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Optimal Control of Multi-battery Energy-aware Systems

Tao Wang and **Christos G. Cassandras**
Division of Systems Engineering
and Center for Information and Systems Engineering
Boston University, renowang@bu.edu, cgc@bu.edu

Abstract— **We study the problem of optimally controlling a set of non-ideal rechargeable batteries that can be shared to perform a given amount of work over some specified time period. We seek to maximize the minimum residual energy among all batteries at the end of this period by optimally controlling the discharging and recharging process at each battery. Modeling a battery as a dynamic system, we adopt a Kinetic Battery Model (KBM) and formulate an optimal control problem under the constraint that discharging and recharging cannot occur at the same time. We show that the optimal solution must result in equal residual energies for all batteries as long as such a policy is feasible. This simplifies the task of subsequently deriving explicit solutions for the problem.**

I. Introduction

With the increasing use and dependence on wireless and mobile devices, batteries are playing a critical role in areas such as communications, automotive, transportation, robotics, and consumer electronics. Due to their limited power capacity, especially for small and light devices, research on energy management of battery-powered systems has become increasingly active. The opportunity to recharge batteries through energy harvesting for small devices or connecting to the grid for electric vehicles adds an extra level of flexibility and power control. Energy-aware systems of this type have been studied with techniques such as Dynamic Voltage Scheduling (DVS) [1], [2], [3] where a battery is modeled as a queueing system [4], usually based on the assumption that the battery is "ideal," i.e., it maintains a constant voltage throughout the discharge process and a constant capacity for all discharge profiles. However, because of the *rate capacity effect* [5] and the *recovery effect* [6], both characterizing real batteries, the voltage as well as energy amount delivered by the battery heavily rest on the discharge profile. Therefore, when dealing with energy optimization, it is necessary to take that into account along with nonlinear variations in a battery's capacity. As a result, there are several proposed models to describe a non-ideal battery. They are broadly classified into models based on Partial Differential Equations (PDE) [7], [8], diffusion-based models [9], [10], [11], and the Kinetic Battery Model (KBM) [12], [13]. Since an efficient battery model in energy-aware systems requires not only accuracy but also computational speed in quantifying battery discharge behaviors under various profiles, the use of PDE-based and diffusion-based models is limited, especially in real-time application settings [14], [13]. In contrast, a KBM combines speed with sufficient accuracy, as reported, for instance, in embedded system applications [12]. It is also suitable for large-scale systems such as wireless sensor networks [15] where batteries are distributed over the nodes in the network.

With this motivation, in [16] we studied an optimal control problem based on a KBM with the added feature of a recharging capability so that the battery may be in either discharging or recharging mode at any time. We showed that an optimal policy maximizing the work performed by the battery over a given time interval with the requirement that its energy is at a desired level at the end of this interval is of bang-bang type with an optimal time to switch from discharging to recharging within the constraints of the problem. In this paper, we are interested in studying systems with *multiple* rechargeable batteries which can be shared in performing a certain amount of work, viewing this as a first step toward battery-powered networked systems with renewable energy. Along these lines, in [4], a dynamic node activation problem in networks of rechargeable sensors is addressed by modeling the battery as a queueing system processing energy tasks. In [17] an optimal control policy is presented for cross-layer resource allocation in wireless networks operating with rechargeable batteries. In [18] advantage is taken of battery energy storage in optimal power flow problems, while in [19] a network resource allocation problem is presented for energy-harvesting sensor platforms with time-varying battery recharging rates. However, in all these cases the battery models used are simple and assume ideal behavior.

In this paper, we use a KBM for multiple batteries that can be shared and are fully rechargeable. We seek to maximize the minimum residual energy among all batteries at the end of a given time interval $[0, T]$ with the requirement that the total battery output should reach a desired level at the end of $[0, T]$, subject to certain rechargeability constraints. We assume that recharging a battery is possible only while it is not being discharged, a requirement which is application-dependent (e.g., it applies to electric vehicles, but not most small wireless devices.) Relaxing this constraint is a special case of the more general problem we have analyzed and leads to a simpler solution. We first prove some properties of an optimal policy, the main one being the fact that it must result in *equal residual energies for all batteries at time T*. This enables us to subsequently derive explicit solutions for

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the problem. As already mentioned, we view this as a first step toward studying similar problems where the batteries are not all shared at a single location, but rather distributed over a network of devices with one or more batteries placed on board and powering each device. We may then address resource allocation and network lifetime maximization problems where the non-ideal nature of the batteries is not only taken into account but also taken advantage of.

In section II, a multi-battery optimal control problem based on a KBM is formulated. Significant properties of the optimal solution are identified and proved in section III. In section IV, with the help of these properties, we provide a full characterization of the optimal control solution. Conclusions and a description of ongoing research are given in section V.

