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Stabilization of Abstract Delay Systems on Banach Lattices using Nonnegative Semigroups

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Abstract— **In this study, we investigate the problem of designing a linear state feedback control to stabilize a class of linear uncertain systems with state delay. We introduce the concept of an abstract delay system that can be used to characterize the behavior of a broad class of mathematical models that include delay differential equations. We examine the stabilization problem of an abstract delay system on a Banach lattice using semigroup theory. To tackle this problem, we take advantage of the properties of a nonnegative semigroup on a Banach lattice. The objective of this paper is to show that the stabilizability conditions obtained by a stability criterion of nonnegative semigroups have a particular geometric configuration with respect to the permissible locations of uncertain entries. For this purpose, we introduce a variable transformation method to eliminate a restrictive assumption such as the nonnegativity of a system.**

I. Introduction

Time delays arise in many dynamical systems because physical, chemical, biological, and economical phenomena depend naturally not only on the present state but also on past occurrences. Delay differential equations are known to be infinite-dimensional systems, while ordinary differential equations without delays are finite-dimensional systems. The control of infinite-dimensional systems is a challenging problem attracting considerable attention in many research fields. Semigroups have become important tools in infinite-dimensional control theory over the past several decades [1]-[2]. The semigroup method is a unified approach to addressing systems that include ordinary differential equations, partial differential equations, and delay differential equations. The behaviors of many dynamical systems including infinite-dimensional systems and finite-dimensional systems can be characterized by semigroup theory. The recent well-developed theory in such a framework has been accumulated in several books [3]-[5]. In this paper, using semigroup theory, we introduce the concept of an abstract delay system that can be used to describe the behavior of a wide class of infinite-dimensional systems.

The linear quadratic control problem for an abstract delay system has been studied in [6]-[8]. Furthermore, the H^∞ control problem for such a system has been examined in [9]. The problems addressed in [6]-[9] have been reduced to finding a solution of the corresponding operator Riccati equation in Hilbert spaces. The feedback stabilizability of an abstract delay system on a Banach space has been investigated in [10]. The analytic approach in [10] is based on the compactness of Banach spaces, while the problem in [6]-[9] is formulated in Hilbert spaces to make use of the properties of the inner product.

In this paper, we study the stabilization problem of an abstract delay system on a Banach lattice, which is a Banach space supplied with an order relation [11]. To tackle this problem, we take advantage of the properties of a nonnegative C_0 semigroup on a Banach lattice. In general, the stability of an abstract delay system cannot be determined by its spectral bound. We show here that the nonnegativity assumption enables us to determine the stability of an abstract delay system simply by examining whether the spectral bound is negative.

In many systems arising from physics, biology, chemistry, and economics, a solution with a nonnegative initial value should remain nonnegative. In fact, there are many systems that satisfy the nonnegativity assumption. Nevertheless, the applicability of the control method is restricted to a class of systems that satisfy the nonnegativity assumption. To remove such a restrictive assumption, we introduce a variable transformation method in this study. Furthermore, using this transformation, we derive sufficient conditions under which an abstract delay system is uniformly exponentially stabilizable. Based on the stability criterion of nonnegative semigroups, we provide stabilizability conditions for linear time-varying uncertain systems with state delay. The objective of this study is to show that the stabilizability conditions obtained by the stability criterion of nonnegative semigroups have a specific geometric configuration with respect to the permissible locations of uncertain entries. It is shown that the stabilizability of the considered system are determined by the localization of uncertain parameters.

This paper is organized as follows. Some notation and terminology are given in Sec. II. Sec. III is devoted to the introduction of an abstract delay system on a Banach lattice and its stability criterion. In Sec. IV, we first derive sufficient conditions for the stabilization of an abstract delay system under the assumption that the system satisfies the nonnegativity. Next, we introduce a variable transformation method to eliminate such a restrictive assumption. In Sec. V, we present fundamental lemmas that are used to derive the main results. The main theorem is provided in Sec. VI. We show that if a linear uncertain system has a particular geometric configuration with respect to the permissible locations of uncertain entries, then the system is uniformly

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exponentially stabilizable however large the given upper bounds of uncertain parameters might be. Finally, some concluding remarks are given in Sec. VII.

II. Notation and Terminology

Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of real numbers and nonnegative real numbers, respectively. Let $\mathbb{N}_+$ denote the set of positive integers. Let X be a Banach space endowed with the operator norm $\|\cdot\|$. Let $\mathcal{L}(X,Y)$ denote the set of all bounded linear operators from a Banach space X to another Banach space Y. Let $\mathcal{L}(X)$ be defined by $\mathcal{L}(X,X)$. Let $I_d \in \mathcal{L}(X)$ denote the identity operator on X.

