lagrange-Dirac dynamical systems. However, in this paper, we will consider this circuit as an interconnected system of two disconnected Lagrange-Dirac dynamical systems. To do this, let us split the circuit into two disconnected systems as in Fig.??, where the parts $s1$ and $s2$ are ports that are intended to be "connected". Let us call the parts $s1$ and $s2$ *shared ports* following [?].

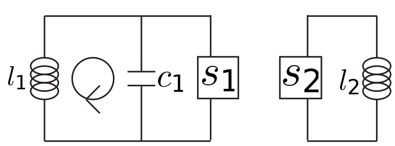

Fig. 2. Two disconnected circuits

We first deal with the circuit on the left with the s_1 part. The configuration subspace is $Q_1 \times Q_s = \mathbb{R}^2 \times \mathbb{R}$ with coordinates $(q_{l_1}, q_{c_1}, q_{s_1})$ or (q_1, q_{s_1}) for short by setting $q_1 := (q_{l_1}, q_{c_1})$. The Lagrangian on $T(Q_1 \times Q_s)$ is given by

$$L_1(q,v) = \frac{l_1}{2} v_{l_1}^2 - \frac{1}{2c_1} q_{c_1}^2,$$

which is degenerate and we have the primary constraints.

$$p_{c_1} = 0, \;\; p_{s_1} = 0 \;\text{ and }\; p_{l_1} = l_1 v_{l_1}.$$

The KCL constraint distribution is denoted by

$$\begin{aligned}\Delta_1 = \{(v_{l_1}, v_{c1}, v_{s_1}) \in T_{(q_1,q_{s_1})}(Q_1 \times Q_s) \\ : v_{l_1} + v_{s_1} - v_{c1} = 0\}\end{aligned} \tag{4}$$

and its annihilator is given by

$$\Delta_1^\circ = \{(\lambda, -\lambda, \lambda) \in T^*_{(q_1,q_{s_1})}(Q_1 \times Q_s)\}.$$

We may assemble the constraint induced Dirac structure

$$D_1 = D_{\Delta_1} \in \daleth(T^*(Q_1 \times Q_s)).$$

Using the local representation given in (??), we can write this Dirac structure, for each $(q_1, q_s, p_1, p_s) \in T^*(Q_1 \times Q_s)$, as

$$\begin{aligned}D_1(q_1,q_s,p_1,p_s) = \{(X,\Lambda) \in \mathbb{T}_{(q_1,q_s,p_1,p_s)}T^*(Q_1 \times Q_s) : \\ X = (\dot q_{l_1}, \dot q_{c_1}, \dot q_{s_1}, \dot p_{l_1}, \dot p_{c_1}, \dot p_{s_1}), \\ \Lambda = (\alpha_{l_1}, \alpha_{c_1}, \alpha_{s_1}, w_{l_1}, w_{c_1}, w_{s_1}), \\ \dot q_{c_1} - \dot q_{l_1} = \dot q_{s_1}, \\ \dot q_{c_1} = w_{c_1}, \dot q_{l_1} = w_{l_1}, \dot q_{s_1} = w_{s_1}, \\ \dot p_{l_1} + \alpha_{l_1} = -\alpha, \}.\end{aligned}$$

In the above, note that $\dot p_{c_1} = 0$ and $\dot p_{s_1} = 0$ will be implied by the consistency condition. Note that we have the splitting

$$T_{(q_1,q_s,p_1,p_s)}T^*(Q_1 \times Q_s) = V_1 \oplus V_s,$$

where

$$V_1 = \{(\dot q_{l_1}, \dot q_{c_1}, 0, \dot p_{l_1}, \dot p_{c_1}, 0)\} \cong T_{(q_1,p_1)}T^*Q_1$$

and

$$V_s = \{(0,0,\dot q_{s_1}, 0, 0, \dot p_{s_1})\} \cong T_{(q_s,p_s)}T^*Q_s$$

so that $D_1(q_1,q_s,p_1,p_s) \in \daleth(V_1 \oplus V_s)$.

