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# On Stability Properties of Time-Varying Systems with Delays

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*Abstract*— **We consider several stability properties in the framework of input-to-state stability for time-varying systems with delays. By following a natural approach to convert a time-varying system to an auxiliary time-invariant system, we obtain several results including Razumikhin-type stability criteria, Razumikhin-Lyapunov functions with non-strict decay rates, and cyclic small-gain theorems for time-varying systems with delays.**

*Index Terms.* **time-varying systems, delays, input-to-state stability, nonlinear small-gain, Razumikhin theorems.**

## I. INTRODUCTION

Time-varying systems with delays are frequently seen in practice (see for instance, [2] and [8] for examples from mechanical systems, models of nuclear reactors, and microbiological systems). In this work, we consider stability properties of time-varying systems with delays in the input-to-state stability framework.

A frequently used tool in stability analysis for systems with delays is the Razumikhin theorem (see [2] and [8]). It was first recognized in [16] that the Razumikhin theorem can be re-stated as a small-gain theorem in the context of input-to-state stability. The significance of the work [16] is that it illustrated how the Razumikhin approach makes many results on robust stability analysis for delay-free systems available to systems with delay. In our recent work [17], it was shown that the Razumikhin method can be extended to stability analysis based on trajectories without involving Lyapunov functions. In this work, we develop results on uniform stability properties for time-varying systems based on Lyapunov-Razumikhin functions with non-strict decay rates.

In the context of stability for systems without delays, one can often find a Lyapunov function with a non-strict (i.e., nonnegative) decay rate, while it is much harder to find one with a strict (i.e., negative) decay rate at every moment. The works [12] and [13] focused on constructing Lyapunov functions with strict decay rates based on Lyapunov functions with non-strict decay rates. Other works, as in [10], [11] and [3], have also focused on stability analysis based on Lyapunov functions with non-strict decay rates. It is thus natural to consider stability properties for time-varying systems with delays based Lyapunov-Krasovskii functionals or Lyapunov-Razumikhin functions with non-strict decay rates.

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Some past work on stability analysis for time-varying systems with delays can be found in [6] and [7], where Lyapunov-like characterizations were obtained for uniform and non-uniform output stability properties. More specifically, in [6] and [7], non-strict Lyapunov-Krasovskii functionals or Lyapunov-Razumikhin functions are associated with non-uniform stability properties, and strict Lyapunov-Krasovskii functionals or Lyapunov-Razumikhin function are associated with uniform stability properties. What distinguishes our work from [6] and [7] is that we focus on obtaining uniform stability properties based on non-strict Lyapunov-Razumikhin functions, much in the spirit of [12] and [3].

Our main approach in dealing with a time-varying system with delays is to convert a system into an auxiliary time-invariant system with an output map defined by the state variables of the original system. This way, many results on output stability properties for time-invariant systems can be applied to time-varying systems. As a consequence of the relationship between ISS of time-varying systems and IOS of the auxiliary system, we have extended our recent work [17] on cyclic small-gain theorems to time-varying systems with delays. Much work has been done on small-gain theorems for systems with delays, see e.g., [5], [4], and [1], where various stability properties of interconnected systems are obtained based on Lyapunov-like functionals or other types of auxilary functions. The small-gain theorems obtained in this work will be based directly on trajectories of the systems.

The rest of the paper is organized as follows. In Section II, we discuss basic definitions of stability properties of systems with delays, and derive some Razumikhin-type criteria. In Section III, we discuss Razumikhin-Lyapunov functions whose decay rates are affected by persistently exciting functions. In Section IV, we obtain cyclic small-gain theorems, and in Section V we discuss how the main results can be proved by exploring the output stability properties of an auxiliary time-invariant system corresponding to a time-varying system.

*Notations.* Throughout this work, we use $|\cdot|$ to denote the Euclidean norm of vectors, and $\|\cdot\|_I$ to denote the $L_\infty$ norm of measurable functions on the interval $I$. For $\phi = (\phi_1, \cdots, \phi_k)$ defined on an interval $I$, we let $\|\phi\|_I = \max_{1\le i\le k} \{\|\phi_i\|_I\}$. A function $\alpha : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is *of class* $\mathcal{K}$ if it is continuous, positive definite, and strictly increasing; and is *of class* $\mathcal{K}_\infty$ if it is also unbounded. A function $\beta : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is said to be *of class* $\mathcal{KL}$ if for each fixed $t \ge 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each fixed $s \ge 0$, $\beta(s, t)$ decreases to 0 as $t \to \infty$. For any $\mathcal{K}$-function

$\kappa$, we say that $\kappa < \mathbf{id}$ if $\kappa(s) < s$ for all $s > 0$.

## II. Time-Varying Systems

Let $\theta > 0$ be given, and let $\mathcal{X} = C([-\theta, 0])$ be the space of continuous functions from $[-\theta, 0]$ to $\mathbb{R}$, equipped with the sup-norm $\|\cdot\|_{[-\theta,0]}$. For a function $q$ defined on some interval $[t_0 - \theta, t_1)$, we define, for each $t \in [t_0, t_1)$, the map $(q)_t$ by $(q)_t(s) = q(t+s)$ for all $s \in [-\theta, 0]$.