The determinant and the transpose of $A \in \mathbb{R}^{n\times n}$ are denoted by $\det(A)$ and A', respectively. Let diag.$\{\cdots\}$ denote a diagonal matrix. Let I denote the identity matrix. For $A, B \in \mathbb{R}^{n\times m}$, every inequality between A and B, such as $A > B$, indicates that it is satisfied componentwise. If $A \in \mathbb{R}^{n\times m}$ satisfies $A \geq 0$, A is called a non-negative matrix. For $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $|A|$ denotes a matrix with $|a_{ij}|$ as its (i,j) entries.

The set of all continuous or piecewise continuous functions with domain $[a,b]$ and range $\mathbb{R}^n$ is denoted by $\mathcal{K}^n[a,b]$ or $\mathcal{L}^n[a,b]$, respectively. We denote it simply by $\mathcal{K}^n$ or $\mathcal{L}^n$ if the domain is $\mathbb{R}$. The notation for a class of functions is introduced below.

Definition 1: Let $\xi(\mu) \in \mathcal{K}^1$ and let $m \in \mathbb{R}$ be a constant. If $\xi(\mu)$ satisfies the conditions

$$\limsup_{|\mu|\to\infty} \left|\frac{\xi(\mu)}{\mu^m}\right| < \infty, \qquad \limsup_{|\mu|\to\infty} \left|\frac{\xi(\mu)}{\mu^{m-a}}\right| = \infty$$

for any positive scalar $a \in \mathbb{R}$, then $\xi(\mu)$ is called a function of order m, and we denote this as follows:

$$Ord(\xi(\mu)) = m.$$

The set of all $\mathcal{K}^1$ functions of order m is denoted by $O(m)$,

$$O(m) = \left\{\xi(\mu) | \xi(\mu) \in \mathcal{K}^1,\ Ord(\xi(\mu)) = m\right\}.$$

We assume that the following relations between $\xi_1(\mu) \in O(m_1)$ and $\xi_2(\mu) \in O(m_2)$ hold:

$$\begin{aligned} &Ord(\xi_1(\mu) \pm \xi_2(\mu)) = \max\{m_1, m_2\}\\ &Ord(\xi_1(\mu) \times \xi_2(\mu)) = m_1 + m_2,\\ &Ord(\xi_1(\mu)/\xi_2(\mu)) = m_1 - m_2. \end{aligned}$$

Definition 2 ([3]): Let $(T(t))_{t\geq 0}$ be a C_0 semigroup on a Banach space X and let $D(\mathcal{A})$ be the subspace of X defined as

$$D(\mathcal{A}) := \left\{ x \in X : \lim_{h\searrow 0} \frac{1}{h}(T(h)x - x) \text{ exists} \right\}.$$

For every $x \in D(\mathcal{A})$, we define

$$\mathcal{A}x := \lim_{h\searrow 0} \frac{1}{h}(T(h)x - x).$$

The operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ is called the generator of the semigroup $(T(t))_{t\geq 0}$. In the following, let $(\mathcal{A}, D(\mathcal{A}))$ denote the operator $\mathcal{A}$ with domain $D(\mathcal{A})$.

Definition 3 ([3]): Let $(\mathcal{A}, D(\mathcal{A}))$ be the generator of a C_0 semigroup $(T(t))_{t\geq 0}$. $\omega_0(\mathcal{A}) := \inf\{\omega \in \mathbb{R} : \exists M > 0$ such that $\|T(t)\| \leq Me^{\omega t}, \forall t \in \mathbb{R}_+\}$ is called the semigroup's growth bound. The set $\rho(\mathcal{A}) := \{\lambda \in \mathbb{C} : \lambda I_d - \mathcal{A}$ is bijective$\}$ is called the resolvent set of $\mathcal{A}$, and the set $\sigma(\mathcal{A}) := \mathbb{C}\backslash\rho(\mathcal{A})$ is called the spectrum of $\mathcal{A}$. For $\lambda \in \rho(\mathcal{A})$, $R(\lambda, \mathcal{A}) := (\lambda I_d - \mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$ at λ. $s(\mathcal{A}) := \sup\{\text{Real part of } \lambda : \lambda \in \sigma(\mathcal{A})\}$ is called the spectral bound of $\mathcal{A}$.