Next we deal with the circuit on the right with the s_2 part. The configuration subspace is $Q_s \times Q_2 = \mathbb{R} \times \mathbb{R}$ with coordinates (q_{s_2}, q_{l_2}). The Lagrangian on $T(Q_s \times Q_2)$ is given by

$$L_2(q_{l_2}, v_{l_2}) = \frac{l_2}{2} v_{l_2}^2,$$

which is degenerate and we have the primary constraints:

$$\dot p_{s_2} = 0 \;\text{ and }\; p_{l_2} = l_2 v_{l_2}.$$

The KCL constraint distribution is given by

$$\Delta_2(q_{s_2}, q_{l_2}) = \{(v_{s_2}, v_{l_2}) : v_{s2} = v_{l2}\}$$

with its annihilator

$$\Delta_2^\circ(q_{s_2}, q_{l_2}) = \{(\delta, -\delta) \in T^*_{(q_{s_2},q_{l_2})}(Q_s \times Q_2)\}.$$

Hence, we may assemble the Dirac structure $D_2 = D_{\Delta_2} \in \daleth(T^*Q_2)$. Using the local expressions given in (??), we can develop a Dirac structure, for each $(q_{s_2}, q_{l_2}, p_{s_2}, p_{l_2}) \in T^*(Q_s \times Q_2)$, as

$$\begin{aligned} D_2(q_{s_2}, q_{l_2}, p_{s_2}, p_{l_2}) = \\ \{(X, \Lambda) \in \mathbb{T}_{(q_{s_2}, q_{l_2}, p_{s_2}, p_{l_2})} T^*(Q_s \times Q_2) : \\ X = (\dot{q}_{s_2}, \dot{q}_{l_2}, \dot{p}_{s_2}, \dot{p}_{l_2}), \\ \Lambda = (\alpha_{s_2}, \alpha_{l_2}, w_{s_2}, w_{l_2}), \\ \dot{q}_{s_2} = w_{s_2},\ \dot{q}_{l_2} = w_{l_2},\ \dot{q}_{s_2} = \dot{q}_{l_2}, \\ \dot{p}_{s_2} + \alpha_{s_2} = -\dot{p}_{l_2} - \alpha_{l_2},\ \dot{p}_{s_2} = 0\}. \end{aligned}$$

Again we have the splitting

$$T_{(q_{s_2}, q_{l_2}, p_{s_2}, p_{l_2})} T^*(Q_s \times Q_2) = V_s' \oplus V_2,$$

where

$$\begin{aligned} V_s' &= \{(\dot{q}_{s_2}, 0, \dot{p}_{s_2}, 0)\} \cong \{(\dot{q}_{s_2}, \dot{p}_{s_2})\} \\ &= T_{(q_{s2}, p_{s2})} T^* Q_s \\ &= V_s \end{aligned}$$

and

$$V_2 = \{(0, \dot{q}_{l_2}, 0, \dot{p}_{l_2})\} \cong \{(\dot{q}_{l_2}, \dot{p}_{l_2}\} = T_{(q_{l_2}, p_{l_2})} T^* Q_2$$

so that $D_2(q_{s2}, q_{l2}, p_{s2}, p_{l2}) \in \daleth(V_s \oplus V_2)$.

With this splitting of $TT^*(Q_1 \times Q_s)$ and $TT^*(Q_s \times Q_2)$ into the transverse distributions V_1, V_s, V_2, we may assemble the Dirac structure $D_{||} = D_1 || D_2$. We find that

$$(\dot{q}_1, \dot{q}_2, \dot{p}_1, \dot{p}_2, \alpha_1, \alpha_2, w_1, w_2) \in D_{||}$$

if and only if there exist $(\dot{q}_s, \dot{p}_s) \in V_s$ and $(\alpha_s, w_s) \in V_s^*$ such that

$$\begin{aligned} (\dot{q}_{l_1}, \dot{q}_{c_1}, \dot{q}_s) = (w_{l_1}, w_{c_1}, w_s) \in \Delta_1, \\ (-\dot{q}_s, \dot{q}_{l_2}) = (-w_s, w_{l_2}) \in \Delta_2, \\ (\dot{p}_{l_1} + \alpha_{l_1}, \dot{p}_{c_1} + \alpha_{c_1}, \dot{p}_s + \alpha_s) \in \Delta_1^\circ, \\ (-\dot{p}_s + \alpha_s, \dot{p}_{l_2} + \alpha_{l_2}) \in \Delta_2^\circ. \end{aligned}$$