II. Problem Statement

We consider N batteries that may be shared to serve a common load, as shown in Fig. 1. Each battery is modeled by

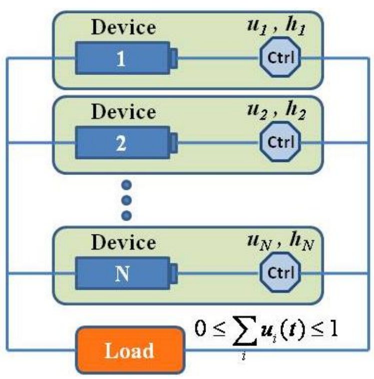

Fig. 1. A multi-battery system

a KBM as in prior work [16]. To briefly review, a KBM views each battery, indexed by $i = 1, \ldots, N$, as consisting of two communicating wells, a "bound-charge well" whose content (energy level) is $b_i(t)$ and an "available-charge well" whose content is $r_i(t)$. The dynamics in such a model are given by (2)-(3) below (further details are given in [16]). In our prior work involving a single battery we sought to control the discharging and recharging processes so as to maximize the battery output over a given interval while maintaining some required residual energy level. However, when dealing with multiple rechargeable batteries we adopt an objective which is motivated by the desire to maximize a system's "lifetime" often viewed as the time until the first battery is depleted (e.g., [20],[15]). Thus, we seek to maximize the minimum residual energy after finishing a prescribed workload within a time interval $[0, T]$. Let S be the battery index set with $|S| = N$, and let $U(t) = (u_1(t), h_1(t), \ldots, u_N(t), h_N(t))^T$, where $u_i(t)$ and $h_i(t)$ for $i = 1, \ldots, N$ denote the instantaneous discharge and recharge rate of battery i respectively. We then formulate the problem as follows:

$$\max_{U(t)} \min_{i \in S} r_i(T) \tag{1}$$

$$\dot{r}_i(t) = -c_1 u_i(t) + k(b_i(t) - r_i(t)) \tag{2}$$

$$\dot{b}_i(t) = c_2 h_i(t) - k(b_i(t) - r_i(t)) \tag{3}$$

$$r_i(t) \geq 0, \quad b_i(t) \leq B \tag{4}$$

$$u_i(t) h_i(t) = 0 \tag{5}$$

$$0 \leq u_i(t) \leq 1, \ 0 \leq h_i(t) \leq 1 \tag{6}$$

$$0 \leq \sum_{i=1}^{N} u_i(t) \leq 1 \tag{7}$$

$$\int_0^T \sum_i u_i(t) dt = Q \tag{8}$$

Here, (2) and (3) capture the battery dynamics through the KBM, where k depends on the battery characteristics and c_1, c_2 are battery-specific influencing factors for discharge and recharge processes, satisfying $c_1 > c_2 \geq 0$ (this indicates that a battery discharges faster than it recharges.) The constraint (5) requires that the discharging and recharging processes cannot occur simultaneously (this can be relaxed, depending on the application) and (6) imposes limits on the corresponding process rates. The state variables $r_i(t), b_i(t)$ are physically constrained as in (4) with $b_i(0) \geq r_i(0)$. The overall load to be served can be supported by either one or multiple batteries at any time as indicated in (7) and consistent with (6). Finally, (8) captures the fact that the load is required to complete a specific amount of work Q within $[0, T]$.

III. Optimal Control Properties

We begin with the solution of (2) and (3) under the assumption that a control policy $\{u_i(t), h_i(t),\ i \in S\}$ is feasible over some interval $[t_1, t_2] \subseteq [0, T]$, including a possible boundary arc where $r_i(\tau) = 0$ or $b_i(\tau) = B$, $\tau \in [t_1, t_2]$. It is straightforward to derive this solution, which is

$$\begin{aligned} r_i(\tau) =& \frac{1}{2}\left[b_i(t_1) + r_i(t_1) - (b_i(t_1) - r_i(t_1))e^{-2k(\tau - t_1)}\right] \\ &- \int_{t_1}^{\tau} \frac{1}{2} c_1 u_i(t)[1 + e^{2k(t-\tau)}]dt \\ &+ \int_{t_1}^{\tau} \frac{1}{2} c_2 h_i(t)[1 - e^{2k(t-\tau)}]dt \end{aligned} \tag{9}$$

$$\begin{aligned} b_i(\tau) =& \frac{1}{2}\left[b_i(t_1) + r_i(t_1) + (b_i(t_1) - r_i(t_1))e^{-2k(\tau - t_1)}\right] \\ &- \int_{t_1}^{\tau} \frac{1}{2} c_1 u_i(t)[1 - e^{2k(t-\tau)}]dt \\ &+ \int_{t_1}^{\tau} \frac{1}{2} c_2 h_i(t)[1 + e^{2k(t-\tau)}]dt \end{aligned} \tag{10}$$

Using $\rho_i(t_1,\tau)$ and $\beta_i(t_1,\tau)$ for the first term in (9) and (10) respectively, we can therefore write

$$r_i(T) = \rho_i(0,T) - \int_0^T c_1 u_i(r)\frac{1+e^{2k(r-T)}}{2}dr + \int_0^T c_2 h_i(r)\frac{1-e^{2k(r-T)}}{2}dr \quad (11)$$

$$b_i(T) = \beta_i(0,T) - \int_0^\tau c_1 u_i(r)\frac{1-e^{2k(r-T)}}{2}dr + \int_0^\tau c_2 h_i(t)\frac{1+e^{2k(r-T)}}{2}dr \quad (12)$$

as long as the feasibility of $\{u_i(t), h_i(t)\}$ over $[0,T]$ is assumed. Since $b_i(0) \geq r_i(0)$, obviously $\beta_i(0,T) > \rho_i(0,T) > 0$.