Definition 4 ([3]): A C_0 semigroup $(T(t))_{t\geq 0}$ with generator $(\mathcal{A}, D(\mathcal{A}))$ is said to be uniformly exponentially stable if $\omega_0(\mathcal{A}) < 0$.

Definition 5 ([11]): A Banach space X is called a Banach lattice if X is supplied with an order relation such that all the following conditions hold:

(i) $f \geq g \Rightarrow f + h \geq g + h$ for all $f, g, h \in X$.
(ii) $f \geq 0 \Rightarrow \lambda f \geq 0$ for all $f \in X$ and $\lambda \in \mathbb{R}_+$.
(iii) $|f| \geq |g| \Rightarrow \|f\| \geq \|g\|$ for all $f, g \in X$.

Definition 6 ([3]): A C_0 semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X is said to be nonnegative if

$$0 \leq x \in X \ \Rightarrow\ 0 \leq T(t)x, \ \text{ for all } t \geq 0.$$

An operator $T(x) \in \mathcal{L}(X)$ on a Banach lattice X is also said to be nonnegative if $T(x) \geq 0$ whenever $0 \leq x \in X$.

III. Preliminaries

In this section, we introduce the concept of an abstract delay system that can be used to describe the behavior of a wide class of dynamical systems. For a Banach space Y and a constant $\tau \in \mathbb{R}_+$, let $C([-\tau, 0], Y)$ denote the set of all continuous functions with domain $[-\tau, 0]$ and range Y. For a Banach space $X := C([-\tau, 0], Y)$, let $\Phi \in \mathcal{L}(X, Y)$ be a delay operator, and let $(\mathcal{B}, D(\mathcal{B}))$ be the generator of a C_0 semigroup on Y. With these notations, an abstract delay system is described by the following equation with an initial function $\varphi : [-\tau, 0] \to Y$:

$$\left\{ \begin{array}{l} \dot{x}(t) = \mathcal{B}x(t) + \Phi(x(t-\tau)) \quad \text{for } t \geq 0,\\ x_0 = \varphi \in X. \end{array} \right. \tag{1}$$

A continuous function $x : [-\tau, \infty) \to Y$ is called a solution of (1) if all the following properties hold:

(i) $x(t)$ is right-sided differentiable at $t = 0$ and continuously differentiable for all $t > 0$.
(ii) $x(t) \in D(\mathcal{B})$ for all $t \geq 0$.
(iii) $x(t)$ satisfies (1).

Let C^r be the set of all r-times continuously differentiable functions. Let $(\mathcal{A}, D(\mathcal{A}))$ be the corresponding delay differential operator on X defined by

$$\begin{aligned} &\mathcal{A}f := \dot{f}, \\ &D(\mathcal{A}) := \{f \in C^1([-\tau, 0], Y) : f(0) \in D(\mathcal{B}) \\ &\qquad \text{and } \dot{f}(0) = \mathcal{B}f(0) + \Phi(f(-\tau))\}. \end{aligned} \tag{2}$$

Lemma 1 ([3]): The operator $(\mathcal{A}, D(\mathcal{A}))$ in (2) generates a C_0 semigroup $(T(t))_{t\geq 0}$ on X.

Lemma 2 ([3]): If $\varphi \in D(\mathcal{A})$, then the function $x : [-\tau, \infty) \rightarrow Y$ defined by

$$x(t) := \begin{cases} \varphi(t) & \text{if } -\tau \le t \le 0, \\ [T(t)\varphi](0) & \text{if } 0 < t, \end{cases}$$

is the unique solution of (1).

In the subsequent discussion, we assume that each Banach space X, Y in (1) is a Banach lattice.

Lemma 3 ([3]): If $\mathcal{B}$ generates a nonnegative C_0 semigroup on Y and the delay operator $\Phi \in \mathcal{L}(X, Y)$ is nonnegative, then the C_0 semigroup $(T(t))_{t \ge 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ in (2) is also nonnegative, and the following equivalence holds:

$$s(\mathcal{A}) < 0 \Leftrightarrow s(\mathcal{B} + \Phi) < 0.$$

Lemma 4 ([3]): Assume that $(T(t))_{t \ge 0}$ is a nonnegative C_0 semigroup with generator $(\mathcal{A}, D(\mathcal{A}))$ on X. Then,

$$s(\mathcal{A}) = \omega_0(\mathcal{A}).$$

The following proposition directly follows from Lemmas 3 and 4.