Consider a time-varying delay system as given below:

$$\dot{x}(t) = f(t, x(t), (x)_t, u(t)), \tag{1}$$

with the initial condition $x(t_0 + s) = \xi(s)$ for $s \in [-\theta, 0]$), where for each $t$, $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$; $\xi$ is a continuous function defined on $[-\theta, 0]$, and the map $f : \mathbb{R} \times \mathbb{R}^n \times \mathcal{X}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is completely continuous (i.e., $f$ is continuous and maps closed and bounded sets to bounded sets), and Lipschitz on compacts, that is, for any compact subset $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathcal{X}^n \times \mathbb{R}^m$, $f$ is Lipschitz on $\Omega$.

The inputs of the system, denoted by $u$, are measurable and locally essentially bounded functions defined on some interval $[t_0, T_u)$. With the assumptions on $f$ stated above, the existence and uniqueness properties hold for (1) (c.f. [2], [17]), that is, for each input $u$, for each $t_0 \geq 0$ and $\xi \in \mathcal{X}^n$, there is a unique solution to the system (1) defined on some maximum interval $[t_0 - \theta, r)$ satisfying the initial condition $x(t_0 + s) = \xi(s)$ for $s \in [-\theta, 0)$. We use $x(t; t_0, \xi, u)$ to denote such a solution. A system is said to be forward complete if for all $t_0 \geq 0, \xi \in \mathcal{X}^n$ and all $u$ defined on $[0, \infty)$, the trajectory $x(t; t_0, \xi, u)$ is defined for all $t \geq 0$.

For a time-varying delay system as in (1), the input-to-state stability properties are defined in the same way as for delay-free systems (refer to [9] for time-varying ISS in the delay-free case).

*Definition 2.1:* A time-varying delay system as in (1) is said to be:

- *uniformly input-to-state stable* (U-ISS) if there exist some $\beta \in \mathcal{KL}$ and $\rho \in \mathcal{K}$ such that the following holds along every trajectory:

$$|x(t + t_0; t_0, \xi, u)| \tag{2}$$
$$\leq \max\left\{\beta\left(\|\xi\|_{[-\theta,0]}, t\right), \rho\left(\|u\|_{[t_0,\infty)}\right)\right\} \ \forall t \geq 0,$$

- *semi-uniform input-to-state stable* (SEMI-UISS) if for some $\sigma$ and $\rho \in \mathcal{K}$, the following two conditions are satisfied along each trajectory:

$$|x(t; t_0, \xi, u)| \leq \max\left\{\sigma\left(\|\xi\|_{[-\theta,0]}\right), \rho\left(\|u\|_{[t_0,\infty)}\right)\right\}, \tag{3}$$

for all $t \geq t_0$, and

$$\overline{\lim_{t\to\infty}} |x(t; t_0, \xi, u)| \leq \rho\left(\|u\|_{[t_0,\infty)}\right). \tag{4}$$

*Remark 2.2:* It was show that in the delay-free case, the SEMI-UISS property is equivalent to the existence of $\beta \in \mathcal{KL}$, $\sigma \in \mathcal{K}$ and $\rho \in \mathcal{K}$ such that

$$|x(t + t_0; t_0, x_0, u)| \tag{5}$$
$$\leq \max\left\{\beta\left(|x_0|, \frac{t}{1 + \sigma(t_0)}\right), \rho\left(\|u\|_{[t_0,\infty)}\right)\right\} \tag{6}$$

for all $t \geq 0$ (see [9]). However, it is still not clear if for systems with delays the SEMI-UISS property is equivalent to an estimation as in (5) with $x_0$ and $|x_0|$ replaced by $\xi$ and $\|\xi\|$ respectively. □

Observe that the type of system represented by (1) also includes systems of the type $\dot{x}(t) = f(t, (x)_t, u(t))$. A main reason for considering a system as in (1) where state variables with delays are presented separately is that it is often convenient to treat the state variables with delays as disturbances as discussed below. For a system as in (1), consider the (state) delay-free system

$$\dot{z}(t) = f(t, z(t), (w)_t, u(t)), \tag{7}$$

where $(w, u)$ is considered the input of the system. Though this system appears to have time-delays, it is in fact delay-free, since it can be presented as

$$\dot{z}(t) = f(t, z(t), u(t)),$$

where for each $t$, $u(t)$ takes values in the infinite dimentional spapce $\mathcal{X}^n \times \mathbb{R}^m$.

