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Position Stabilization of a Stewart Platform: High-Order Sliding Mode Observers Based Approach

Liset Fraguela Cuesta, Leonid Fridman and Vladimir V. Alexandrov

Abstract— **The problem of robust exact control for a Stewart platform with smooth bounded unknown inputs is considered. This platform has three degrees of freedom and it is used as a remote surveillance device. We consider high-order sliding mode observers to provide both theoretical exact observation and unknown input identification. In this paper, a methodology is proposed to select the most adequate control strategy for unknown external perturbation identification. The results obtained are illustrated by simulations.**

I. INTRODUCTION

A. Antecedents and Motivation

The present work is associated with the so-called Stewart platform, which is a closed cinematic chain robot, and it is the most important example of a totally parallel manipulator [15], understanding as such the robot that possess two bodies, one fixed and the other mobile, which are connected between them by several arms. Typically, every arm is controlled by an actuator. Stewart platform has, therefore, a parallel configuration of six degrees of freedom composed by two rigid bodies connected by six prismatic actuators [4], [17]. The largest rigid body is named the base, and the mobile body is called the mobile platform. The application of this type of robots is useful when we are looking for load capacity, good dynamic performance and / or precision in the positioning.

Our goal is to stabilize Stewart platform with three degrees of freedom around a wished position when we do not have complete information about the initial conditions and the permanent disturbance that affect this platform. Of these uncertainties it is only known that they belong to a convex and bounded set. In addition, there are two restrictions concerning the permanent disturbance: one regarding its value, the other to the value of its integral.

Our specific application consists in an aerostatic balloon, easy to manipulate, that it is moored to earth by a cable of approximately 400 meters of length. The base platform is connected to this balloon and a video camera is fixed to the mobile platform to keep under surveillance a specific area of approximately 20 square kilometers (see Fig. 1). This device offers a wide range of applications in missions of surveillance, such as monitoring, rescue operations, intelligence, traffic control, recognition, among others.Since the base platform is over the mobile one we will name this platform: *inverted Stewart platform* and we will denote it with the letter $\boldsymbol{P}$ for further references. We can observe that,

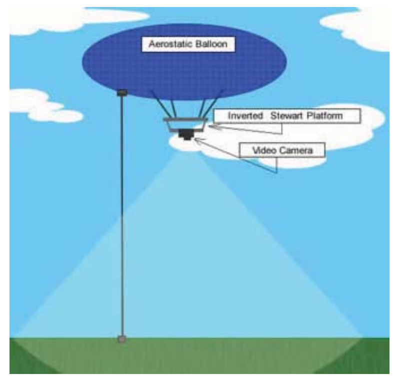

Fig. 1. Scheme of the remote surveillance device

due to the type of application, our platform $\boldsymbol{P}$ is permanently under the action of the force of the wind. Therefore, we will work with the wind's acceleration as our permanent disturbance.

Another characteristic of our implementation is that we have only information available about positions but not about velocities. In this situation we need to reconstruct such velocities in order to design a robust control with respect the external perturbation to be able to stabilize a wished position. High-order sliding mode observers are commonly used for observation of dynamical systems and unknown input identification (see [2], [3], [7] and [16]).

B. Main Contribution

In this paper a robust control is designed to stabilize a Stewart Platform used as a remote surveillance device when only positions are available. High-Order Sliding Mode (HOSM) observer is applied to reconstruct the velocities and external perturbation identification from the position measurements. First we will use a second-order sliding mode observer that ensures the finite time convergence to the value of observed velocities without filtration but for uncertainties identification the realization of the observer produces high switching frequencies making necessary the application of a filter. Then, a third-order sliding mode observer is applied

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L. Fraguela is with Engineering Faculty, Division of Electrical Engineering, Department of Control, Universidad Nacional Autónoma de México, 04510, México, D.F. lisetfrc@siu.buap.mx

L. Fridman is with Dept. Automatic Control CINVESTAV, Instituto Politécnico Nacional, México D.F., AP-14740. On leave on: Engineering Faculty, Division of Electrical Engineering, Department Control Engineering and Robtics, Universidad Nacional Autónoma de México, 04510, México, D.F. lfridman@servidor.unam.mx

V.V. Alexandrov is with Mathematics and Physics Faculty, Benemérita Universidad Autónoma de Puebla, Puebla, México, vladimiralexandrov366@hotmail.com

giving us a theoretical exact estimation of the external perturbation without filtration.

Some specific contributions are enumerated below:

1) We propose a third order sliding mode observer to reconstruct the velocities and exact external perturbation identification of platform ***P*** when only positions are available.
2) Such states reconstruction and perturbation identification permits to design controller compensating uncertainties after convergence time.
3) We compare the perturbation identification using second and third order sliding mode observers.

C. Structure of the Paper

In section II, a model of platform ***P*** is presented and the control law is formulated. Section III is devoted to the state observation. In section IV are analyzed two methods to identify unknown external perturbation. Subsection IV-A deals with the standard procedure where a high frequency component will appears and subsection IV-B an extended order approach will be applied to eliminate that component. In section V a nominal control is design and section VI deals with the numerical illustration of the platform ***P***.