Proposition 1: Under the assumption that $\mathcal{B}$ generates a nonnegative C_0 semigroup on Y and the delay operator $\Phi \in \mathcal{L}(X, Y)$ is nonnegative, the C_0 semigroup $(T(t))_{t \ge 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ in (2) is uniformly exponentially stable if and only if the spectral bound $s(\mathcal{B} + \Phi) < 0$.

Note that the equality in Lemma 4 might not hold in general. This means that a C_0 semigroup $(T(t))_{t \ge 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ is not necessarily uniformly exponentially stable even if the spectral bound is negative, i.e., $s(\mathcal{A}) < 0$. It can be seen from Proposition 1 that the nonnegativity assumption enables us to determine the stability of an abstract delay system simply by examining the spectral bound.

IV. Stabilization of Abstract Delay Systems

Let $(\mathcal{C}, D(\mathcal{C}))$ be the generator of a C_0 semigroup on a Banach lattice Y. For a Banach lattice $X := C([-\tau, 0], Y)$, let $\Phi \in \mathcal{L}(X, Y)$ be a delay operator. In this section, we consider the stabilization problem of an abstract delay system described by

$$\begin{cases} \dot{x}(t) = \mathcal{C}x(t) + \Phi(x(t-\tau)) + \mathcal{D}u(t), \\ x_0 = \varphi \in X, \end{cases} \tag{3}$$

where $u(t) : t \in \mathbb{R}_+ \rightarrow Y$ is the control input, and $(\mathcal{D}, D(\mathcal{D}))$ is the generator of a C_0 semigroup on Y.

Assumption 1: Φ is assumed to be nonnegative.

Next, we consider the feedback stabilization problem of (3). Let $u(t)$ be given by

$$u(t) = \mathcal{G}x(t), \tag{4}$$

where $(\mathcal{G}, D(\mathcal{G}))$ is the generator of a C_0 semigroup on Y. Substituting (4) into (3) yields

$$\dot{x}(t) = (\mathcal{C} + \mathcal{D}\mathcal{G})x(t) + \Phi(x(t-\tau)). \tag{5}$$

Considering

$$\mathcal{B} = (\mathcal{C} + \mathcal{D}\mathcal{G}), \tag{6}$$

we see that the resulting closed-loop system (5) can be rewritten as (1).

Definition 7: System (3) is said to be uniformly exponentially stabilizable if there exists $u(t)$ in (4) such that the equilibrium point $x = 0$ of the resulting closed-loop system (5) is uniformly exponentially stable.

Next, we state the following proposition that directly follows from Proposition 1.

Proposition 2: If there exists $\mathcal{G}$ such that $(\mathcal{C} + \mathcal{D}\mathcal{G})$ generates a nonnegative C_0 semigroup and

$$s(\mathcal{C} + \mathcal{D}\mathcal{G} + \Phi) < 0 \tag{7}$$

is satisfied, then system (3) is uniformly exponentially stabilizable.

Proof: Under the assumption that Φ is nonnegative and $(\mathcal{C} + \mathcal{D}\mathcal{G})$ generates a nonnegative C_0 semigroup, we see from Proposition 1 that the resulting closed-loop system (5) is uniformly exponentially stable if $s(\mathcal{C}+\mathcal{D}\mathcal{G}+\Phi) < 0$ holds. ■

In the above, we examined the stabilization problem of an abstract delay system (3) under Assumption 1. In many systems arising from physics, biology, chemistry, and economics, a solution with a nonnegative initial value should remain nonnegative. In fact, there are many systems that satisfy the nonnegativity assumption. Nevertheless, the applicability of the stabilization condition in Proposition 2 is restricted to a class of systems that satisfy Assumption 1. To remove such a restrictive assumption, we propose a variable transformation method below. We also consider system (3) here, but Φ is not assumed to be nonnegative in the subsequent discussion. In the following, we investigate the stabilization problem of system (3) whose $\mathcal{C}$ and Φ are not necessarily nonnegative. To tackle this problem, we introduce a linear variable transformation

$$w = \mathcal{V}^{-1}(x), \tag{8}$$

where $\mathcal{V}$ is a linear bijective operator, i.e., $w = \mathcal{V}^{-1}(x)$ uniquely exists, and $w = 0$ whenever $x = 0$, i.e., $\mathcal{V}^{-1}(0) = 0$.

Substituting (4) and (8) into (3), we have

$$\dot{w}(t) = \mathcal{V}^{-1}(\mathcal{C} + \mathcal{D}\mathcal{G})\mathcal{V}w(t) + \mathcal{V}^{-1}\Phi\mathcal{V}w(t-\tau). \tag{9}$$

Note that the stabilization problem of system (3) at $x = 0$ has been reduced to stabilizing system (9) at $w = 0$. Therefore, we see that the following statement directly follows from Proposition 2.