The time-varying $z$-system represented by (7) is U-ISS if for some $\beta \in \mathcal{KL}$, some $\kappa, \rho \in \mathcal{K}$, it holds that

$$|z(t + t_0; t_0, z_0, w, u)| \leq \max\Big\{\beta(z_0, t), \tag{8}$$
$$\kappa\left(\|w\|_{[t_0-\theta,\infty)}\right), \rho\left(\|u\|_{[t_0,\infty)}\right)\Big\} \ \forall t \geq 0.$$

Note that if (8) holds along each $z$-trajectory, then the following holds along each $x$-trajectory *on its maximum interval*:

$$|x(t + t_0; t_0, \xi, u)| \leq \max\{\beta(|\xi(0)|, t), \tag{9}$$
$$\kappa\left(\|x\|_{[t_0-\theta,t)}\right), \ \rho\left(\|u\|_{[t_0,\infty)}\right)\}.$$

As defined in [9], we say that a delay-free system as in (7) is SEMI-UISS if for some $\mathcal{K}$-functions $\sigma, \kappa, \rho$, the following hold:

$$|z(t; t_0, z_0, w, u)| \leq \max\Big\{\sigma(|z_0|), \tag{10}$$
$$\kappa(\|w\|_{[t_0-\theta,\infty)}), \ \rho(\|u\|_{[t_0,\infty)})\Big\} \quad \forall t \geq t_0,$$

and

$$\overline{\lim_{t\to\infty}} |z(t; t_0, z_0, w, u)| \tag{11}$$
$$\leq \max\Big\{\kappa(\|w\|_{[t_0-\theta,\infty)}), \ \rho\left(\|u\|_{[t_0,\infty)}\right)\Big\}.$$

Similar to the time-invariant case (see [17]), we have the following Razumikhin-type result:

*Theorem 1:* Consider system (1).

($a$.) The system is U-ISS if the corresponding delay-free $z$-system (7) satisfies a U-ISS estimation as in (8) with $\kappa < \mathbf{id}$.

($b$.) The system is SEMI-UISS if the corresponding delay-free $z$-system (7) satisfies the SEMI-UISS estimations as in (10)–(11) with $\kappa < \mathbf{id}$. ■

The following is a slightly more general result than part (a.) of Theorem 1:

*Lemma 2.3:* System (1) is U-ISS if and only if for some $\beta \in \mathcal{KL}$, $\kappa, \rho \in \mathcal{K}$ with $\kappa < \mathbf{id}$, the U-ISS-like estimation (9) holds on the maximum interval along each trajectory.

We view Lemma 2.3 as a more general result than part (a) of Theorem 1 for the following reason. If property (8) holds, then, with $w(t) = x(t; t_0, \xi, u)$ defined on the maximum interval of $x(t; t_0, \xi, u)$, one sees that (9) holds. However, it is still not clear whether under the assumpution that (9) holds, condition (8) would hold for *all* inputs $w$.

## III. LYAPUNOV FUNCTIONS

One of the central approaches used in stability analysis for systems with delays is via the construction of Razumikhin-type Lyapunov functions (see e.g., [2] and [16]).

A $C^1$-function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an ISS-Razumikhin-Lyapunov function for a system as in (1) if the following holds:

- there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|p|) \leq V(t,p) \leq \overline{\alpha}(|p|) \quad \forall\, p \in \mathbb{R}^n, \tag{12}$$

- there exist $\alpha \in \mathcal{K}$, $\kappa, \chi \in \mathcal{K}_\infty$ with $\kappa < \mathbf{id}$, such that for all $\xi \in \mathcal{X}^n$, and all $u \in \mathbb{R}^m$, the following holds:

$$V(t, \xi(0)) \geq \max\{ \max_{s\in[-\theta,0]} \{\kappa(V(t+s, \xi(s))\},\ \chi(|u|)\} \Longrightarrow \dot{V}(t, \xi(0)) \leq -\alpha(V(t, \xi(0))),$$

where $\dot{V}(t,\xi(0)) = \frac{d}{d\tau}\big|_{\tau=0^+} V(t+\tau, x(t+\tau; t, \xi, u))$.

The Razumikhin-type Lyapunov theorem (see [16]) states that if a system as in (1) admits a Razumikhin-Lyapunov function, then the system is U-ISS. When treating the state variables as disturbances, this result can be stated as in the following.

*Proposition 3.1:* Consider system (1). Assume that the corresponding delay-free system (7) admits a $C^1$ ISS-Lyapunov function $V$ satisfying (12), and for some $\kappa, \chi, \alpha \in \mathcal{K}_\infty$, the following holds for all $z \in \mathbb{R}^n, w \in \mathcal{X}^n$ and $u \in \mathbb{R}^m$:

$$V(t,z) \geq \max\{ \max_{s\in[-\theta,0]} \{\kappa(V(t+s, w(s)))\},\ \rho(|u|)\} \Rightarrow D_tV(t,z) + D_zV(t,z)f(t,z,w,u) \leq -\alpha(V(t,z)).$$

If the small-gain condition $\kappa < \mathbf{id}$ holds, then the system (1) is U-ISS. □