II. Model Description and Problem Statement

A. Plant Model

Let us consider the inverted Stewart platform which consists of a base platform and a mobile one, both with the shape of an equilateral triangle, of sides a and b, $a > b$, respectively. The vertexes of the base are joined to the corresponding vertexes of the mobile platform by actuators of lengths, l_i $(i = 1, 2, 3)$, variable and bounded. These actuators are fastened to the base platform at the points A_i $(i = 1, 2, 3)$ by cylindrical joints with axes of rotation perpendicular to the segment $\overline{A_i A_0}$ $(i = 1, 2, 3)$, forming the respective angles γ_i, and they are connected to the mobile platform in the corresponding points B_i $(i = 1, 2, 3)$ by spherical joints (see Fig. 2). The type of joints used to connect the platforms, both base and mobile, through the actuators allow us to reduce the six original degrees of freedom to three: two rotations (α and β) and one translation (h). α and β are the rotations of the center of mass of the mobile platform (B_0) respect the axis y' and x', respectively and h is the vertical position of B_0 respect to A_0, as is shown in Figure 2. We also possess a GPS device placed in the center of mass of the base platform (A_0, see Fig. 2) to measure its position with respect to the ground and angle sensors positioned in the mobile platform to measure the aforementioned angles. The position of the center of mass of the mobile platform (B_0) can be recovered using the position of A_0 and the geometry of platform ***P***. We shall assume that the measurement errors of the GPS device are negligible for our application. The position to be stabilized is $\alpha = \beta = 0$ and $h = h_0$. To obtain the equations that describe the movement of the platform ***P*** we use Lagrange's equations considering the independent variables α, β and h as the generalized coordinates. Then, to investigate the behavior of the system in a neighborhood of the wished position ($\alpha = \beta = 0$ and $h = h_0$) we introduce a small deviation in the coordinates: $\triangle\alpha = \alpha$, $\triangle\beta = \beta$ and $\triangle h = h - h_0$, and we obtain the linear model of the platform ***P*** that is valid in a vicinity of the wished position. This linear model in deviation is normalized using the following adimensional magnitudes, $x_1 = \triangle\alpha$, $x_2 = dx_1/d\tilde{\tau}$, $x_3 = \triangle\beta$, $x_4 = dx_3/d\tilde{\tau}$, $x_5 = \triangle h$, $x_6 = dx_5/d\tilde{\tau}$, where

$$\tilde{\tau} = t\sqrt{g_r/h_0}, \tag{1}$$

and $\tilde{\tau}$ is the adimensional time, t is time in seconds and g_r is the gravity acceleration. This way we obtain the following linear time-invariant model in deviations for first approximation with uncertainties, of the platform ***P*** [13],

$$\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) + g(w, x(t)); \;\; x(0) = x^0, \\ y(t) = Cx(t), \\ w \in V = \{V \subset \mathbb{R}^r : \dot{v} = w, |v_i(t)| \le v_0, |w_i(t)| \le w_0\} \end{array} \tag{2}$$

where $x(t) = (x_1, \cdots, x_6)^\top$ is the state vector, $u(t) \in \mathbb{R}^m$ $(m = 3)$ is the control law, $y(t) \in \mathbb{R}^p$ $(1 \le p < n)$ $(p = 3)$ is the output of the system and w is the permanent perturbation, representing the wind's acceleration ($r = 3$). There exist two kinds of influences of the external disturbance on the platform ***P***: the general (normal) resonance and the parametric resonance. The vector x^0 is supposed to be unknown but belonging to a given ball, that is, $\left\|x^0\right\| \le \mu$, where $||\cdot||$ is the Euclidian norm.

The matrices A and B are described below.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & a_3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ -b_1 & -b_1 & 2b_1 \\ 0 & 0 & 0 \\ b_2 & -b_2 & 0 \\ 0 & 0 & 0 \\ b_3 & b_3 & b_3 \end{pmatrix}$$

$$\begin{array}{ll} a_1 = [\dfrac{b^2\cos^2\gamma_0 - b(a-b)}{6r_y^2}], & b_1 = \dfrac{bh_0}{6\sqrt{3}r_y^2} \\ a_2 = -[\dfrac{b^2\cos^2\gamma_0 + b(a-b)}{6(h_0^2 + r_x^2)}], & b_2 = \dfrac{bh_0}{6(h_0^2 + r_x^2)} \\ a_3 = -\cos^2\gamma_0, & b_3 = -1/3; \end{array}$$

where γ_0 is the angle between the actuators and the base platform when the mobile platform is in the wished position, r_x and r_y are radius of inertia. The matrix C indicates which system parameters we measure and it is selected as,