Proposition 3: If there exist $\mathcal{G}$ and $\mathcal{V}$ such that all the following conditions are satisfied, then system (3) is uniformly exponentially stabilizable.

(i) $\mathcal{V}^{-1}(\mathcal{C}+\mathcal{D}\mathcal{G})\mathcal{V}$ generates a nonnegative C_0 semigroup.
(ii) $\mathcal{V}^{-1}\Phi\mathcal{V}$ is nonnegative.
(iii) $s(\mathcal{V}^{-1}(\mathcal{C} + \mathcal{D}\mathcal{G} + \Phi)\mathcal{V}) < 0$.

It is usual that the stability of a system is guaranteed by Lyapunov theory. Hence, the stabilization problem of dynamical systems is usually reduced to finding a Lyapunov function. On the one hand, we have examined the stabilization problem of abstract delay systems on the basis of the stability criteria with a nonnegative C_0 semigroup. Note that our problem has been reduced to not finding a Lyapunov function, but finding a variable transformation such that all the conditions in Proposition 3 are satisfied.

V. Fundamental Lemmas

In this section, we provide some preliminary lemmas that are useful for deriving the main results. The following lemma enables the verification of whether a given matrix generates a nonnegative C_0 semigroup.

Lemma 5: A real matrix $A \in \mathbb{R}^{n\times n}$ generates a nonnegative C_0 semigroup $(T(t))_{t\geq 0} := e^{tA}$ if and only if every off-diagonal entry of A is nonnegative.

Proof: (Necessity) Suppose that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is the generator of $(T(t))_{t\geq 0}$, then

$$A = \lim_{h\searrow 0} \frac{e^{hA} - I}{h} \tag{10}$$

holds. Let f_{ij} denote the (i,j)-th entry of e^{hA}. Then, it follows from (10) that

$$a_{ij} = \begin{cases} \lim_{h\searrow 0} \frac{f_{ij}}{h} & \text{for } i \neq j, \\ \lim_{h\searrow 0} \frac{f_{ij}-1}{h} & \text{for } i = j. \end{cases} \tag{11}$$

Assume that $f_{ij} \geq 0$ for all i, j, then we obtain

$$\begin{array}{ll} a_{ij} \geq 0 & \text{for } i \neq j, \\ a_{ij} \in \mathbb{R} & \text{for } i = j. \end{array} \tag{12}$$

Therefore, we see that if A generates a nonnegative C_0 semigroup $(T(t))_{t\geq 0}$, then $a_{ij} \geq 0$ for all $i \neq j$.

(Sufficiency) Suppose that every off-diagonal entry of A is nonnegative, then we can find $\delta \in R$ such that

$$P_\delta := A + \delta I \geq 0. \tag{13}$$

Note that if $P_\delta \geq 0$, then

$$e^{tP_\delta} = \sum_{k=0}^{\infty} \frac{tP_\delta^k}{k!} \geq 0 \ \text{ for all } \ t \geq 0.$$

Considering

$$e^{-t\delta I} = \text{diag.}\{e^{-t\delta}, \cdots, e^{-t\delta}\},$$

we obtain

$$\begin{aligned} e^{tA} &= e^{\{t(A+\delta I)-t\delta I\}} = e^{tP_\delta} e^{-t\delta I} \\ &= e^{tP_\delta} e^{-t\delta} \geq 0 \ \text{ for all } \ t \geq 0. \end{aligned} \tag{14}$$

Therefore, we see that if every off-diagonal entry of A is nonnegative, then A generates a nonnegative C_0 semigroup $(T(t))_{t\geq 0}$. ■

The following lemma was proved in [3].

Lemma 6 ([3]): For a nonnegative C_0 semigroup with generator $\mathcal{A}$, the following properties are equivalent for $\mu \in \rho(\mathcal{A})$:

(i) $s(\mathcal{A}) < \mu$.
(ii) $R(\mu, \mathcal{A}) \geq 0$.

Using Lemma 6, we can state the following lemma that is useful for evaluating whether condition (iii) of Proposition 3 is satisfied.

Lemma 7: Assume that $A \in \mathbb{R}^{n\times n}$ generates a nonnegative C_0 semigroup, then the following assertions are equivalent.

(i) $s(A) < 0$.
(ii) $(-A)^{-1} \geq 0$.