Let us denote an optimal control policy by $\{u_i^*(t), h_i^*(t), i \in S\}$. We can immediately observe that $\{0,0\}$ for all $i \in S$ cannot be an optimal policy, i.e., a policy that maximizes $\min_{i\in S} r_i(T)$. This follows from the constraint (5) and the fact that $\{0,0\}$ in (11) is dominated by any control $\{0, h_i(t)\}$ with $h_i(t) > 0$ which would give a larger value for $r_i(T)$. Moreover, (8) requires $u_i(t) > 0, h_i(t) = 0$ for some i and over some interval $[t_1,t_2] \subseteq [0,T]$. Thus, at least some $i \in S$ must include $u_i(t) > 0$; for the remaining $i \in S$ an optimal control would be $\{0, h_i(t)\}$ with $h_i(t) > 0$. Therefore, an optimal control for any $i \in S$ has the property that either $u_i^*(t) > 0, h_i^*(t) = 0$ or $u_i^*(t) = 0, h_i^*(t) > 0$ (with $h_i^*(t) = 1$ when $b_i(t) < B$).

The main result in this section (Theorem 1) is that, under optimal control, all $r_i^*(T)$, $i \in S$, are equal provided there is at least one feasible policy under which all $r_i(T)$, $i \in S$, are equal. In order to establish this result, we will make use of a *perturbed* policy $\{u_i'(t), h_i'(t),\ i \in S\}$ relative to any feasible one $\{u_i(t), h_i(t),\ i \in S\}$. We define such a policy by perturbing two of the controls indexed by i and $j \neq i$ respectively as follows:

$$\begin{cases} u_i'(t) = u_i(t),\ h_i'(t) = h_i(t) & t \in [0,T]/[\tau_i, \tau_i+\Delta_i] \\ u_i'(t) = u_i(t) - \Delta u_i,\ h_i'(t) = 0 & t \in [\tau_i, \tau_i+\Delta_i] \end{cases} \quad (13)$$

$$\begin{cases} u_j'(t) = u_j(t),\ h_j'(t) = h_j(t) & t \in [0,T]/[\tau_j, \tau_j+\Delta_j] \\ u_j'(t) = u_j(t) + \Delta u_j,\ h_j'(t) = 0 & t \in [\tau_j, \tau_j+\Delta_j] \end{cases} \quad (14)$$

where Δu_i, Δu_j, Δ_i and Δ_j are all positive constants. For notational convenience, we shall refer to the perturbed optimal control for i above as $\pi^-[u_i, \tau_i, \Delta_i, \Delta u_i]$ and the one for j as $\pi^+[u_j, \tau_j, \Delta_j, \Delta u_j]$. In simple terms, under $\pi^-[u_i, \tau_i, \Delta_i, \Delta u_i]$ the discharging control $u_i(t)$ is reduced by $\Delta u_i > 0$ over an interval $[\tau_i, \tau_i+\Delta_i]$ and $u_j(t)$ is increased by $\Delta u_j > 0$ over an interval $[\tau_j, \tau_j+\Delta_j]$; in both cases, the recharging control over these intervals is 0 to satisfy (5) and the controls remain unchanged over the rest of $[0,T]$. Assuming for the moment the feasibility of $\{u_i'(t), h_i'(t)\}$ and $\{u_j'(t), h_j'(t)\}$, let $\Delta r_i(t) = r_i'(t) - r_i(t)$, $\Delta b_i(t) = b_i'(t) - b_i(t)$ and observe that for any $t \in [\tau_i, T]$ it follows from (9)-(10):

$$\Delta r_i(t) = \begin{cases} \frac{1}{2}c_1\Delta u_i \int_{\tau_i}^{t}[1+e^{2k(r-t)}]dr & t < \tau_i+\Delta_i \\ \frac{1}{2}c_1\Delta u_i \int_{\tau_i}^{\tau_i+\Delta_i}[1+e^{2k(r-t)}]dr & t \geq \tau_i+\Delta_i \end{cases} \quad (15)$$

$$\Delta b_i(t) = \begin{cases} \frac{1}{2}c_1\Delta u_i \int_{\tau_i}^{t}[1-e^{2k(r-t)}]dr & t < \tau_i+\Delta_i \\ \frac{1}{2}c_1\Delta u_i \int_{\tau_i}^{\tau_i+\Delta_i}[1-e^{2k(r-t)}]dr & t \geq \tau_i+\Delta_i \end{cases} \quad (16)$$

and note that $\Delta r_i(t) > \Delta b_i(t) > 0$. Similarly, for any $t \in [\tau_j, T]$,