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

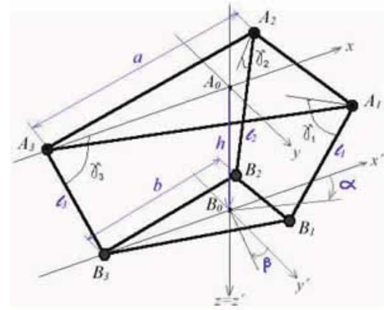


Fig. 2. Geometric scheme of the platform ***P***

The plant in (2) operates under matched uncertainties, that is,

$$g(w(t),x(t)) = B\gamma(w(t),x(t)) = (0,g_1,0,g_2,0,g_3)^\top; \quad (4)$$

$$\begin{aligned}
g_1 &= \left(\frac{h_0 b}{2\sqrt{3}r_y^2}\right) w_x x_1,\\
g_2 &= -\left(\frac{h_0^2}{h_0^2+r_x^2} - \frac{2h_0^4}{(h_0^2+r_x^2)^2}\right) w_y x_5 - \left(\frac{h_0^2}{h_0^2+r_x^2}\right) w_y,\\
g_3 &= -w_y x_3;
\end{aligned}$$

where the wind acceleration has the form $w(t) = (w_x(t), w_y(t), w_z(t))^\top$. But, due to our application consists in a Stewart Platform connected to a balloon that is moored to the ground, we assumed that the influence of $w_z(t)$ can be of no consequence, in other words, we are going to consider $w_z(t) = 0$.

The nominal part of the system dynamics is represented by the function

$$F(x(t),u(t)) = Ax(t) + Bu(t),$$

while the uncertainties and perturbations are concentrated in $g(w,x,t)$. The solutions to system (2) are understood in Filippov's sense [11]. The nominal function $F(x,u)$ and the uncertainty $g(w,x,t)$ are Lebesgue-measurable and uniformly bounded in any compact region of the state-space x. This means that the space of "real" mechanical system variables is bounded. The tasks are to design an observation algorithm to obtain the values of x_2, x_4, x_6 and an identification algorithm to get the system uncertainties having only the knowledge of the states x_1, x_3, x_5.

B. Control Challenge

Here, a compensation control law is designed based on the estimated states and the identification of the system uncertainties. Consider the nominal system

$$\dot{x}_0(t) = Ax_0(t) + Bu_0(t). \quad (5)$$

The control design problem is to design a control law that, providing that $x(0) = x_0(0)$, guaranties the identity $x(t) = x_0(t)$ for all $t \geq 0$. By comparing (2) and (5) it is clear that the control design achieved only if the equivalent control is equal to the negative of the uncertainty. Thus, the control objective can be formulated in the following terms: design the control law $u(t)$ to be

$$u(t) = u_0(t) + u_1(t) \quad (6)$$

where the control $u_1 \in \mathbb{R}^m$ is the control part guarantying the compensation of the unknown matched uncertainty $g(w(t),x(t))$ (see [9]) and $u_0 \in \mathbb{R}^m$ is the nominal control part designed for the system (5). In this paper u_0 is a control function minimizing the worst possible scenario in the sense of some LQ-index, as is shown in section V. For u_1 we used a high-order sliding mode observer to identify the perturbation $g(w(t),x(t))$ (section IV) in order to obtain $u_1 = -g(w(t),x(t))$.

III. State Observation

For state observation let us use the second-order sliding mode based observer of the form

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + \alpha_{21}|x_1-\hat{x}_1|^{1/2} sign(x_1-\hat{x}_1)\\
\dot{\hat{x}}_2 &= F_1(x_1,\hat{x}_2,u_1(x_2)) + \alpha_{11} sign(x_1-\hat{x}_1)\\
\dot{\hat{x}}_3 &= \hat{x}_4 + \alpha_{22}|x_3-\hat{x}_3|^{1/2} sign(x_3-\hat{x}_3)\\
\dot{\hat{x}}_4 &= F_2(x_3,\hat{x}_4,u_2(x_4)) + \alpha_{12} sign(x_3-\hat{x}_3)\\
\dot{\hat{x}}_5 &= \hat{x}_6 + \alpha_{23}|x_5-\hat{x}_5|^{1/2} sign(x_5-\hat{x}_5)\\
\dot{\hat{x}}_6 &= F_3(x_5,\hat{x}_6,u_3(x_2,x_4,x_6)) + \alpha_{13} sign(x_5-\hat{x}_5)
\end{aligned} \quad (7)$$

where $\hat{x}_i$ are the estimation of x_i $(i = 1,\cdots,6)$. The constant gains α_{1j} and α_{2j} $(j = 1,2,3)$ are the correction factors designed for convergence of estimation error for each couple of coordinates (x_1,x_2), (x_3,x_4) and (x_5,x_6) [14].