Proof: Taking $\mu = 0$ in (i) and (ii) of Lemma 6, we see that $s(A) < 0 \Leftrightarrow R(0, A) \geq 0$. Considering that $R(0, A) = (0 - A)^{-1} = (-A)^{-1}$, we see from Lemma 6 that

$$s(A) < 0 \Leftrightarrow (-A)^{-1} \geq 0.$$

This completes the proof. ■

Next, we introduce the notation for some matrix operations. Let $A_{i-1} \in \mathbb{R}^{(i-1)\times(i-1)}$, $b \in \mathbb{R}^{i-1}$, $c' \in \mathbb{R}^{i-1}$, $d \in \mathbb{R}$ be given from a matrix $A_i \in \mathbb{R}^{i\times i}$ in the following form:

$$A_i = \left(\begin{array}{c|c} \boxed{A_{i-1}} & \boxed{b} \\ \boxed{c} & \boxed{d} \end{array} \right). \tag{15}$$

Let $f : \mathbb{R}^{i\times i} \mapsto \mathbb{R}^{(i-1)\times(i-1)}$ be an operator defined for $i \geq 2, d \neq 0$ as

$$f(A_i) = A_{i-1} - bd^{-1}c. \tag{16}$$

For convenience, we introduce the following notation:

$$\begin{aligned} f^0(A_i) &= A_i \ \in \mathbb{R}^{i\times i}, & (17) \\ f^1(A_i) &= f(A_i) \ \in \mathbb{R}^{(i-1)\times(i-1)}, & (18) \\ f^2(A_i) &= f(f(A_i)) \ \in \mathbb{R}^{(i-2)\times(i-2)}, & (19) \\ &\vdots \\ f^{i-1}(A_i) &= \underbrace{f(f(\cdots f(A_i)\cdots))}_{i-1} \in \mathbb{R}. & (20) \end{aligned}$$

Using the above notation, we can state the following lemma.

Lemma 8: Assume that $A \in \mathbb{R}^{n\times n}$ generates a nonnegative C_0 semigroup. If all diagonal elements of $f^p(-A)$ are positive for all $p = 0, \cdots, (n-1)$, then $s(A) < 0$.

Proof: It is well known [12] that $(-A)^{-1} \geq 0$ if and only if all principal minors of $-A$ are positive. Considering that

$$\det(A_i) = \det(d)\det(A_{i-1} - bd^{-1}c), \tag{21}$$

we see that if all principal minors of $f^{i+1}(-A)$ are positive, then all principal minors of $f^i(-A)$ are positive. Using this relation recursively under the assumption of $f^{n-1}(-A) > 0 \in \mathbb{R}$, we see that all principal minors of $f^0(-A) = -A$ are positive. It follows from Lemma 7 that if all principal minors of $-A$ are positive, then $s(A) < 0$. This completes the proof. ■

From Lemma 8, we obtain the following lemma using the notation in Definition 1.

Lemma 9: Let $k \in \mathbb{N}_+$ satisfy $1 < k < n$. Let $A \in \mathbb{R}^{n\times n}$ be a diagonal matrix all of whose entries are negative, and let $B \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Let A, B be decomposed into four block matrices as

$$A = \left(\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right), \quad B = \left(\begin{array}{c||c} \boxed{B_{11}} & \boxed{B_{12}} \\ \hline \boxed{B_{21}} & \boxed{B_{22}} \end{array}\right), \quad (22)$$

where all entries of each block matrix are functions of μ of the same order. Both A_{11} and B_{11} are $k \times k$ matrices. For $i = 1, 2$ and $j = 1, 2$, all entries of A_{ii} and B_{ij} belong to $O(a_{ii})$ and $O(b_{ij})$, respectively. For sufficiently large μ, if

$$a_{11} > b_{11}, \;\; a_{22} > b_{22}, \;\; a_{11} > b_{12} - a_{22} + b_{21}, \quad (23)$$

then $s(A + B) < 0$.

Proof: The proof is obtained by a straightforward computation of $f^p(-A-B)$ $(p = 0, \cdots, n-1)$ in Lemma 8. ■

Lemma 9 is useful for the evaluation of stabilizability condition (iii) of Proposition 3, and is used in the proof of the main theorem.