$$\Delta r_j(t) = \begin{cases} -\frac{1}{2}\int_{\tau_j}^{t}\big[c_1\Delta u_i[1+e^{2k(r-t)}] \\ \quad +c_2 h_j(r)[1-e^{2k(r-t)}]\big]dr & t < \tau_j+\Delta_j \\ -\frac{1}{2}\int_{\tau_j}^{\tau_j+\Delta_j}\big[c_1\Delta u_i[1+e^{2k(r-t)}] \\ \quad +c_2 h_j(r)[1-e^{2k(r-t)}]\big]dr & t \geq \tau_j+\Delta_j \end{cases} \quad (17)$$

$$\Delta b_i(t) = \begin{cases} -\frac{1}{2}\int_{\tau_j}^{t}\big[c_1\Delta u_i[1-e^{2k(r-t)}] \\ \quad +c_2 h_j(r)[1+e^{2k(r-t)}]\big]dr & t < \tau_j+\Delta_j \\ -\frac{1}{2}\int_{\tau_j}^{\tau_j+\Delta_j}\big[c_1\Delta u_i[1-e^{2k(r-t)}] \\ \quad +c_2 h_j(r)[1+e^{2k(r-t)}]\big]dr & t \geq \tau_j+\Delta_j \end{cases} \quad (18)$$

and note that $\Delta r_j(t) < \Delta b_j(t) < 0$. Regarding the feasibility of $\{u_i'(t), h_i'(t)\}$ and $\{u_j'(t), h_j'(t)\}$, we need to satisfy all problem constraints. This can be accomplished under certain conditions, as expressed in the next two lemmas. (Due to space limitations, only proofs of Lemma 3 and Theorem 2 are given. The others can be found in [21].)

Lemma 1: Let $\{u_i(t), h_i(t)\}$, $\{u_j(t), h_j(t)\}$ be controls for i,j in a feasible policy. If $r_j(t) > 0$ for all $t \in [0,T]$ under this policy, then the following conditions ensure that there are feasible perturbed controls $\{u_i'(t), h_i'(t)\}$, $\{u_j'(t), h_j'(t)\}$:

(C1) There exists an interval $[\tau_i, \tau_i+\Delta_i]$ with $u_i(t) > 0$, $t \in [\tau_i, \tau_i+\Delta_i]$.

(C2) There exists an interval $[\tau_j, \tau_j+\Delta_j]$ such that $\sum_{k\in S} u_k(t) < 1$, $t \in [\tau_j, \tau_j+\Delta_j]/[t_1,t_2]$, where $[t_1,t_2] = [\tau_i, \tau_i+\Delta_i] \cap [\tau_j, \tau_j+\Delta_j]$ and $[\tau_i, \tau_i+\Delta_i]$ satisfies **(C1)**.

Under certain conditions, **(C2)** in Lemma 1 can be relaxed and the result requires only **(C1)** as expressed in the following corollary.

Corollary 1: For the setting of Lemma 1, suppose $t_i = t_j$ and $\Delta_i = \Delta_j$. Then, the result holds under **(C1)**.

Before establishing our main result, we need one more lemma as follows which ensures the existence of some j with $r_j(t) > 0$ whenever $\sum_{i\in S} u_i(t) = 1$.

Lemma 2: Suppose $\sum_{i\in S} u_i(t) = 1$ over some interval $[t_1,t_2] \subset (0,T]$. Then, among all j with $u_j(t) > 0$ over $[t_1, t_1+\epsilon] \subseteq [t_1,t_2]$ for some $\epsilon > 0$, there exists at least one with $r_j(t) > 0$ over $[t_1, t_1+\epsilon]$.

Theorem 1: Let Π be the set of feasible policies for the problem (1)-(8). If there exists $\pi_0 \in \Pi$ under which $r_i(T) = r_j(T)$ for all $i,j \in S$, then there exists an optimal policy $\pi^* \in \Pi$ such that $r_i^*(T) = r_j^*(T)$ for all $i,j \in S$.

We will now tackle the situation where there exists no $\pi_0 \in \Pi$ under which $r_i(T) = r_j(T)$ for all $i,j \in S$. Let us start by defining $\bar{r}_i(T)$ as the maximum reachable value in (11) based on the initial condition $\rho_i(0,T)$ and setting $h_i(t)$ to its maximum feasible value subject to $b_i(t) \le B$. Let

$$\bar{r}_L(T) = \min_{i \in S}\{\bar{r}_i(T)\} \tag{19}$$

$$L = \operatorname*{argmin}_{i \in S}\{\bar{r}_i(T)\} \tag{20}$$

We will show in Theorem 2 that $\bar{r}_L(T)$ is the optimal value of the objective function in (1). We will accomplish this with the help of the following lemma.

Lemma 3: If there exists no feasible policy $\pi_0 \in \Pi$ such that $r_i(T) = r_j(T)$ for all $i, j \in S$, then under an optimal control policy π^*, there exists $k \in S$ such that $r_k^*(T) > \bar{r}_L(T)$.