Equations of estimation error take the form

$$\begin{aligned}
\dot{\tilde{x}}_1 &= \tilde{x}_2 - \alpha_{21}|x_1-\hat{x}_1|^{1/2} sign(x_1-\hat{x}_1)\\
\dot{\tilde{x}}_2 &= \tilde{F}_1(x_1,x_2,\hat{x}_2,u_1(x_2)) + g_1 - \alpha_{11} sign(x_1-\hat{x}_1)\\
\dot{\tilde{x}}_3 &= \tilde{x}_4 - \alpha_{22}|x_3-\hat{x}_3|^{1/2} sign(x_3-\hat{x}_3)\\
\dot{\tilde{x}}_4 &= \tilde{F}_2(x_3,x_4,\hat{x}_4,u_2(x_4)) + g_2 - \alpha_{12} sign(x_3-\hat{x}_3)\\
\dot{\tilde{x}}_5 &= \tilde{x}_6 - \alpha_{23}|x_5-\hat{x}_5|^{1/2} sign(x_5-\hat{x}_5)\\
\dot{\tilde{x}}_6 &= \tilde{F}_3(x_5,x_6,\hat{x}_6,u_3(x_2,x_4,x_6)) + g_3 - \alpha_{13} sign(x_5-\hat{x}_5)
\end{aligned} \quad (8)$$

where $\tilde{F}_j(x_j,x_{j+1},\hat{x}_{j+1}) = F_j(x_j,x_{j+1},u_j) - F_j(x_j,\hat{x}_{j+1},u_j)$, $||g_j(w,x,t)|| \leq g_j^+$ and $\tilde{x}_i = x_i - \hat{x}_i$ $(j = 1,2,3;\ i = 1,\cdots,6)$. Due to the boundedness assumption it is possible to find an upper bound for each couple of coordinates such that,

$$|\tilde{F}_j(x_j,x_{j+1},\hat{x}_{j+1},u_j) + g_j| < f_j^+. \quad (9)$$

Theorem 1: [5] Suppose that condition (9) holds for system (2), and parameters of the observer (7) are selected according to

$$\begin{aligned}
\alpha_{2j} &> f_j^+\\
\alpha_{1j} &> \sqrt{\frac{2}{\alpha_{2j}}\frac{(\alpha_{2j}+f_j^+)(1+p_j)}{(1-p_j)}}
\end{aligned} \quad (10)$$

where p_j are constants to be chosen $0 < p_j < 1$. Then the observer (7) ensures the convergence of the estimated states $(\hat{x}_j, \dot{\hat{x}}_j)$ to the real value of the states $(x_j, \dot{x}_j)$ after a first time transient and there exists a time constant t_0 such that for all $t > t_0$, $(\hat{x}_j, \hat{x}_{j+1}) = (x_j, x_{j+1})$.

The proof of Theorem 1 is given in [5].

IV. Equivalent Output Injection Analysis: Perturbation Identification

In this section we present two approaches to identify the unknown matched perturbation $g(w(t),x(t))$ by means of a second-order and a third-order sliding mode observers. The realization of the second-order sliding mode observer produce high switching frequencies thus calling for the application of a low-pass filter that in turn introduces time delay. This time delay, coupled with a discontinuous term, produces chattering. Meanwhile, the third-order sliding mode observer produce a continuous term, and no filtration is required to obtain the equivalent output injection. This way, given the finite time convergence of the differentiator, we are able to reconstruct in finite time the equivalent output injection. These methods are explained more thoroughly in Sections IV-A and IV-B.

A. Standard Procedure

The finite time convergence to the second-order sliding mode set ensures that there exists a time constant $t_0 > 0$ such that for all $t > t_0$ the following identities holds (see [6] and [12])

$$\begin{aligned}
0 &\equiv \dot{\tilde{x}}_2 \equiv \tilde{F}_1(x_1,x_2,\hat{x}_2,u_1) + g_1 - \alpha_{11} sign(x_1-\hat{x}_1)\\
0 &\equiv \dot{\tilde{x}}_4 \equiv \tilde{F}_2(x_3,x_4,\hat{x}_4,u_2) + g_2 - \alpha_{12} sign(x_3-\hat{x}_3)\\
0 &\equiv \dot{\tilde{x}}_6 \equiv \tilde{F}_3(x_5,x_6,\hat{x}_6,u_3) + g_3 - \alpha_{13} sign(x_5-\hat{x}_5).
\end{aligned}$$

Notice that $\tilde{F}_j(x_j,x_{j+1},\hat{x}_{j+1},u_j) = 0$ because $\hat{x}_{j+1} = x_{j+1}$ $(j = 1,2,3)$. Then the equivalent output injection $z_{eq} = (0, z_{eq_1}, 0, z_{eq_2}, 0, z_{eq_3})^\top$ is given by the following terms

$$\begin{aligned}
z_{eq_1} &\equiv \alpha_{11} sign(x_1-\hat{x}_1) \equiv g_1\\
z_{eq_2} &\equiv \alpha_{12} sign(x_3-\hat{x}_3) \equiv g_2\\
z_{eq_3} &\equiv \alpha_{13} sign(x_5-\hat{x}_5) \equiv g_3.
\end{aligned} \quad (11)$$

It was mentioned before that the term $g = (0,g_1,0,g_2,0,g_3)^\top$ is the external perturbation that affects the platform $\boldsymbol{P}$.

Theoretically, the equivalent output injection is the result of an infinite switching frequencies of the discontinuous term $\alpha_{1j} sign(x_j - \hat{x}_j)$. Nevertheless, the realization of the observer produces high (finite) switching frequency making necessary the application of a filter. To eliminate the high frequency component we will use the low-pass filter of the form

$$\tau\dot{\tilde{z}}_{eq}(t) = z_{eq}(t) - \tilde{z}_{eq}(t) \tag{12}$$

where $\tau \in \mathbb{R}$ and $\Delta \ll \tau \ll 1$ (Δ is the sampling step).