We now invoke the ILDA equations using $D_{||}$. As in (**??**), if we restrict $p \in P$ where $P = \mathbb{F}(L_1 + L_2)$, then we are restricting $\dot{p}_s = 0$ as the consistency condition. Substituting this into the local formulas, we obtain

$$\begin{aligned} (\dot{q}_{l_1}, \dot{q}_{c_1}, \dot{q}_s) = (w_{l_1}, w_{c_1}, w_s) \in \Delta_1, \\ (-\dot{q}_s, \dot{q}_{l_2}) = (-w_s, w_{l_2}) \in \Delta_2, \\ (\dot{p}_{l_1} - \frac{\partial L_1}{\partial q_{l_1}}, \dot{p}_{c_1} - \frac{\partial L_1}{\partial q_{c_1}}, \alpha_s) \in \Delta_1^\circ, \\ (\alpha_s, \dot{p}_{l_2} - \frac{\partial L_2}{\partial q_{l_2}}) \in \Delta_2^\circ, \end{aligned}$$

which follows from Δ_1 and Δ_2 that we can eliminate the variable $\dot{q}_s$ and replace the first two conditions with the condition $(\dot{q}_1, \dot{q}_2) \in \Delta$, where

$$\Delta = \{(v_{l_1}, v_{c_1}, v_{l_2}) : v_{l_1} - v_{c_1} - v_{l_2} = 0\} \subset T(Q_1 \times Q_2).$$

Similarly we may eliminate the variable α_s and use the constraint

$$(\dot{p}_1 + \alpha_1, \dot{p}_2 + \alpha_2) \in \Delta^\circ \subset T^*(Q_1 \times Q_2).$$

It is no coincidence that Δ is the KCL distribution for the connected circuit depicted in figure **??**.

Thus, we can develop the ILDA equations

$$\begin{aligned} \dot{q}_{l1} - \dot{q}_{c1} - \dot{q}_{l2} = 0, \\ l_1 \ddot{q}_{l1} = \frac{1}{c_1} q_{c1} = l_2 \ddot{q}_{l2}. \end{aligned}$$

VI. Bowtie Interconnection

The *Dirac tensor product*, $\boxtimes$, was developed by Gaultieri in [**?**] in the study of complex geometry [3]. It was later found in [**?**] and [**?**] that $\boxtimes$ could be used as an alternative means of interconnecting Dirac structures, where we used the symbol $\bowtie$ instead, and called it the Bowtie product. The interconnection appears rather different from composition in that it comes from another Dirac structure that we call the *interconnection Dirac structure*.

A. Direct Sums of Dirac Structures

Denote the set of Dirac structures on a manifold M_i by $\daleth(M_i)$. Let $D_1 \in \daleth(M_1)$ and $D_2 \in \daleth(M_2)$ be Dirac structures. Let $M = M_1 \times M_2$ with the projections $\mathrm{pr}_i : M \to M_i$. The tangent map is $T\mathrm{pr}_i : TM \to TM_i$ and $T^*\mathrm{pr}_i : T^*M_i \to T^*M$ its dual. Then, we can define the *Dirac direct sum* of D_1 and D_2 by

$$\begin{aligned} D_1 \oplus D_2 = \{((v_1, v_2), \tilde{\Lambda}_1 + \tilde{\Lambda}_2) \in \mathbb{T}M_1 \times M_2 : \\ (X_1, \Lambda_1) \in D_1,\ \ (X_2, \Lambda_2) \in D_2, \\ \tilde{\Lambda}_1 = T^*\mathrm{pr}_1(\Lambda_1),\ \ \tilde{\Lambda}_2 = T^*\mathrm{pr}_2(\Lambda_2)\}. \end{aligned}$$