By converting the stability analysis for a delay system to one for the corresponding delay-free system, many tools for robust stability analysis, in the ISS context for delay-free systems, are made available to the delay system. However, finding a Lyapunov function for a time-varying system, even in the delay-free case, with a decay rate that is uniform in the intial time can be a hard task. Analogous to the work in [12] and [3], we consider a type of Lyapunov function with a weaker requirement on the decay rate. A $C^1$ function $V$ is called a $p$-Lyapunov function for (7) if

(a.) for some $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, it holds that

$$\underline{\alpha}(|z|) \leq V(t,z) \leq \overline{\alpha}(|z|) \quad \forall\, z \in \mathbb{R}^n; \tag{13}$$

(b.) for some $\kappa, \chi, \alpha \in \mathcal{K}_\infty$ and some locally integrable function $p$ defined on $[0, \infty)$, the following holds for all $z \in \mathbb{R}^n, w \in \mathcal{X}^n$ and $u \in \mathbb{R}^m$:

$$V(t,z) \geq \max\{ \max_{s\in[-\theta,0]} \{\kappa(V(t+s, w(s)))\},\ \rho(|u|)\} \Rightarrow D_tV(t,z) + D_zV(t,z)f(t,z,w,u) \leq -p(t)\alpha(V(t,z)). \tag{14}$$

*Theorem 2:* Consider system (1). Assume that the corresponding delay-free system (7) admits a $p$-Lyapunov function $V$ satisfying (13)–(14). Suppose the following holds:

(a.) $p(t) \geq 0$ a.e., and there exist $\delta > 0$ and $r > 0$ such that

$$\int_t^{t+\delta} p(s)\, ds \geq r \quad \forall\, t \geq 0;$$

(b.) (small-gain condition): $\kappa < \mathbf{id}$.

Then system (1) is U-ISS. ∎

One may weaken the condition on $p(\cdot)$ to get the following result on SEMI-UISS which extends a result in [9] to systems with delays (see also [6] and [7] for related results on non-uniform ISS-like properties associated with Lyapunov-Krasovskii functionals):

*Theorem 3:* Consider system (1). Assume that the corresponding delay-free system (7) admits a $p$-Lyapunov function $V$ satisfying (13)–(14). Suppose the following conditions hold:

(a.) $p(t) \geq 0$ a.e., and

$$\int_0^\infty p(s)\, ds = \infty; \tag{15}$$

(b.) (small-gain condition): $\kappa < \mathbf{id}$.

Then system (1) is SEMI-UISS. ∎

To illustrate the meaning of the above two theorems, we consider the following examples (with $u \equiv 0$ for simplicity):

$$\dot{x}(t) = -\sin^2 t(x(t) + bx(t-\theta)); \tag{16}$$

and

$$\dot{x}(t) = -\frac{1}{1+t}\left(x(t) + b\int_{t-\theta}^{t} x(s)\, ds\right). \tag{17}$$

For system (16), the corresponding delay-free $z$-system is

$$\dot{z}(t) = -\sin^2 t(z(t) + bw(t-\theta)); \tag{18}$$

and for system (17), the corresponding delay-free $z$-system is

$$\dot{z}(t) = -\frac{1}{1+t}\left(z(t) + b\int_{t-\theta}^{t} w(s)\, ds\right). \tag{19}$$

Let $V(z) = z^2$. Then, for system (18), it holds that

$$D_zVf(z,w) = -2\sin^2 t \cdot z(z + bw(t-\theta)),$$

and hence, for any $0 < c < 1$, one has the following:

$$V(z) \geq \max_{s\in[-\theta,0]} \kappa_0 \cdot V(w(s)) \Rightarrow D_zV(z)f(z, w(-\theta)) \leq -2(1-c)\sin^2 t \cdot V(z),$$

where $\kappa_0 = (\frac{|b|}{c})^2$. Since

$$\int_t^{t+\delta} \sin^2 s\, ds = \frac{\delta}{2} - \frac{1}{4}\sin 2(t+\delta) + \frac{1}{4}\sin 2t,$$

the function $p(t) := 2(1-c)\sin^2 t$ satisfies property (a) in Theorem 2 with $\delta = 2$ and $r = 1-c$. By Theorem 2, one sees that the $x$-system (16) is uniformly-GAS whenever $|b| < 1$.

For system (19), with $V(z) = z^2$, one has

$$\begin{aligned} D_z V(z,w) &= -\frac{2z}{1+t}\left(z + b\int_{-\theta}^{0} w(s)\,ds\right) \\ &\le -\frac{2z}{1+t}\left(z + b\theta\,\|w\|\right) \end{aligned}$$

Consequently, for any $0 < c < 1$, it holds that

$$V(z) \ge \max_{s\in[-\theta,0]} \kappa_1 \cdot V(w(s))$$
$$\Rightarrow D_z V(z) f(z, w(-\theta)) \le -2(1-c)\frac{1}{1+t}\cdot V(z),$$

where $\kappa_1 = (\theta\,|b|\,/c)^2$. It can be seen that the function $p(t) := \frac{2(1-c)}{1+t}$ satisfies property (a) of Theorem 3. Hence, the system (17) (which has distributed delays) is semi-uniformly GAS.