It is possible to rewrite z_{eq} as result of a filtering process in the following form

$$z_{eq}(t) = \tilde{z}_{eq}(t) + \varepsilon(t) \tag{13}$$

where $\varepsilon(t) \in \mathbb{R}^6$ is the difference caused by the filtration and $\tilde{z}_{eq}(t)$ is the filtered version of $z_{eq}(t)$.

Nevertheless, as it is shown in [18] and [8]

$$\lim_{\substack{\tau\to 0\\ \Delta/\tau\to 0}} \tilde{z}_{eq}(\tau, \Delta) = z_{eq}(t),$$

then, it is possible to assume that the equivalent output injection is equal to the output of the filter.

B. Extended Order Approach

Let the second, fourth and sixth equation of (2) be differentiated. As we know $F_j(x_j, x_{j+1}, u_j)$ $(j = 1, 2, 3)$ is linear then it is smooth and we will assume that g is also smooth. The extensions of the system forces the new requirement that $\dot{F}_j(x_j, x_{j+1}, u_j) + \dot{g}_j$ $(j = 1, 2, 3)$ is bounded. If this new requirement is satisfied, it is possible to apply the third-order sliding mode observer [10]:

$$\begin{aligned}
&\dot{\hat{x}}_1 = \hat{x}_2 + \alpha_{31}|x_1 - \hat{x}_1|^{2/3} sign(x_1 - \hat{x}_1)\\
&\dot{\hat{x}}_2 = F_1(x_1, \hat{x}_2, u_1) + \alpha_{21}|\dot{\hat{x}}_1 - \hat{x}_2|^{1/2} sign(\dot{\hat{x}}_1 - \hat{x}_2) + \hat{z}_1\\
&\dot{\hat{z}}_1 = \alpha_{11} sign(\dot{\hat{x}}_1 - \hat{x}_2)\\
&\dot{\hat{x}}_3 = \hat{x}_4 + \alpha_{32}|x_3 - \hat{x}_3|^{2/3} sign(x_3 - \hat{x}_3)\\
&\dot{\hat{x}}_4 = F_2(x_3, \hat{x}_4, u_2) + \alpha_{22}|\dot{\hat{x}}_3 - \hat{x}_4|^{1/2} sign(\dot{\hat{x}}_3 - \hat{x}_4) + \hat{z}_2\\
&\dot{\hat{z}}_2 = \alpha_{12} sign(\dot{\hat{x}}_3 - \hat{x}_4)\\
&\dot{\hat{x}}_5 = \hat{x}_6 + \alpha_{33}|x_5 - \hat{x}_5|^{2/3} sign(x_5 - \hat{x}_5)\\
&\dot{\hat{x}}_6 = F_3(x_5, \hat{x}_6, u_3) + \alpha_{23}|\dot{\hat{x}}_5 - \hat{x}_6|^{1/2} sign(\dot{\hat{x}}_5 - \hat{x}_6) + \hat{z}_3\\
&\dot{\hat{z}}_3 = \alpha_{13} sign(\dot{\hat{x}}_5 - \hat{x}_6)
\end{aligned} \tag{14}$$

where $\hat{x}_i$ are the estimation of x_i $(i = 1, \cdots, 6)$. The constant gains α_{1j}, α_{2j} and α_{3j} $(j = 1, 2, 3)$ are the correction factors designed for convergence of the estimation error for each couple of coordinates (x_1, x_2), (x_3, x_4) and (x_5, x_6) [14].

Equations of error estimation takes the form

$$\begin{aligned}
&\dot{\tilde{x}}_1 = \tilde{x}_2 - \alpha_{31}|x_1 - \hat{x}_1|^{2/3} sign(x_1 - \hat{x}_1),\\
&\dot{\tilde{x}}_2 = \tilde{F}_1(x_1, x_2, \hat{x}_2, u_1) + g_1 -\\
&\quad - \alpha_{21}|\dot{\hat{x}}_1 - \hat{x}_2|^{1/2} sign(\dot{\hat{x}}_1 - \hat{x}_2) - \hat{z}_1,\\
&\dot{\tilde{x}}_3 = \tilde{x}_4 - \alpha_{32}|x_3 - \hat{x}_3|^{2/3} sign(x_3 - \hat{x}_3)\\
&\dot{\tilde{x}}_4 = \tilde{F}_2(x_3, x_4, \hat{x}_4, u_2) + g_2\\
&\quad - \alpha_{22}|\dot{\hat{x}}_3 - \hat{x}_4|^{1/2} sign(\dot{\hat{x}}_3 - \hat{x}_4) - \hat{z}_2,\\
&\dot{\tilde{x}}_5 = \tilde{x}_6 - \alpha_{33}|x_5 - \hat{x}_5|^{2/3} sign(x_5 - \hat{x}_5)\\
&\dot{\tilde{x}}_6 = \tilde{F}_3(x_5, x_6, \hat{x}_6, u_3) + g_3 -\\
&\quad - \alpha_{23}|\dot{\hat{x}}_5 - \hat{x}_6|^{1/2} sign(\dot{\hat{x}}_5 - \hat{x}_6) - \hat{z}_3.
\end{aligned} \tag{15}$$

