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Piecewise polynomial Lyapunov functions for stability and nonlinear $\mathcal{L}_{2m}$-gain computation of saturated uncertain systems

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Abstract— **In this paper we provide sufficient conditions for regional asymptotic and nonlinear $\mathcal{L}_{2m}$ gain estimation of linear systems subject to saturations and/or deadzones, based on piecewise polynomial Lyapunov functions. By using sum-of-squares relaxations, these conditions are formulated in terms of linear matrix inequalities for the global case, and bilinear ones for the regional case. The results are provided for both the cases with and without structured parametric uncertainties. Example studies are used to comparatively illustrate the proposed techniques.**

I. Introduction

Stability and performance of linear systems subject to saturations and/or deadzones can be addressed using convex optimization tools that have been recently made available by way of the increasingly powerful computational abilities of modern computer systems. In particular, quadratic Lyapunov conditions for assessing exponential stability can be derived using the well known circle criterion and the Linear Matrix Inequalities (LMI) machinery [1]. It is only in recent years that nonquadratic Lyapunov functions have been suggested for the stability and performance analysis in the presence of saturations. In particular, [2] revisited the existing quadratic conditions providing a complete characterization of the nonlinear algebraic loop possibly arising from nonzero feedthrough terms and also proposed two nonquadratic Lyapunov functions. Later on, an alternative constuction was given in [3], where each nonlinearity was regarded as implicitly defining 3 partitions of the state space. The resulting regions implicitly defined via the nonlinear algebraic loop mentioned above, were used to define a piecewise quadratic construction, possibly leading to nonconvex Lyapunov functions. This construction was shown in [3] to lead to improved results as compared to the previous nonquadratic tools of [2].

Polynomial Lyapunov functions have been the subject of intensive investigations during the last decade, motivated by a number of analysis and design problems relevant to control systems (see e.g., [4, 5]). When using this class of Lyapunov functions, the conditions arising from Lyapunov theory can usually be formulated as positivity tests on suitable multivariable polynomials. A convex relaxation of the problem above consists in testing whether a polynomial is a Sum Of Squares (SOS), which can be cast as an LMI feasibility problem [6, 7]. Although polynomial Lyapunov functions have been proposed for many different classes of uncertain and/or nonlinear systems (see [8] and references therein), their potential has not been exploited yet to deal with stability of saturated systems.

In this paper we consider the problem of the estimation of the domain of attraction of saturating systems. By taking into account exogenous inputs we develop conditions allowing to compute the reachable set under a class of disturbances with bounded $\mathcal{L}_{2m}$ norm. Then considering a performance output we are able to compute an estimate of the regional $\mathcal{L}_{2m}$ gain, so that an estimate of the nonlinear $\mathcal{L}_{2m}$ gain curve [9] can be suitably sampled and constructed. The analysis problems allowing to define regional properties of the system are based on modified sector conditions using polynomial multipliers [10] and are converted into optimization problems subject to LMI constraints for the global case and BMI constraints for the regional case. Moreover, we exploit the power arising from the use of polynomial positivity relaxations to characterize regional robust stability properties in the presence of parametric uncertainties. Using the functions first proposed in [3], our result extend the preliminary results in [11] where only global stability results were addressed.

The paper is structured as follows. In Section II we introduce some preliminary standard notions about SOS polynomials. In Section III we state our results for the case without uncertainty. In Section IV we address structured uncertainties. Finally, in Section V we present numerical examples.