Observe that the function $\frac{1}{1+t}$ slows down the decay rate of the trajectories of system (17). It can, in fact, be shown rigorously that (17) is not uniformly-GAS, due to this "slowing down" effect.

## IV. Interconnected Systems

Consider now a time-varying interconnected system with delays

$$\begin{aligned} \dot{x}_1(t) &= f_1\left(t, x_1(t), (x_1)_t, (\widehat{v}_1)_t, u_1(t)\right), \\ \dot{x}_2(t) &= f_2\left(t, x_2(t), (x_2)_t, (\widehat{v}_2)_t, u_2(t)\right), \\ &\vdots \\ \dot{x}_k(t) &= f_k\left(t, x_k(t), (x_k)_t, (\widehat{v}_k)_t, u_k(t)\right), \end{aligned} \tag{20}$$

where $\widehat{v}_i = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots v_n)^T$, together with $u_i$, is treated as the input of the $x_i$-subsystem. To consider the U-ISS property of the large-scale system (20)

$$v_i(t) = g_i(x(t)), \tag{21}$$

where $g_i$ is a $C^1$ function satisfying $|g_i(x)| \le |x_i|$, we impose the following U-ISS estimation on each $x_i$-subsystem:

$$\begin{aligned} &|x_i(t+t_0; t_0, \xi_i, \widehat{v}_i, u_i)| \\ &\le \max_{i\neq j}\{\beta_i(\|\xi_i\|, t), \gamma_{ij}(\|v_j\|, \rho_i(\|u_i\|)\}. \end{aligned} \tag{22}$$

For the SEMI-UISS property for (20)-(21), we impose the following SEMI-UISS estimations on each $x_i$-subsystem:

$$\begin{aligned} &|x_i(t; t_0, \xi_i, \widehat{v}_i, u_i)| \\ &\le \max_{i\neq j}\{\sigma_i(\|\xi_i\|),\ \gamma_{ij}(\|v_j\|),\ \rho_i(\|u_i\|)\} \end{aligned} \tag{23}$$

for all $t \ge t_0$, and

$$\overline{\lim_{t\to\infty}}\, |x_i(t; t_0, \xi_i, \widehat{v}_i, u_i)| \le \max_{i\neq j}\{\gamma_{ij}(\|v_{ij}\|),\ \rho_i(\|u_i\|)\} \tag{24}$$

where $\sigma, \kappa_i, \gamma_{ij}, \rho_i \in \mathcal{K}$.

The collection of gain functions $\{\gamma_{ij}\}$ is said to satisfy the (cyclic) small-gain condition if for each $r = 2, \ldots, k$:

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \cdots \circ \gamma_{i_r i_1} < \mathbf{id}, \tag{25}$$

for $1 \le i_j \le k$, where $i_j \neq i_{j'}$ if $j \neq j'$. The set of the small-gain conditions given by (25) can be more succinctly stated as: the composition of the gain functions along every closed cycle is a contraction.

As in the time-invariant case, we have the following small-gain theorem:

*Theorem 4:* Consider the large scale system (20) with the interconnection (21). Assume that each $x_i$-subsystem satisfies the U-ISS estimation (22) (the SEMI-UISS estimations (23)-(24) respectively). Suppose that the cyclic small-gain condition holds for $\{\gamma_{ij}\}$. Then the interconnected system (20)-(21) is U-ISS (SEMI-UISS, respectively). ∎

To consider a Razumikhin-type small-gain theorem, we associate to each $x_i$-subsystem a delay-free $z_i$-subsystem:

$$\dot{z}_i(t) = f_i\left(t, z_i(t), (w_i)_t, (\widehat{v}_i)_t, u_i(t)\right).$$

For the U-ISS property of (20)–(21), we consider the U-ISS property for each $z_i$-subsystem

$$\begin{aligned} |z_i(t+t_0)| \le \max_{i\neq j}\{&\beta_i(|z_i(t_0)|, t), \\ &\kappa_i(\|w_i\|), \gamma_{ij}(\|v_{ij}\|, \rho_i(\|u_i\|)\}. \end{aligned} \tag{26}$$

To consider the SEMI-UISS property for system (20)–(21), we assume the following SEMI-UISS estimations for each $z_i$-system:

$$\begin{aligned} |z_i(t)| \le \max_{i\neq j}\{&\sigma_i(|z_i(t_0)|), \\ &\kappa_i(\|w_i\|),\ \gamma_{ij}(\|v_{ij}\|),\ \rho_i(\|u_i\|)\}, \end{aligned} \tag{27}$$

for all $t \ge t_0$, and

$$\overline{\lim_{t\to\infty}}\, |z(t)| \le \max_{i\neq j}\{\kappa_i(\|w_i\|),\ \gamma_{ij}(\|v_{ij}\|),\ \rho_i(\|u_i\|)\} \tag{28}$$

where $\sigma, \kappa_i, \gamma_{ij}, \rho_i \in \mathcal{K}$.