After convergence of the differentiator, the equalities $\dot{\hat{x}}_2 = \dot{x}_2$, $\dot{\hat{x}}_4 = \dot{x}_4$ and $\dot{\hat{x}}_6 = \dot{x}_6$ hold, the following expressions are equal to zero:

$$\begin{aligned}
&\tilde{F}_1(x_1, x_2, \hat{x}_2, u_1) + g_1 - \alpha_{21}|\dot{\hat{x}}_1 - \hat{x}_2|^{1/2} sign(\dot{\hat{x}}_1 - \hat{x}_2) - \hat{z}_1\\
&\tilde{F}_2(x_3, x_4, \hat{x}_4, u_2) + g_2 - \alpha_{22}|\dot{\hat{x}}_3 - \hat{x}_4|^{1/2} sign(\dot{\hat{x}}_3 - \hat{x}_4) - \hat{z}_2\\
&\tilde{F}_3(x_5, x_6, \hat{x}_6, u_3) + g_3 - \alpha_{23}|\dot{\hat{x}}_5 - \hat{x}_6|^{1/2} sign(\dot{\hat{x}}_5 - \hat{x}_6) - \hat{z}_3.
\end{aligned} \tag{16}$$

The third term of (16) is equal to zero as a result of the differentiator convergence, then it is possible to obtain the equivalent output injection (in our case perturbation identification) as:

$$\begin{aligned}
\hat{z}_1 &= g_1\\
\hat{z}_2 &= g_2\\
\hat{z}_3 &= g_3.
\end{aligned} \tag{17}$$

In this case, $\hat{z}_1$, $\hat{z}_2$ and $\hat{z}_3$ are continuous terms, and no filtration is required to obtain the equivalent output injection. This is an important fact, because given the finite time convergence of the differentiator, we are able now to reconstruct the perturbation in finite time. Moreover, the variables $\hat{z}_1$, $\hat{z}_2$ and $\hat{z}_3$ are not affected by any filtration process, hence they are a theoretical exact estimation of g_1, g_2 and g_3, respectively.

V. Case of Study: Min-max Stabilization of Platform *P*

Let us consider for the nominal system (5) the control u_0 as a control with linear output feedback:

$$u_0 = Kx \tag{18}$$

where $K \in Q = \{Q \subset \mathbb{R}^{m\times n} | Re(\lambda_i) \le -k_0, k_0 > 0\}$ $(m = 3)$ and λ_i $(i = 1, \cdots, n)$ are the eigenvalues of the matrix $A_1(K) := A + BK$. Matrix K is described below

$$K = \begin{pmatrix} 0 & k_{11} & 0 & k_{12} & 0 & k_{13}\\ 0 & k_{21} & 0 & k_{22} & 0 & k_{23}\\ 0 & k_{31} & 0 & k_{32} & 0 & k_{33} \end{pmatrix} \tag{19}$$

In view of (18) and (19), the dynamic equations for the state x have the form:

$$\dot{x}(t) = A_1(K)x(t),\ x(0) = x^0 \tag{20}$$

The elements of K are denoted as k_{ij} $(i = 1, \cdots, m;\ j = 1, \cdots, p)$ and are known as coefficients of stabilization. Thus, the *min-max problem* consists in finding the values of k_{ij} which satisfy the following evaluation criterium:

$$J(K) = \max_{|x(0)|\le\mu} \int_0^\infty \left(x^\intercal Gx\right)dt \to \min_{K\in Q}, \tag{21}$$

where $G = G^\top \ge 0$, in our case we use G as the identity matrix of dimension n.

Physically this means that given the worst initial conditions it minimizes the deviations in time of the system parameters and this way achieves an asymptotically stable behavior. For our application, as a remote surveillance device, it is of great importance, not only to decrease the angles deviations but also its velocities because we need the movement of the camera to be slow in order to capture better images.

Thus, the control law solving (21) for (20) has the form:

$$u_0\left(x\left(t\right)\right) \equiv u_0^*\left(x\left(t\right)\right) = K^*x(t)$$

Let us reduce the optimal stabilization problem (21) to a nonlinear programming problem (see [1]). For that let us consider the differential and matrix equation:

$$\dot{Z} = A_1^\intercal Z + ZA_1, \quad Z(0) = G, \tag{22}$$

The general solution of (22) has the form,

$$Z(t) = e^{A_1^\intercal t} G e^{A_1 t}. \tag{23}$$

For any $K \in Q$ the integral $\int_0^\infty Z(t)dt$ converges, and therefore it is possible to integrate (22).

Thus we have:

$$\begin{aligned}
A_1^\intercal \int_0^\infty Z(t)dt + \int_0^\infty Z(t)dtA_1 = \int_0^\infty \dot{Z}(t)dt =\\
= Z(\infty) - Z(0) = -G.
\end{aligned}$$

Notice that $Z(\infty)$ is zero because we are considering that the matrix A_1 is such that the real part of its eigenvalues are negative.