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II. Preliminaries

Let us first introduce the notation adopted in the paper. Given a vector $x \in \mathbb{R}^n$, $x^{\{m\}} \in \mathbb{R}^{\sigma_{n,m}}$ denotes a vector containing all monomials $x_1^{i_1} \cdots x_n^{i_n}$ such that $i_1 + \ldots + i_n = m$, where $\sigma_{n,m} = \binom{n+m-1}{n-1}$ is the number of monomials of degree m in n variables. Given two vectors x, y, $x \otimes y$ denotes the Kronecker product of x and y. $\mathbb{P}$ is the set of real polynomials, $\mathbb{P}^{n\times m}$ is the set of $n \times m$ matrices of real polynomials, $\mathbb{P}^{n\times n}_{diag}$ is the set of $n \times n$ polynomial diagonal matrices. For $\Pi(\xi) \in \mathbb{P}^{n\times n}$, $\Pi(\xi) \geq 0$ means that $\Pi(\xi)$ is positive semidefinite for every value taken by the variables ξ in the polynomial entries of $\Pi(\xi)$. $\Sigma^{n\times m}$ denotes the set of $n\times m$ matrices of polynomials whose entries are SOS, while $\Sigma^{n\times n}_{diag}$ is the set of $n \times n$ SOS diagonal matrices. For a generic set $\mathcal{D}$, co($\mathcal{D}$) denotes its closed convex hull. For a polytope $\mathcal{Q}$, Ver[$\mathcal{Q}$] denotes the vertices of $\mathcal{Q}$. Given a vector $x \in \mathbb{R}^n$ $diag(x)$ is a diagonal $n \times n$ matrix whose diagonal entries are the elements of x. Given matrices $M_1 \in \mathbb{R}^{n\times n}$ and $M_2 \in \mathbb{R}^{m\times m}$, blkdiag$(M_1, M_2)$ is a block-diagonal $M \in \mathbb{R}^{nm\times nm}$ matrix. The $\mathcal{L}_{2m}$-norm of a signal $x(t)$ is defined as

$$\|x(t)\|_{2m} = \left(\int_0^\infty \left(x^T(t)x(t)\right)^m dt\right)^{\frac{1}{2m}}$$

the $\mathcal{L}_{2m}$-gain from $z(t)$ to $w(t)$ is denoted γ_{2m} and is given by $\gamma_{2m} = \|z(t)\|_{2m}/\|w(t)\|_{2m}$. The subscript of γ may be omitted whenever m can be inferred from the context.

III. Stability via piecewise polynomial Lyapunov functions

Generally a system with saturations or deadzones can be described in the following compact form:

$$\dot{x} = Ax + B_q q + B_w w \tag{1a}$$
$$y = C_y x + D_{yq} q + D_{yw} w \tag{1b}$$
$$z = C_z x + D_{zq} q + D_{zw} w \tag{1c}$$
$$q = \mathrm{dz}(y) \tag{1d}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^p$, $y \in \mathbb{R}^d$, $w \in \mathbb{R}^r$, and all the matrices are real matrices of appropriate dimensions. The deadzone function $\mathrm{dz}(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is defined as $\mathrm{dz}(y) = y - \mathrm{sat}(y)$, for all $y \in \mathbb{R}^d$, where $\mathrm{sat}(\cdot)$ is a symmetric vector saturation function with saturation levels given by the vector $\overline{u} \in \mathbb{R}^d$, $\overline{u}_i > 0$, $i = 1, \ldots, d$. In particular $\mathrm{sat}_{\overline{u}_i}(u_i) = sign(u_i)\min(|u_i|, |\overline{u}_i|), sat(u) = [\mathrm{sat}_{\overline{u}_1}(u_1) \ \ldots \ \mathrm{sat}_{\overline{u}_d}(u_d)]^T$ For system (1) we assume the following well posedness condition.

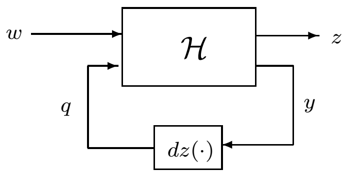

Fig. 1. Representation of a system with deadzone.

Assumption 3.1: The nonlinear algebraic loop in (1) is well posed, namely for any $\zeta \in \mathbb{R}^d$, there exists a unique value y satisfying the nonlinear equation $y - D_{yq}\mathrm{dz}(y) = \zeta$.

Based on the results in [2], Assumption 3.1 is equivalent to verifying that $det(I - D_{yq}\Delta_\phi) > 0$ for all $\Delta_\phi \in \mathcal{D}_\phi$, where

$$\mathcal{D}_\phi := \{\Delta_\phi = diag(k_1, \ldots, k_d), k_i \in \{0, 1\}, i = 1, \ldots, d\} \quad (2)$$

which amounts to checking the determinant of 2^d matrices.

In the rest of this section we will first, in Section III-A, extend to the polynomial case some sector-like conditions given in [3]. Then, in Section III-B, based on these conditions, we will provide sufficient asymptotic stability conditions for (1) based on piecewise polynomial Lyapunov functions and we will extend the tools to establish a finite global $\mathcal{L}_{2m}$ gain from w to z for (1). Finally, in Section III-C we present the tools to establish local $\mathcal{L}_{2m}$ gain from w to z.