Below we present a small-gain theorem when various stability properties are imposed on the state-delay-free $z_i$-subsystems:

*Theorem 5:* Consider the interconnected system (20)–(21). Assume for each $x_i$-subsystem, the corresponding $z_i$-system satisfies the U-ISS estimation (26) (the SEMI-UISS estimations (27)-(28) respectively). Suppose $\kappa_i < \mathbf{id}$ for each $i$ and the cyclic small-gain condition holds for $\{\gamma_{ij}\}$. Then the interconnected system (20)-(21) is U-ISS (SEMI-UISS, respectively). ∎

## V. Input-to-Output Stability Properties

A time-varying system as in (1) can be treated as a time-invariant system with an output map as in the following:

$$\begin{aligned} \dot{x}(t) &= f(\lambda(t),\ x(t),\ (x)_t,\ (v)_t,\ u(t)) \\ \dot{\lambda}(t) &= 1, \\ y(t) &= x(t), \end{aligned} \tag{29}$$

where the initial conditions are $(x)_0 = \xi$ (i.e., $x(t) = \xi(t)$ for $t \in [-\theta, 0]$) and $(\lambda)_0 = t_0$. (Note: We define $\lambda(s) := t_0$ for all $s \in [-\theta, 0]$. Instead of defining it to be a constant function, one could use $\lambda(s) := t_0 - s$ for all $s \in [-\theta, 0]$.) This motivates the consideration of input-to-output stability properties for the more general systems with outputs.

Consider the following time-invariant system with an output map:

$$\begin{aligned}\dot{x}(t) &= f\Big(x(t),\,(x)_t,\,(v)_t,\,u(t)\Big),\\ y(t) &= h(x(t)),\end{aligned} \tag{30}$$

where $f : \mathbb{R}^n \times \mathcal{X}^n \times \mathcal{X}^p \times \mathbb{R}^m \to \mathbb{R}^n$ is completely continuous and Lipschitz on compacts, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous, and $h(0) = 0$. For system (30), we use $x(t; \xi, v, u)$ to denote the solution of the system with initial value $x(t) = \xi(t)$ on $[-\theta, 0]$ and the input $(v, u)$, and we denote the output $h(x(t; \xi, v, u))$ by $y(t; \xi, v, u)$. We follow [15] to define the following output stability properties.

*Definition 5.1:* A forward complete system as in (30) is said to satisfy:

• *the state-independent-input-to-output-stable* (SI-IOS) property, if there exist some $\beta \in \mathcal{KL}$, $\gamma, \rho \in \mathcal{K}$ such that

$$\begin{aligned}|y(t;\xi,v,u)| \le \max\Big\{&\beta\left(\|y\|_{[-\theta,0]},\,t\right),\\ &\gamma\left(\|v\|_{[-\theta,\infty)}\right),\rho\left(\|u\|_{[0,\infty)}\right)\Big\}\ \ \forall t\ge 0,\end{aligned} \tag{31}$$

• the *output Lagrange global stability* (OL-GS) property, if there are $\mathcal{K}$-functions $\sigma^y(\cdot)$, $\sigma^v(\cdot)$ and $\sigma^u(\cdot)$ such that

$$\begin{aligned}|y(t;\xi,v,u)| \le \max\Big\{&\sigma^y\left(\|y\|_{[-\theta,\ 0]}\right),\\ &\sigma^v\left(\|v\|_{[-\theta,\infty)}\right),\ \sigma^u\left(\|u\|_{[0,\infty)}\right)\Big\}\ \forall t\ge 0,\end{aligned} \tag{32}$$

• the *output asymptotic gain* (O-AG) property, if there are $\mathcal{K}$-functions $\gamma^v(\cdot)$ and $\gamma^u(\cdot)$ such that

$$\begin{aligned}&\overline{\lim_{t\to\infty}}\,|y(t;\xi,v,u)|\\ &\quad\le \max\Big\{\gamma^v\left(\|v\|_{[-\theta,\infty)}\right),\gamma^u\left(\|u\|_{[0,\infty)}\right)\Big\}.\end{aligned} \tag{33}$$

For more detailed studies of these output stability properties, we refer the reader to [15] and [17].

A system as in (30) is said to be *unboundedness observable* (UO) if for every input $(v, u)$ and on any finite interval $[t_0,\ t)$ on which $x(t)$ is defined, $x(t)$ stays bounded on $[t_0,\ t)$ whenever $(y(t), v(t), u(t))$ is bounded on $[t_0,\ t)$. In particular, a forward complete system is UO.

It can be seen that the auxiliary system (29) associated with a time-varying system as in (1) is always UO.

The following result illustrates the relation between the stability properties of a time-varying system (1) and the output stability properties of its auxiliary system (29).