Then it is possible to affirm that the matrix,

$$P = \int_0^\infty Z(t)dt \tag{24}$$

is the solution of the matrix equation

$$A_1^{\intercal} P + P A_1 = -G. \tag{25}$$

As we mentioned before, $G = I_n$ (I_n denotes the identity matrix with dimension n) (25) is called Lyapunov Equation and its solution is a symmetrical positive defined matrix.

Then, the functional $J(K)$ can be rewritten as:

$$\max_{|x(0)|\le\mu} \int_0^{\infty} (\Sigma_{j=1}^{n} x_j^2) dt = \max_{|x(0)|\le\mu} x^{\intercal}(0) P x(0). \tag{26}$$

On the other hand, for any symmetrical definite positive matrix the following inequality fulfilled:

$$x^{\intercal}(0) P x(0) \le \mu^2 \nu_{max}, \tag{27}$$

where ν_{max} is the maximum value of the ν_i $(i = 1, \cdots, n)$, roots of the characteristic equation,

$$\det(\nu I_n - P) = 0. \tag{28}$$

Among all the initial conditions, $|x(0)| \le \mu$, there exists one for which the equality is reached in (27). Consequently the functional can be expressed the following way:

$$\max_{|x(0)|\le\mu} \int_0^{\infty} (\Sigma_{j=1}^{n} x_j^2(t)) dt = \mu^2 \nu_{max}.$$

Thus way we can reduce the min-max problem (21) to the following extremal problem of finite dimension:

$$\mu^2 \nu_{max} \rightarrow \min_{K\in Q}. \tag{29}$$

Besides, from Theorem 1 , the estimated state $\hat{x}$ is used to realize the control u_0, i.e., the control u_0 should be designed as:

$$u_0(t) = K^* \hat{x}(t) \tag{30}$$

with $\hat{x}(t)$ being designed as in (14).

VI. Application of HOSM-Observer to Platform *P*

Let us consider the following structural dimensions for our platform ***P***: $a = 0.5m$; $b = 0.3m$; $g_r = 9.81m/s^2$; $h_0 = 0.2m$; $\gamma_0 = 60$ and $m = 3kg$ (see Fig. 2). Then, the system (2) becomes:

$$\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= -1.875x_1 - 3.464(u_{01} + u_{11}) + 5.196 w_x x_1 \\
\dot{x_3} &= x_4 \\
\dot{x_4} &= -0.3433x_3 - 0.2105(u_{02} + u_{12}) + 0.576 w_y x_5 - 0.842 w_y \\
\dot{x_5} &= x_6 \\
\dot{x_6} &= -0.25x_5 - 0.3333(u_{03} + u_{13}) - w_y x_3 \\
y &= (x_1, x_3, x_5)^{\top}
\end{aligned} \tag{31}$$

where $w_x(t) = w_y(t) = 0.1 + 0.5\sin t$, $u_0 = (u_{01}, u_{02}, u_{03})^{\top}$ is the nominal control and $u_1 = (u_{11}, u_{12}, u_{13})^{\top}$ compensate the external perturbation; $g = (0, 5.196 w_x x_1, 0, 0.576 w_y x_5 - 0.842 w_y, 0, -w_y x_3)'$.

The vector state x consists of six state variables: $x_1 = \alpha - \alpha_0$, $x_3 = \beta - \beta_0$, $x_5 = (h - h_0)/h_0$, x_2, x_4 and x_6 represents the velocity of x_1, x_3 and x_5, respectively. The wished position that we want to stabilize is $(0, 0, 0, 0, 1, 0)^{\top}$.

We design the nominal control u_0 in (18) for a particular $K \in \tilde{Q}$, where $\tilde{Q} = \{K \in Q | k_{11} = k_{21}, k_{12} + k_{22} = 2k_{32}, k_{13} = k_{23} = k_{33}\}$. Then, u_0 is given by the following terms,

$$\begin{aligned}
u_{01} &= 0.6843x_2 \\
u_{02} &= 4.6767x_4 \\
u_{03} &= 1.368x_2 + 7.0149x_4 + 3.36x_6.
\end{aligned}$$

The initial conditions are considered as $x(0) = [0.15, -0.4, 0.2, 0.5, 0.35, 0.55]^{\top}$; and as consequence we have $y(0) = [0.15, 0.2, 0.35]^{\top}$. The gains for the second-order sliding mode observer are: $\alpha_{2j} = a_2 L^{1/2}$ and $\alpha_{1j} = a_1 L$. For the third-order observer we also have $\alpha_{3j} = a_3 L^{1/3}$, where we selected $L = 1$, $a_3 = 1.9$, $a_2 = 1.5$ and $a_1 = 1.1$. Fig. 3 shows the comparison between the errors for the estimation of the velocities using second and third order sliding mode observers. The convergence time is smaller when the second-order sliding mode observer is used but if we consider in Fig. 3 the convergence of third-order observer for time $\tilde{\tau} = 2$ (see (1)), which is equivalent to $t = 0.3$ s, then we can affirm that the convergence is fast