Proof: We will use a contradiction argument and assume that under π^* we have $r_i^*(T) \le \bar{r}_L(T)$ for all $i \in S$. We have already established that in an optimal policy we have $u_i^*(t) > 0, h_i^*(t) = 0$ or $u_i^*(t) = 0, h_i^*(t) > 0$. Therefore, if $r_L^*(T) = \bar{r}_L(T)$ we have $\int_0^T u_i^*(t)dt > 0$ for all $i \in S/\{L\}$, and if $r_L^*(T) < \bar{r}_L(T)$ we have $\int_0^T u_i^*(t)dt > 0$ for all $i \in S$. Since $r_i^*(T) = \bar{r}_L(T)$ for all $i \in S$ is excluded by the assumption that a policy π_0 is not feasible, let us define two sets

$$S_1 = \{i : r_i^*(T) < \bar{r}_L(T)\}, \quad S_2 = \{i : r_i^*(T) = \bar{r}_L(T)\}$$

Note that regardless of whether $L \in S_1$ or $L \in S_2$, we have $\int_0^T u_j^*(t)dt > 0$ for all $j \in S_1$. Next, there are two cases to consider.

First, suppose $S_2 \neq \emptyset$. Then, we can perturb the controls of all $j \in S_1$ and all $k \in S_2$ to increase $r_j(T)$ and decrease $r_k(T)$ respectively. Since $\bar{r}_L(T) = \min_{i \in S}\{\bar{r}_i(T)\}$, it is feasible for each $r_j(T)$ to increase and reach the value $r_j'(T) = \bar{r}_L(T)$. Moreover, from (11), $r_i(T)$ can be continuously perturbed for all $i \in S$. Therefore, we can fix a value $r_i'(T) < \bar{r}_L(T)$ which is attainable by *all* $i \in S$. This contradicts the assumption that π_0 does not exist. Consequently, it is not possible to satisfy $r_i^*(T) \le \bar{r}_L(T)$ for all $i \in S$ under π^* and it follows that $r_k^*(T) > \bar{r}_L(T)$ for some $k \in S$.

Second, suppose $S_2 = \emptyset$, i.e., $S_1 = S$. Then, we can always find some $l = \arg\max_{i \in S}\{r_i^*(T)\}$ and similarly perturb the controls of all $j \in S/\{l\}$ and of l so as to increase $r_j(T)$ and decrease $r_l(T)$ through (11). Since $\int_0^T u_i^*(t)dt > 0$ for all $i \in S$, it follows that $r_l^*(T)$ is not the smallest value that l can reach. In addition, each $r_j(T)$ can be increased to $\bar{r}_L(T)$ since $r_j^*(T) \le \bar{r}_L(T)$. Thus, we can fix a value $r_i'(T) < \bar{r}_L(T)$ which is attainable by *all* $i \in S$. This again contradicts the assumption that π_0 does not exist. Consequently, it is not possible to satisfy $r_i^*(T) \le \bar{r}_L(T)$ for all $i \in S$ under π^* and it follows that $r_k^*(T) > \bar{r}_L(T)$ for some $k \in S$. ■

Theorem 2: If there exists no feasible policy $\pi_0 \in \Pi$ such that $r_i(T) = r_j(T)$ for all $i, j \in S$, then the optimal value of the objective function is $\bar{r}_L(T)$ in (19).

Proof: We will use a contradiction argument. Assume the optimal value of the objective function is $r^* < \bar{r}_L(T)$. Let us define three sets

$$S_1 = \{i : r_i^*(T) < \bar{r}_L(T)\}, \quad S_2 = \{i : r_i^*(T) = \bar{r}_L(T)\},$$
$$S_3 = \{i : r_i^*(T) > \bar{r}_L(T)\}$$

By Lemma 3 , $S_3 \neq \emptyset$. On the other hand, if $S_1 = \emptyset$, then it directly contradicts the assumption $r^* < \bar{r}_L(T)$. Therefore, $S_1 \neq \emptyset$ in the following argument. Since we have established that in an optimal policy we have $u_i^*(t) > 0, h_i^*(t) = 0$ or $u_i^*(t) = 0, h_i^*(t) > 0$ and in view of the definition of $\bar{r}_L(T)$ in (19), we have $\int_0^T u_j^*(t)dt > 0$ for all $j \in S_1$. We can now proceed similar to the argument used in Cases 1 and 2 in the proof of Theorem 1. We can always find some $l \in S_3$ to increase $r_j(T)$ for all $j \in S_1$ through perturbations $\pi^-[u_j, \tau_j, \Delta_j, \Delta u_j]$ for all $j \in S_1$ and decrease $r_l(T)$ through $\pi^+[u_l, \tau_j, \Delta_j, \Delta u_l]$ for $l \in S_3$ as long as $r_j'(T) \le \bar{r}_L(T)$ and $l \in S_3$. If $r_l'(T)$ decreases to a value $r_l'(T) = \bar{r}_L(T)$, i.e., $l \in S_2$, and not all $r_j'(T)$ increase to $r_j'(T) = \bar{r}_L(T)$, i.e., $S_1 \neq \emptyset$, then we can select some other $m \in S_3$ to repeat the process. By Lemma 3 , S_3 will never be empty. However, we will eventually reach the point where all $r_j'(T) = \bar{r}_L(T)$ for all $j \in S_1$, thus emptying S_1. Then, we contradict the assumption that $r^* < \bar{r}_L(T)$ and thus prove the theorem. ■

IV. Optimal Control Characterization

In this section, we provide a characterization and structure of the optimal solution that exploits the two main results in Section III. A detailed explicit solution requires further analysis because of the possibility of singular, as well as boundary, arcs in the optimal state trajectories and will be provided in a forthcoming paper.