We state without proof that $D_1 \oplus D_2 \in \daleth(M)$. For the case in which M_1 and M_2 are vector spaces, we may consider

$$T_m^* M \cong T_{m_1}^* M_1 \times T_{m_2}^* M_2$$

for each $m = (m_1, m_2) \in M$. The direct sum of D_1 and D_2 may be written as

$$\begin{aligned} D_1 \oplus D_2 = \{((X_1, X_2), (\Lambda_1, \Lambda_2)) \in \mathbb{T}M : \\ (X_1, \Lambda_1) \in D_1,\ \ (X_2, \Lambda_2) \in D_2\}. \end{aligned}$$

B. The Bowtie Product

Note that $D = D_1 \oplus D_2$ does not alter the dynamics generated on M_1 and M_2 since these Dirac structures do not intersect each other. To do this, we introduce another Dirac structure $D_{\mathrm{int}} \in \daleth(M)$ for interconnecting D_1 and D_2. The bowtie product of D and D_{int} is given by

$$\begin{aligned} D \bowtie D_{\mathrm{int}} = \{(v, \alpha) \in \mathbb{T}M :\ \exists \beta \in T^*M \text{ such that} \\ (v, \alpha + \beta) \in D,\ (v, -\beta) \in D_{\mathrm{int}}\}. \end{aligned}$$

If we let $d : M \hookrightarrow M \times M$ be the diagonal embedding, then a more elegant definition of bowtie can be given as:

$$D \bowtie D_{\mathrm{int}} = d^*(D \times D_{\mathrm{int}})$$

where d^* denotes the pullback of $\daleth(M \times M)$ to $\daleth(M)$ as in [**?**]. $D \bowtie D_{\mathrm{int}}$ is a Dirac structure if the set $\mathrm{pr}_{TM}(D \bowtie D_{\mathrm{int}})$ is a distribution with constant rank [**?**].

[3]We appreciate H. Bursztyn for pointing out this.

C. Example: Port–Interconnection using $\bowtie$

Consider again the circuits depicted in Fig. **??**. We may use the same configuration manifolds Q_1, Q_s, Q_2, Dirac structures D_1 and D_2 and Lagrangians L_1 and L_2 as in Section **??**. Let $Q = Q_1 \times Q_s \times Q_s \times Q_2$ and let $\Delta_{12} \subset TQ$ be the distribution

$$\Delta_{12}(q) = \{(v_{l_1}, v_{c_1}, v_s, -v_s, v_{l_2})\}$$

for each $q \in Q$ and we may lift Δ_{12} to be a distribution Δ_{int} on T^*Q

$$\Delta_{\text{int}}(q,p) = T\pi_Q^{-1}(\Delta_{12}(q)),$$

where $\pi_Q : T^*Q \to Q;\ (q,p) \mapsto q$. Set

$$D_{\text{int}} = \Delta_{\text{int}} \oplus \Delta_{\text{int}}^{\circ}$$

and define

$$D_{\bowtie} = (D_1 \oplus D_2) \bowtie D_{\text{int}}.$$

Then the ILDA equations for $L = L_1 + L_2$ are given by for $(q,v,p) \in \mathbb{T}(T^*Q)$

$$(X(q,v,p), \mathbf{d}E(q,v,p)|_{TP}) \in D_{\bowtie}(q,p),$$

where

$$X(q,v,p) = (\dot{q}_{l_1}, \dot{q}_{c_1}, \dot{q}_{s_1}, \dot{q}_{s_2}, \dot{q}_{l_2}, \dot{p}_{l_1}, \dot{p}_{c_1}, \dot{p}_{s_1}, \dot{p}_{s_2}, \dot{p}_{l_2})$$

and

$$\mathbf{d}E(q,v,p)|_{TP} = (0, \frac{1}{c_1}q_{c_1}, 0, 0, 0, v_{l_1}, v_{c_1}, v_{s_1}, v_{s_2}, v_{l_2}).$$

By the definition of $\bowtie$, there exists $\beta \in T^*Q$ such that:

$$(X, dE|_{TP} + \beta) \in D_1 \oplus D_2, \tag{5}$$
$$(X, -\beta) \in D_{\text{int}}. \tag{6}$$