VI. MAIN RESULTS

In this section, we consider the following system defined on $x \in \mathbb{R}^n$ for $t \in [0, \infty)$ as

$$\dot{x}(t) = A_0 x(t) + \Delta A_1(t)x(t) + \Delta A_2(t)x(t-\tau) + bu(t), \quad (24)$$

with an initial curve $\phi \in \mathcal{L}^n[-\tau, 0]$, where n is a fixed positive integer. $A_0 \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are known constant matrices. Furthermore, $\Delta A_1(t)$ and $\Delta A_2(t) \in \mathbb{R}^{n\times n}$ are uncertain coefficient matrices and may vary with $t \in [0, \infty)$. It is assumed that all entries of $\Delta A_1(t)$ and $\Delta A_2(t)$ are piecewise continuous functions and are uniformly bounded, i.e., for nonnegative constant matrices $\Delta\bar{A}_1$ and $\Delta\bar{A}_2 \in \mathbb{R}^{n\times n}$, they satisfy

$$|\Delta A_1(t)| \le \Delta\bar{A}_1, \quad |\Delta A_2(t)| \le \Delta\bar{A}_2, \quad (25)$$

for all $t \ge 0$. The upper bound of each entry can independently take an arbitrarily large value, but each is assumed to be known. Other variables are as follows: $u(t) \in \mathbb{R}$ is a control variable, and τ is a constant time delay. τ can be arbitrarily large and is not necessarily assumed to be known.

Assumption 2: Because the system must be controllable, we assume that the pair (A_0, b) of the nominal system is a controllable pair as follows:

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \quad (26)$$

Now, we consider the stabilization problem of system (24) based on the stabilizability criteria in Proposition 3. We investigate whether it is possible to construct a linear state feedback control

$$u(t) = g'x(t) \quad (27)$$

such that the equilibrium point $x = 0$ of the resulting closed-loop system is uniformly exponentially stable, using a constant vector $g \in \mathbb{R}^n$. Consider a linear transformation such that

$$w(t) = V^{-1}x(t), \quad (28)$$

using a real nonsingular matrix $V \in \mathbb{R}^{n\times n}$. Substituting (27) and (28) into (24), the following equation is obtained.

$$\begin{aligned} \dot{w}(t) &= V^{-1}(A_0 + bg')Vw(t) + V^{-1}\Delta A_1(t)Vw(t) \\ &\quad + V^{-1}\Delta A_2(t)Vw(t-\tau). \end{aligned} \quad (29)$$

Because of Assumption 2, it is possible to choose $g \in \mathbb{R}^n$ so that all the eigenvalues of $(A_0 + bg')$ are real, negative and distinct. Let g be as such. In addition, let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be such eigenvalues of $(A_0 + bg')$. Let V in (28) be the Vandermonde matrix constructed using $\lambda_1, \lambda_2, \cdots, \lambda_n$ such that

$$V = \begin{pmatrix} 1 & 1 & . & . & . & 1 \\ \lambda_1 & \lambda_2 & . & . & . & \lambda_n \\ {\lambda_1}^2 & {\lambda_2}^2 & . & . & . & {\lambda_n}^2 \\ . & . & . & . & . & . \\ {\lambda_1}^{n-1} & {\lambda_2}^{n-1} & . & . & . & {\lambda_n}^{n-1} \end{pmatrix}. \quad (30)$$

This V is well known to be nonsingular in view of the above assumption. Let Λ be defined by

$$\Lambda := V^{-1}(A_0 + bg')V = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}. \quad (31)$$

Let H_1, H_2, and H be defined by

$$H_1 := \left|V^{-1}\right| \Delta\bar{A}_1 \left|V\right|, \quad (32)$$

$$H_2 := \left|V^{-1}\right| \Delta\bar{A}_2 \left|V\right|, \quad (33)$$

$$H := H_1 + H_2. \quad (34)$$

Then, it follows from (29) that

$$\dot{w}(t) \le (\Lambda + H_1)w(t) + H_2 w(t-\tau). \quad (35)$$

Noting that every off-diagonal entry of $(\Lambda + H_1)$ is nonnegative, we see from Lemma 5 that $(\Lambda + H_1)$ generates a nonnegative C_0 semigroup. It is obvious that H_2 is nonnegative. Consequently, using Proposition 3 and the modulus semigroup theory [13], we can state the following proposition.

Proposition 4: If there exist g and V such that

$$s(\Lambda + H) < 0 \quad (36)$$

is satisfied, then system (24) is uniformly exponentially stabilizable.

Next, we introduce a set of matrices $\Omega \subset \mathbb{R}^{n\times n}$ to show the main result.