If there exists no feasible policy $\pi_0 \in \Pi$ such that $r_i(T) = r_j(T)$ for all $i, j \in S$, we can directly determine the optimal objective function value by Theorem 2. Therefore, let us concentrate on the case where $\pi_0 \in \Pi$ exists. Then, by Theorem 1, we can add a terminal state constraint to the problem without affecting its solution:

$$r_i(T) = r_j(T), \quad \forall i, j \in S \tag{21}$$

so that in the original objective function (1) we have $\min_{i \in S} r_i(T) = r_i(T)$ for any $i \in S$. Since $\max_{U(t)} r_i(T) = \max_{U(t)} \sum_{i=1}^N r_i(T)$ in light of (21), we can rewrite (1) as

$$\min_{U(t)} \; -\sum_{i=1}^N r_i(T)$$

As for the integral constraint (8), we define an additional state variable $q(t)$ and replace (8) by

$$\dot{q}(t) = \sum_{i=1}^N u_i(t), \quad q(0) = 0, \quad q(T) = Q \tag{22}$$

Now the original max-min problem becomes a typical state-constrained optimal control problem with terminal state constraints. However, we have totally $2N+1$ states and the problem is not easy to solve if N is large. In order to have a clear picture of the solution, we start with $N = 2$ and index the batteries so that $\rho_1(0,T) \geq \rho_2(0,T)$. Accordingly, the control is $U(t) = (u_1(t), h_1(t), u_2(t), h_2(t))^T$. Moreover, referring to (11) and (21), we require $U(t)$ to satisfy the feasibility condition:

$$\rho_1(0,T) - \rho_2(0,T) = \int_0^T \left(c_1(u_1(r) - u_2(r)) \frac{1 + e^{2k(r-T)}}{2} - c_2(h_1(r) - h_2(r)) \frac{1 - e^{2k(r-T)}}{2} \right) dr \quad (23)$$

Based on the definition of Π in Theorem 1, we denote the set of feasible policies in Π satisfying (21) by Π_0. Subject to the control constraints (5)-(8), no feasible solution exists if $\rho_1(0,T) - \rho_2(0,T) > \bar{\alpha}$ where $\bar{\alpha}$ is determined from (23):

$$\bar{\alpha} = \max_{\pi \in \Pi_0} \int_0^T \left(c_1(u_1(r) - u_2(r)) \frac{1 + e^{2k(r-T)}}{2} - c_2(h_1(r) - h_2(r)) \frac{1 - e^{2k(r-T)}}{2} \right) dr$$

Since $(1 + e^{2k(t-T)})$ is monotonically increasing in t, $\bar{\alpha}$ is attained by letting $u_1(t) = 0$ over $[0, T - Q)$ and $u_1(t) = 1$ over $[T - Q, T]$, $h_1(t) = 0$, $u_2(t) = 0$, $h_2(t) = 1$ over $[0, T]$:

$$\bar{\alpha} = \int_{T-Q}^T c_1 \frac{1 + e^{2k(r-T)}}{2} dr + \int_0^T c_2 \frac{1 - e^{2k(r-T)}}{2} dr$$

Then, $\rho_1(0,T) - \rho_2(0,T) \leq \bar{\alpha}$ must be satisfied to ensure a feasible solution.

In order to obtain an explicit optimal control $U^*(t)$, we proceed as in [16] by first analyzing the unconstrained case in which (4) is relaxed and the optimal state trajectories for both batteries consist of an interior arc over the entire interval $[0, T]$. Let $\mathbf{x}(t) = (r_1(t), b_1(t), r_2(t), b_2(t), q(t))^T$ and $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t))^T$ denote the state and costate vector respectively. The Hamiltonian for this problem is then

$$\begin{aligned} &H(\mathbf{x}, \lambda, u_1, h_1, u_2, h_2) = \lambda(t)^T \dot{\mathbf{x}}(t) \\ &= [-c_1\lambda_1(t) + \lambda_5(t)]u_1(t) + c_2\lambda_2(t)h_1(t) \\ &+ [-c_1\lambda_3(t) + \lambda_5(t)]u_2(t) + c_2\lambda_4(t)h_2(t) \\ &+ k[\lambda_1(t) - \lambda_2(t)][b_1(t) - r_1(t)] \\ &+ k[\lambda_3(t) - \lambda_4(t)][b_2(t) - r_2(t)] \end{aligned} \quad (24)$$