By the structure of D_{int}, the second condition implies β is of the form

$$\beta = (0, 0, \beta_s, \beta_s, 0, 0, 0, 0, 0, 0) \in \Delta_{\text{int}}^{\circ}$$

and

$$(\dot{q}, \dot{p}) \in \Delta_{\text{int}}. \tag{7}$$

Using this, equation (**??**) can be written locally as

$$\begin{aligned}
(\dot{q}_{l_1}, \dot{q}_{c_1}, \dot{q}_{s_1}) &\in \Delta_1,\\
(\dot{q}_{s_2}, \dot{q}_{l_2}) &\in \Delta_2,\\
(\dot{p}_{l_1}, \dot{p}_{c_1} + \frac{1}{c_1}q_{c_1}, \dot{p}_{s_1} + \beta_s) &\in \Delta_1^{\circ},\\
(\dot{p}_{s_2} + \beta_s, \dot{p}_{l_2}) &\in \Delta_2^{\circ},\\
(\dot{q}_{l_1}, \dot{q}_{c_1}, \dot{q}_{s_1}, \dot{q}_{s_2}, \dot{q}_{l_2}) &= (w_{l_1}, w_{c_1}, w_{s_1}, w_{s_2}, w_{l_2}).
\end{aligned}$$

Equation (**??**) implies $\dot{q}_{s_1} = -\dot{q}_{s_2}$. Recall the primary constraint set

$$p \in P = \mathbb{F}L(\Delta_1 \oplus \Delta_2)$$

induces the consistency conditions $\dot{p}_s = \dot{p}'_s = 0$ and the local expressions gives

$$\begin{aligned}
\dot{q}_{l1} - \dot{q}_{c1} + \dot{q}_{s1} &= 0,\\
\dot{q}_{l2} &= \dot{q}_{s2},\\
\dot{q}_{s1} &= -\dot{q}_{s2},\\
l_1\ddot{q}_{l1} = \frac{1}{c_1}q_{c1} &= \beta = l_2\ddot{q}_{l2}.
\end{aligned}$$

Eliminating $\dot{q}_{s1}, \dot{q}_{s2}$ and β, we can recover the same equations as in Section **??**.

VII. Composition as a projection of the Bowtie Interconnection

In this section we will show the link between the bowtie interconnection and the composition of Dirac structures, where we will focus on the case of linear Dirac structures on vector spaces. Let V_1, V_2, V_s be vector spaces as in section §**??**. Let $V = V_1 \times V_s \times V_s \times V_2$ and $\bar{V} = V_1 \times V_2$. Before going into details, we establish an important fact on the projection from V to $\bar{V}$.

Lemma 1: Given a natural projection $\Psi : V \to \bar{V}$ as $(v_1, v_s, v'_s, v_2) \mapsto (v_1, v_2)$. The mapping $\Psi^\dagger : \bar{V}^* \to V^*$ dual to Ψ is given by

$$\Psi^\dagger(\alpha_1, \alpha_2) = (\alpha_1, 0, 0, \alpha_2) \in V^*.$$

Proof: We have that:

$$\begin{aligned}
\langle \Psi^\dagger(\alpha_1, \alpha_2), (v_1, v_s, v'_s, v_2)\rangle &= \langle(\alpha_1, \alpha_2), \Psi(v_1, v_s, v'_s, v_2)\rangle\\
&= \langle(\alpha_1, \alpha_2), (v_1, v_2)\rangle\\
&= \langle\alpha_1, v_1\rangle + \langle\alpha_2, v_2\rangle\\
&= \langle(\alpha_1, 0, 0, \alpha_2), (v_1, v_s, v'_s, v_2)\rangle
\end{aligned}$$

for arbitrary $(v_1, v_s, v'_s, v_2) \in V$ ■

Next, we recall the push forward map associated to a Dirac structure.

Definition 1: Let $f : V \to W$ be a linear map and let D be a linear Dirac structure on V. The push forward of D to W by f is the set

$$f_*D = \{(f(v), \alpha) \in \mathbb{T}W : (v, f^\dagger(\alpha)) \in D\}$$

It is worth noting that the push forward of a Dirac structure is itself a Dirac structure as in [**?**].

We are now ready to provide the link between the bowtie product and the composition of Dirac structures.