*Proposition 5.2:* A time-varying system with delays as in (1) satisfies the U-ISS property (SEMI-UISS property, respectively) if and only if its auxiliary time-invariant system (29) with output $y = x$ satisfies the SI-IOS property (OL-GS and O-AG properties, respectively).

To prove Proposition 5.2, we associate with a time-varying system

$$\dot{x}(t) = f(t, (x)_t, (w)_t), \tag{34}$$

(where $f$ is completely continuous, Lipschitz on compacts, and $(w)_t = ((v)_t, u(t))$, the auxiliary system defined by

$$\begin{aligned}\dot{q}(t) &= f(\lambda(t), (q)_t, (w)_t)\\ \dot{\lambda}(t) &= 1, \quad y(t) = q(t)\end{aligned} \tag{35}$$

We denote by $x(t; t_0, \xi, w)$ the trajectory of (34) with the initial condition $x(t_0 + s) = \xi(s)$ on $[-\theta, 0]$ and the input $w$; and by $(q(t; (\lambda_0, \xi), w), \lambda_0 + t)$ the trajectory of (35) with the initial condition $(q(s), \lambda(s)) = (\xi(s), \lambda_0)$ on $[-\theta, 0]$.

Proposition 5.2 follows from the the following property:

$$x(t + t_0; t_0, \xi, T_{t_0} w) = q(t; (t_0, \xi), w) \quad \forall\, t \ge 0, \tag{36}$$

where $T_{t_0} w(t) = w(t - t_0)$.

Proposition 5.2 allows one to treat a time-varying system as a time-invariant system, and as a result, various stability results for time-invariant systems can be extended to time-varying systems. Theorems 4 and 5 in the previous section thus follow directly from the small-gain theorems provided in [17].

### A. A Sketched Proof of Theorem 1

Part (a.) of Theorem 1 follows from Lemma 2.3. To prove Lemma 2.3, we consider the following result on the SI-IOS property of a time-invariant system as in (30):

*Proposition 5.3:* A UO system as in (30) is SI-IOS if and only if for some $\beta \in \mathcal{KL}$, $\kappa, \gamma, \rho \in \mathcal{K}$ with $\kappa < \mathbf{id}$ such that the following holds on the maximum interval along each trajectory:

$$\begin{aligned}|y(t;\xi,v,u)| \le \max\Big\{&\beta\left(\|y\|_{[-\theta,0]},\,t\right),\\ &\kappa\left(\|y\|_{[-\theta,t)}\right),\gamma\left(\|v\|_{[-\theta,\infty)}\right),\rho\left(\|u\|_{[0,\infty)}\right)\Big\}.\end{aligned} \tag{37}$$

Then the system satisfies the SI-IOS property.

Proposition 5.3 can be proved by following the same ideas as in the proof of Theorem 1 in [17].

### B. A Sketched Proof of Theorem 2

For a system as in (1), we assume that its corresponding delay-free system (7) admits a $p$-Lyapunov function $V$ satisfying (13)–(14) with $\kappa < \mathbf{id}$.

First of all, by Proposition 13 in [14], one sees that there exists a $\mathcal{K}_\infty$ function $\lambda$ which is $C^1$ on $(0, \infty)$ such that $\lambda'(s)\alpha(s) \ge \lambda\circ\alpha(s)$ for all $s > 0$. Let $\mathcal{W}(t, z) = \lambda(V(t, z))$. It then follows that

$$\begin{aligned}&\mathcal{W}(t,z) \ge \max\{\max_{s\in[-\theta,0]}\{\kappa_\lambda(\mathcal{W}(t+s,w(s)))\},\rho_\lambda(|u|)\}\\ &\quad\Rightarrow\ D_t\mathcal{W}(t,z) + D_z\mathcal{W}f(t,z,w,u) \le -p(t)\mathcal{W}(t,z),\end{aligned}$$

where $\kappa_\lambda = \lambda \circ \kappa \circ \lambda^{-1}$, $\rho_\lambda = \lambda \circ \rho$.

Let $t_0, \xi, w, u$ be given, and let $z(t)$ denote the corresponding trajectory. By some standard arguments, one can show that

$$\begin{aligned}\mathcal{W}(t+t_0, z(t+t_0)) \le \max\Big\{&\mathcal{W}(t_0, z(t_0))e^{-\int_{t_0}^{t+t_0} p(s)ds},\\ &\kappa_\lambda(\|\mathcal{W}(s,w(s))\|_{[t_0-\theta,\infty)}),\ \rho_\lambda(\|u\|_{[t_0,\infty)})\Big\}\end{aligned}$$

for all $t \ge 0$. Note that property (15) implies that for some $b > 0$ and $M > 0$ (for more details, see [3]), it holds that

$$e^{-\int_{t_0}^{t+t_0} p(s)ds} \le Me^{-bt}.$$

It then follows that

$$\mathcal{W}(t+t_0,z(t+t_0)) \le \max\Big\{\beta_0(\mathcal{W}(t_0,z_0),t)\\ \kappa_\lambda(\|\mathcal{W}(s,w(s))\|_{[t_0-\theta,\infty)}),\ \rho_\lambda(\|u\|_{[t_0,\infty)})\Big\},$$