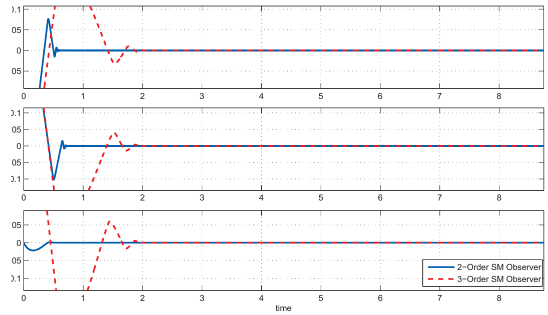
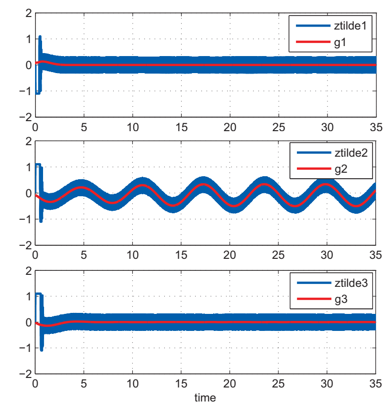
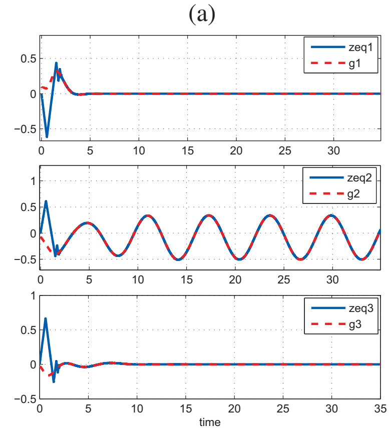


Fig. 3. Comparison of velocities estimation error ($e = \tilde{x}$): 2-order and 3-order sliding mode observers



(a)



(b)

Fig. 4. Comparison of perturbation identification: (a) standard procedure ($ztilde = \tilde{z}_{eq}$ as in (12)); (b) extended order approach ($zeq = z_{eq}$ as in (17))

enough for our application of platform $\boldsymbol{P}$. In Fig. 4 we compare the

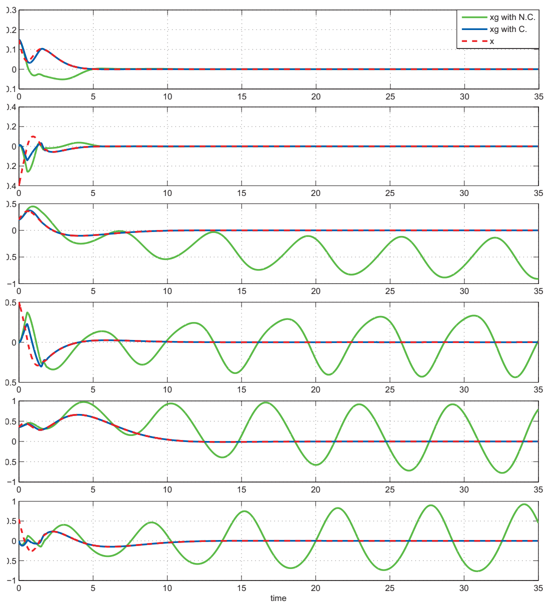


Fig. 5. Comparison of velocities and their estimations ($xg = \hat{x}$) for the extended order approach when compensation (C) control $u = u_0 + u_1$ is used or only the nominal control (NC) is applied

exact reconstruction of the perturbation obtained by the extended order approach (b) and the external perturbation identification using the standard method (a) with a low-pass filter with sampling step $\triangle = 10^{-4}$ and $\tau = \triangle^{1/2}$. In Fig. 4 (b) it is clear that even when the signal presents abrupt changes, the exact method provides a good reconstruction of the perturbation whereas the standard procedure (Fig. 4 (a)), even with the filtration process, the perturbation identification presents the chattering problem. Another element to take into account is the order of the error: for the extended order approach it is of 10^{-8} whereas in the standard procedure it is of 10^{-4}.

The comparison between the reconstructed velocities when a control with perturbation compensation is used, i.e., $u = u_0 + u_1$ ($u_{1i} = -z_{eq_i}$, $i = 1, 2, 3$) and when only the nominal control is applied is shown in Fig. 5 (note that the value 35 of the sample time represents 5 seconds approximately).

VII. Conclusions

Two high-order sliding mode observers providing theoretically finite time state reconstruction and perturbation identification for the platform $\boldsymbol{P}$ were presented. These observers provide two possible methods for external perturbation identification: the standard procedure and the extended order approach. The first one needs a filtration process and the second one provides, after convergence time, theoretical exact perturbation identification. Meanwhile, the second-order observer has smaller convergence time (0.07 seconds) for state reconstruction but the third-order observer has small enough convergence time (0.3 seconds) for our application to platform $\boldsymbol{P}$. For these reasons, the information obtained for the third-order sliding mode observer is the most suitable for reconstruction of velocities and perturbation compensation in the control law of platform $\boldsymbol{P}$.

VIII. Acknowledgments

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