Theorem 1: Let $\Delta_{\text{int}} = \{(v_1, v_s, -v_s, v_2) \in V\}$ and let $D_{\text{int}} = \Delta_{\text{int}} \oplus \Delta_{\text{int}}^{\circ}$. For linear Dirac structures D_1 on $V_1 \times V_s$ and D_2 on $V_s \times V_2$, set

$$D_{\bowtie} = (D_1 \oplus D_2) \bowtie D_{\text{int}}$$

and set also

$$D_{||} = D_1 || D_2.$$

Then, one has

$$D_{||} = \Psi_* D_{\bowtie}.$$

Proof: First observe that $\Delta_{\text{int}}^{\circ} = \{(0, \alpha_s, \alpha_s, 0) \in V^*\}$. We also observe:

$$\begin{aligned}
\Psi_* D_{\bowtie} = \{&(\Psi(v_1, v_s, v'_s, v_2), \alpha_1, \alpha_2) :\\
&(v_1, v_s, v'_s, v_2, \Psi^\dagger(\alpha_1, \alpha_2)) \in D_{\bowtie}\}
\end{aligned}$$

Using the facts that $\Psi(v_1, v_s, v_s', v_2) = (v_1, v_2)$ and $\Psi^\dagger(\alpha_1, \alpha_2) = (\alpha_1, 0, 0, \alpha_2) \in V^*$, it follows

$$\Psi_* D_{\bowtie} = \{(v_1, v_2, \alpha_1, \alpha_2) : \exists v_s, v_s' \in V_s \text{ such that } (v_1, v_s, v_s', v_2, \alpha_1, 0, 0, \alpha_2) \in D_{\bowtie}\}.$$

By definition of the Bowtie product, it follows

$$\begin{aligned}\Psi_* D_{\bowtie} = \{&(v_1, v_2, \alpha_1, \alpha_2) : \exists v_s, v_s' \in V_s \text{ and } \\ &\exists \beta \in V^* \text{ such that} \\ &(v_1, v_s, v_s', v_2, \alpha_1 + \beta_1, \beta_s, \beta_s', \alpha_2 + \beta_2) \in D_1 \oplus D_2, \\ &(v_1, v_s, v_s', v_2, -\beta_1, -\beta_s, -\beta_s', -\beta_2) \in D_{\mathrm{int}}\}\end{aligned}$$

Utilizing the fact that:

$$(v_1, v_s, v_s', v_2, -\beta_1, -\beta_s, -\beta_s', -\beta_2) \in D_{\mathrm{int}}$$

if and only if $v_s = -v_s'$ and $\beta_s = \beta_s', \beta_1 = 0, \beta_2 = 0$, we may restate the above as

$$\Psi_* D_{\bowtie} = \{(v_1, v_2, \alpha_1, \alpha_2) : \exists v_s \in V_s, \alpha_s \in V_s^* \text{ such that } (v_1, v_s, -v_s, v_2, \alpha_1, \beta_s, \beta_s, \alpha_2) \in D_1 \oplus D_2\}.$$

Finally, we have

$$\Psi_* D_{\bowtie} = \{(v_1, v_2, \alpha_1, \alpha_2) : \exists v_s \in V_s, \alpha_s \in V_s^* \text{ such that } (v_1, v_s, \alpha_1, \beta_s) \in D_1, \ (-v_s, v_2, \beta_s, \alpha_2) \in D_2\},$$

but this is nothing but the composed Dirac structure, $D_{||}$. ■

VIII. Conclusions

In this paper, we have shown a link between the composition of Dirac structures and the bowtie interconnection of Dirac structures. The bowtie product provides an alternative to the composition since the latter may be obtained by projecting out the shared variable component. However, for the design of nonlinear control systems on manifolds, one may need to observe some internal variables, which are given by the shared variables. Additionally, an interconnection may not easily manifest a shared variable splitting, yet be easily representable as an interconnection Dirac structure. In such cases, there exists some advantage to using the bowtie interconnection. Having established an inclusive correspondence of composed Dirac structure as a projection of a bowtie interconnected Dirac structure, we can claim that the bowtie interconnection can at least solve the same problems as interconnection by composition.

Additionally, much of the theory on stabilization and achievable Casimers studied in [**?**] for port-Hamiltonian systems using composition will be able to be applied in the context of Lagrange-Dirac dynamical systems using the bowtie-product.

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