The costate equations $\dot{\lambda} = -\frac{\partial H}{\partial \mathbf{x}}$ give

$$\begin{aligned} &\dot{\lambda}_1(t) = k(\lambda_1(t) - \lambda_2(t)),\ \dot{\lambda}_2(t) = -k(\lambda_1(t) - \lambda_2(t)) \\ &\dot{\lambda}_3(t) = k(\lambda_3(t) - \lambda_4(t)),\ \dot{\lambda}_4(t) = -k(\lambda_3(t) - \lambda_4(t)) \\ &\dot{\lambda}_5(t) = 0 \end{aligned} \quad (25)$$

and, due to (21) and (22), we must satisfy $\lambda(T) = \frac{\partial \Phi(\mathbf{x}(T))}{\partial \mathbf{x}}$ where $\Phi(\mathbf{x}(T)) = -r_1(T) - r_2(T) + \nu_1(r_1(T) - r_2(T)) + \nu_2(q(T) - Q)$ and ν_1, ν_2 are unknown multipliers, so that

$$\begin{aligned} &\lambda_1(T) = -1 + \nu_1, \quad \lambda_2(T) = 0 \\ &\lambda_3(T) = -1 - \nu_1, \quad \lambda_4(T) = 0, \quad \lambda_5(T) = \nu_2 \end{aligned} \quad (26)$$

Solving (25) with the boundary conditions (26), we get

$$\begin{cases} \lambda_1(t) = \frac{\nu_1 - 1}{2}[1 + e^{2k(t-T)}] \\ \lambda_2(t) = \frac{\nu_1 - 1}{2}[1 - e^{2k(t-T)}] \\ \lambda_3(t) = \frac{-\nu_1 - 1}{2}[1 + e^{2k(t-T)}] \\ \lambda_4(t) = \frac{-\nu_1 - 1}{2}[1 - e^{2k(t-T)}] \\ \lambda_5(t) = \nu_2 \end{cases} \quad (27)$$

Looking at (24), we define the switching functions $s_1(t), s_2(t)$ and $s_3(t), s_4(t)$ corresponding to $u_1(t), h_1(t)$ and $u_2(t), h_2(t)$ respectively:

$$\begin{aligned} &s_1(t) = -c_1\lambda_1(t) + \lambda_5(t), \quad s_2(t) = c_2\lambda_2(t) \\ &s_3(t) = -c_1\lambda_3(t) + \lambda_5(t), \quad s_4(t) = c_2\lambda_4(t) \end{aligned} \quad (28)$$

and apply the Pontryagin minimum principle:

$$H(\mathbf{x}^*, \lambda^*, u_i^*, h_i^*) = \min_{(u_i, h_i)} H(\mathbf{x}, \lambda, u_i, h_i) \quad (29)$$

where $u_i^*(t), h_i^*(t)$ for $i = 1, 2$, $t \in [0, T)$, denote the optimal controls. We can then see that

$$u_1^*(t) = \begin{cases} 1 & s_1(t) < 0 \\ 0 & s_1(t) > 0 \end{cases},\ h_1^*(t) = \begin{cases} 1 & s_2(t) < 0 \\ 0 & s_2(t) > 0 \end{cases}$$
$$u_2^*(t) = \begin{cases} 1 & s_3(t) < 0 \\ 0 & s_3(t) > 0 \end{cases},\ h_2^*(t) = \begin{cases} 1 & s_4(t) < 0 \\ 0 & s_4(t) > 0 \end{cases}$$

Singular cases may arise when $\nu_2 = 0$ and $\nu_1 = 1$ or -1, making $s_1(t) = s_2(t) = 0$ or $s_3(t) = s_4(t) = 0$ respectively. Let us proceed by setting these aside for the time being. Given the constraint $u_i(t)h_i(t) = 0$, as well as the already excluded $u_i^*(t) = h_i^*(t) = 0$, we can set $h_i^*(t) = 1 - u_i^*(t)$ in this unconstrained case and rewrite $H(\mathbf{x}, \lambda, u_i, h_i)$ as follows:

$$\begin{aligned} &H(\mathbf{x}, \lambda, u_i, h_i) = \sigma_1(t)u_1(t) + \sigma_2(t)u_2(t) + c_2\lambda_2(t) \\ &+ c_2\lambda_4(t) + k[\lambda_1(t) - \lambda_2(t)][b_1(t) - r_1(t)] \\ &+ k[\lambda_3(t) - \lambda_4(t)][b_2(t) - r_2(t)] \end{aligned} \quad (30)$$

where $\sigma_1(t) = -c_1\lambda_1(t) + \lambda_5 - c_2\lambda_2(t)$, $\sigma_2(t) = -c_1\lambda_3(t) + \lambda_5 - c_2\lambda_4(t)$ are the new switching functions of u_1, u_2 respectively. Using (27), σ_1, σ_2 become

$$\sigma_1(t) = \frac{1 - \nu_1}{2} \left[c_1 + c_2 + (c_1 - c_2)e^{2k(t-T)} \right] + \nu_2 \quad (31)$$
$$\sigma_2(t) = \frac{1 + \nu_1}{2} \left[c_1 + c_2 + (c_1 - c_2)e^{2k(t-T)} \right] + \nu_2 \quad (32)$$