where $\beta_0(r,t) = Mre^{-bt}$. Observe that $\kappa_\lambda$ still satisfies the small-gain condition $\kappa_\lambda < \mathbf{id}$. By the same idea in the proof of Theorem 1 in [17], one can show that there exist some $\hat{\beta} \in \mathcal{KL}, \hat{\rho} \in \mathcal{K}$ such that, for all $t \ge 0$,

$$\mathcal{W}(t+t_0,x(t+t_0)) \le \max\Big\{\hat{\beta}(\overline{\alpha}(\|\xi\|),t),\ \hat{\rho}(\|u\|_{[t_0,\infty)})\Big\}$$

From this one concludes that system (1) is U-ISS.

### C. A Sketched Proof of Theorem 3

First consider a trajectory $z(t)$ of the delay-free system (7) with the initial data $(t_0,\xi)$ and input $(w,u)$. As in the proof of Theorem 1, one can show the following:

$$\mathcal{W}(t+t_0,z(t+t_0)) \le \max\Big\{\mathcal{W}(t_0,z(t_0))e^{-\int_{t_0}^{t+t_0} p(s)ds},\\ \kappa_\lambda(\|\mathcal{W}(s,w(s))\|_{[t_0-\theta,\infty)}),\ \rho_\lambda(\|u\|_{[t_0,\infty)})\Big\}$$

for all $t \ge 0$. From this, we get the following:

$$\mathcal{W}(t+t_0,z(t+t_0)) \le \max\Big\{\mathcal{W}(t_0,z(t_0)),\\ \kappa_\lambda(\|\mathcal{W}(s,w(s))\|_{[t_0-\theta,\infty)}),\ \rho_\lambda(\|u\|_{[t_0,\infty)})\Big\} \quad (38)$$

for all $t \ge 0$, and

$$\overline{\lim_{t\to\infty}}\, \mathcal{W}(t+t_0,z(t+t_0)) \quad (39)\\ \le \max\{\kappa_\lambda(\|\mathcal{W}(s,w(s))\|_{[t_0-\theta,\infty)}),\ \rho_\lambda(\|u\|_{[t_0,\infty)})\}.$$

Note that (38)–(39) hold for all inputs $(w,u)$, and by causality, (38)–(39) can be rewriten as

$$\mathcal{W}(t+t_0,z(t+t_0)) \le \max\Big\{\mathcal{W}(t_0,z(t_0)),\\ \kappa_\lambda(\|\mathcal{W}(s,w(s))\|_{[t_0-\theta,t)}),\ \rho_\lambda(\|u\|_{[t_0,\infty)})\Big\} \quad (40)$$

for all $t \ge 0$, and

$$\overline{\lim_{t\to\infty}}\, \mathcal{W}(t+t_0,z(t+t_0)) \le \max\Big\{\rho_\lambda(\|u\|_{[t_0,\infty)}),\\ \overline{\lim_{t\to\infty}}\kappa_\lambda(\|\mathcal{W}(s,w(s))\|_{[t-\theta,\infty)})\Big\}. \quad (41)$$

To get the stability property for the $x$-system (1), consider a trajectory $x(t)$ with the initial data $(t_0,\xi)$ and the input $u$. Property (40) implies the following on the maximum interval of $x(\cdot)$:

$$\mathcal{W}(t+t_0,x(t+t_0)) \le \max\Big\{\mathcal{W}(t_0,x(t_0)),\\ \kappa_\lambda(\|\mathcal{W}(s,x(s))\|_{[t_0-\theta,t_0+t)}),\ \rho_\lambda(\|u\|_{[t_0,\infty)})\Big\}.$$

One can then conclude that the maximum interval of $x(\cdot)$ is $[0,\infty)$, and, for some $\sigma \in \mathcal{K}$.

$$\mathcal{W}(t+t_0,x(t+t_0))\\ \le \max_{s\in[-\theta,0]}\Big\{\sigma(\mathcal{W}(t_0+s,\xi(s))),\rho_\lambda(\|u\|_{[t_0,\infty)})\Big\}$$

for all $t \ge 0$. Moreover,

$$\overline{\lim_{t\to\infty}}\, \mathcal{W}(t+t_0,x(t+t_0)) \le \rho_\lambda\left(\|u\|_{[t_0,\infty)}\right).$$

The SEMI-UISS property follows readily.

## VI. Conclusion

In this work, we have shown that the U-ISS and SEMI-UISS properties of a time-varying system can be obtained by studying the robust stability properties of a delay-free system resulted by treating the state variables with delays as disturbances. Along the line of Razuminkhin-type results, we show that the existence of Razumikhin-Lyapunov functions whose decay rates are affected by persistently exciting functions are sufficient for stability properties. By converting a time-varying system to a time-invariant auxiliary system, we extend our recent work on cyclic small-gain theorems to time-varying systems with delays.

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