Definition 8: Let k be an integer satisfying $0 \le k \le n$. For this k, let $\Omega(k) = \{E = (e_{ij}) \in \mathbb{R}^{n\times n}\}$ be a set of matrices with the following properties.

(i) If $1 \le j \le k+1$, then $e_{ij} = 0$ for $j-1 \le i \le 2k-j+1$.
(ii) If $k+2 \le j \le n$, then $e_{ij} = 0$ for $2k-j+1 \le i \le j-1$.

A roughly geometric interpretation of a matrix belonging to $\Omega(k)$ is shown below, where $*$ denotes an uncertain entry that may take an arbitrarily large value.

$$
\begin{array}{c}
\begin{array}{ccccccccccccc} & & & & & & & \text{k+1} & & & & & \\ & & & & & & & \downarrow & & & & & \end{array} \\
\left(\begin{array}{ccccccccccccc}
0 & 0 & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * & * & * & * & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & * & * & * & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & * & * & * & * & * & * & * & * & * & * & *
\end{array}\right) \in \Omega(k)
\end{array}
$$

Now, we state the main theorem.

Theorem 1: Assume that

$$(\Delta\bar{A}_1 + \Delta\bar{A}_2) \in \Omega(k) \tag{37}$$

is satisfied for fixed k, then system (24) is uniformly exponentially stabilizable.

Proof: According to Proposition 4, the existence of g and V that assure $s(\Lambda + H) < 0$ is investigated in the remainder of this section. On evaluating the existence of g and V, it is important how to choose the eigenvalues λ_i $(i = 1, \cdots, n)$ of $(A^0 + bg')$.

Here, let μ be a positive number and let α_i $(i = 1, \cdots, n)$ be all negative numbers that are different from one another. Let μ be chosen as much larger than all entries of $\Delta\bar{A}_1$ and $\Delta\bar{A}_2$. Let α_i $(i = 1, \cdots, n)$ be used for distinguishing eigenvalues from one another. The proper way of choosing λ_i $(i = 1, \cdots, n)$ is shown below.

$$\left\{ \begin{array}{ll} \lambda_i = \alpha_i \mu^{-1} & (i = 1, ..., k), \\ \lambda_i = \alpha_i \mu & (i = k+1, ..., n). \end{array} \right. \tag{38}$$

To complete the proof of Theorem 1, we should show that if we choose λ_i $(i = 1, \cdots, n)$ as in (38), then V constructed by such λ_i assures $s(\Lambda + H) < 0$. Considering that

$$\left\{ \begin{array}{ll} \lambda_i \in O(-1) & (i = 1, ..., k), \\ \lambda_i \in O(1) & (i = k+1, ..., n), \end{array} \right. \tag{39}$$

it turns out from the careful calculation that H in (34) is decomposed into four block matrices such as (22). All entries of each block matrix are functions of μ of the same order.

$$H = \left(\begin{array}{c|c} \boxed{-2} & \boxed{2k-1} \\ \hline \boxed{-2k} & \boxed{0} \end{array} \right). \tag{40}$$

Taking into account the fact that Λ is a negative diagonal matrix in which all diagonal entries belong to $O(-1)$ (from the first to kth entry) or $O(1)$ (from the $(k+1)$th to nth entry), we obtain

$$-1 > -2, \;\; 1 > 0, \;\; -1 > (2k-1) - 1 + (-2k) = -2. \tag{41}$$

According to Lemma 9, it is clear from inequalities (41) that $s(\Lambda + H) < 0$. This completes the proof. ■

VII. Conclusion

In this study, we examined the stabilization problem of linear time-varying uncertain systems with state delay using semigroup theory. We first introduced the concept of an abstract delay system and investigated the stabilization problem of an abstract delay system on a Banach lattice using the properties of a nonnegative C_0 semigroup. We derived sufficient conditions for the stabilization of an abstract delay system under the assumption that the system satisfies the nonnegativity. To remove such a restrictive assumption, we next introduced a variable transformation method for stabilizing an abstract delay system without the nonnegativity assumption. Using the stabilizability criterion of nonnegative semigroups, we provided the stabilizability conditions for linear time-varying uncertain systems with state delay, in which the stabilizability of the systems can be determined by the localization of uncertain parameters. In other words, the stabilizability conditions obtained here can be verified simply by examining the locations of uncertain entries in given system matrices. Once a system satisfies the stabilizability conditions, a stabilizing controller can be constructed, irrespective of the given bounds of uncertain variations. We can redesign the controller for improving robustness simply by modifying the design parameter μ when the uncertain parameters exceed the upper bounds given beforehand.

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