Thus, to minimize (30), the optimal control on the interior arc is

$$\begin{cases} u_i^*(t) = 0,\ h_i^*(t) = 1 & \text{if } \sigma_i(t) > 0 \\ u_i^*(t) = 1,\ h_i^*(t) = 0 & \text{if } \sigma_i(t) < 0 \end{cases} \quad (33)$$

for $i = 1, 2$. We immediately observe in (33) that $u_1^*(t) = u_2^*(t) = 1$ when $\sigma_1(t) < 0$ and $\sigma_2(t) < 0$, which violates the constraint (7). In this case, (i) $u_1^*(t) = 1, u_2^*(t) = 0$ if $\sigma_1(t) < \sigma_2(t) < 0$; (ii) $u_1^*(t) = 0, u_2^*(t) = 1$ if $\sigma_2(t) < \sigma_1(t) < 0$; and (iii) either $u_1^*(t) = 1, u_2^*(t) = 0$ or $u_1^*(t) = 0, u_2^*(t) = 1$ if $\sigma_1(t) = \sigma_2(t) < 0$. Correspondingly, $h_i^*(t) = 1 - u_i^*(t)$, $i = 1, 2$. In other words, the optimal control in the interior arc depends on the sign of $\sigma_1(t) - \sigma_2(t)$ when $\sigma_1(t) < 0$, $\sigma_2(t) < 0$. By (31) and (32),

$$\sigma_1(t) - \sigma_2(t) = -\nu_1 \left(c_1 + c_2 + (c_1 - c_2)e^{2k(t-T)} \right)$$

Therefore, along with (33), the optimal control can be summarized as

$$U^*(t) = (0,1,0,1)^T \text{ if } \sigma_1(t) > 0,\ \sigma_2(t) > 0 \quad (34)$$

$$U^*(t) = (0,1,1,0)^T \text{ if } \begin{array}{c} \sigma_2(t) < 0 < \sigma_1(t) \text{ or} \\ \sigma_2(t) < \sigma_1(t) < 0 \end{array} \quad (35)$$

$$U^*(t) = (1,0,0,1)^T \text{ if } \begin{array}{c} \sigma_1(t) < 0 < \sigma_2(t) \text{ or} \\ \sigma_1(t) < \sigma_2(t) < 0 \end{array} \quad (36)$$

Note that by (31)-(32), $\sigma_1(t) = \sigma_2(t)$ when $\nu_1 = 0$, but one can see that $\sigma_1(t) = \sigma_2(t) = 0$ is not possible for any finite-length time interval. Thus, when $\sigma_1(t) = \sigma_2(t)$, we only need to consider the solution with $\sigma_1(t) = \sigma_2(t) > 0$ or $\sigma_1(t) = \sigma_2(t) < 0$. The solution to the former is given in (34) and for the latter it is either $(1,0,0,1)^T$ or $(0,1,1,0)^T$ as already analyzed earlier for the case where $\sigma_1(t) < 0$ and $\sigma_2(t) < 0$.

Moreover, in view of (31)-(32), $\sigma_i(t) = 0$ over $[0,T]$ for $i = 1$ or 2 when $\nu_1 = 1$ or -1 and $\nu_2 = 0$. This is a singular case implying $(u_i^*(t), h_i^*(t))$ can be any feasible value on the singular arc. Even though the optimal solution is still subject to the constraint (8) and (21), we can clearly recognize the presence of non-unique solutions to the multi-battery optimal control problem when (i) $\sigma_1(t) = \sigma_2(t) < 0$ or (ii) $\sigma_i(t) = 0$, $i = 1$ or 2. A complete explicit solution requires analyzing all possible values of the unknown constants ν_1, ν_2 and studying the case where any one of the constraints in (4) becomes active. As already mentioned, this final analysis will be included in a forthcoming paper.

V. Conclusions

We have used a Kinetic Battery Model (KBM) to study the problem of optimally controlling the discharge and recharge processes of multiple non-ideal batteries so as to maximize the minimum residual energy among all the batteries at the end of a given time period $[0,T]$ while performing a prescribed amount of work Q over this period. We have shown that the optimal policy has the property that the residual energies of all batteries are equal at T as long as such a policy is feasible. This helps transform the original max-min optimization problem into a typical optimal control problem with terminal state constraints. Moreover, through the analysis of the $N = 2$ case, exploiting this property, we can characterize the optimal policy and show that it is generally not unique.

Our ongoing work is to complete the entire solution of this problem with $N = 2$ and extend it to $N > 2$. Future work aims at extending this approach to problems where the batteries are not all shared at a single location, but rather distributed over a network of devices with one or more batteries placed on board and powering each device. Thus, we will tackle resource allocation and network lifetime maximization problems where a non-ideal battery model is employed.

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