A. Sector conditions for the deadzones and algebraic-loop

Let us denote by $u(x)$ the solution y of the nonlinear algebraic loop in (1) when $w = 0$. Since the algebraic loop is well posed by Assumption 3.1, the function $u(x)$ is well defined and corresponds to a piecewise affine function defined on 3^d regions of $\mathbb{R}^d$, satisfying the following equation for all $x \in \mathbb{R}^n$:

$$u(x) - D_{yq}\mathrm{dz}(u(x)) = C_y x. \quad (3)$$

In the facts listed below, we will generalize sector-like conditions given in [3] and highlight their extension to the use of polynomial multipliers. For the sake of generality, we will present the results referring to a generic vector ξ representing the variables of the polynomial multipliers. Moreover, to simplify the exposition, we will use the notation $\theta = \mathrm{dz}(u(x))$ and $q = \mathrm{dz}(y)$.

Fact 3.1: Given any polynomial diagonal matrix $\xi \mapsto \Pi(\xi) \in \mathbb{P}^{d\times d}_{diag}$ such that $\Pi(\xi) \geq 0$ for all ξ, then

$$\mathrm{dz}(\psi)^T\Pi(\xi)(\psi - \mathrm{dz}(\psi)) \geq 0 \ \forall\xi, \ \forall\psi \in \mathbb{R}^d. \quad (4)$$

When focusing on global properties, Fact 3.1 provides the following polynomial constraints, which hold for all ξ and for all x, q, w, θ satisfying (1):

$$\begin{aligned} \Phi_1(\Pi_1(\xi)) &= q^T\Pi_1(\xi)\{C_y x + (D_{yq} - I_d)q + D_{yw}w\} \geq 0, \\ \Phi_2(\Pi_2(\xi)) &= \theta^T\Pi_2(\xi)\{C_y x + (D_{yq} - I_d)\theta\} \geq 0, \end{aligned} \quad (5)$$

where $\Pi_1(\xi), \Pi_2(\xi) \in \mathbb{P}^{d\times d}_{diag}$ satisfy $\Pi_1(\xi) \geq 0$, $\Pi_2(\xi) \geq 0$. Fact 3.1 states that $\mathrm{dz}(\cdot)$ belongs to the sector $[0, I]$ whereas the following fact ([12], [13]) establishes a sector condition that only holds regionally, that is, in the set where a given function $h(x)$ satisfies $h(x) = \mathrm{sat}(h(x))$.

Fact 3.2: Given a function $h(x)$ and any polynomial diagonal matrix function $\xi \mapsto \Pi(\xi) \in \mathbb{P}^{d\times d}_{diag}$ such that $\Pi(\xi) \geq 0$ for all ξ, then

$$\mathrm{dz}(\psi)^T\Pi(\xi)(\psi - \mathrm{dz}(\psi) - h(x)) \geq 0,$$

$\forall\xi$, $\forall\psi \in \mathbb{R}^d$, $\forall x \in \mathbb{R}^n$ such that $h(x) = \mathrm{sat}(h(x))$.

When describing the deadzone, extra information can be drawn from the time derivatives of y and $\mathrm{dz}(y)$, whenever they exist. This information can be obtained by observing that, by denoting $\dot{u} = du/dt$ and $\phi(x, w) = d(\mathrm{dz}(u))/dt = d\theta/dt$,

$$\phi_i(x, w) = \begin{cases} 0, & \text{if } |u_i| < \bar{u}_i \\ \dot{u}_i, & \text{if } |u_i| > \bar{u}_i \end{cases} \quad (6)$$

Note that $\phi_i(x, w)$ may not exist where $u_i = \pm\bar{u}_i$. Condition (6) can consequently be described in terms of polynomial constraints as explained in the next fact.

Fact 3.3: Given any polynomial diagonal matrix $\xi \mapsto \Pi(\xi) \in \mathbb{P}^{d\times d}_{diag}$, the following equalities hold almost everywhere

$$\begin{cases} \phi(x, w)^T\Pi(\xi)\{\dot{u} - \phi(x, w)\} \equiv 0 \\ \theta^T\Pi(\xi)\{\dot{u} - \phi(x, w)\} \equiv 0 \end{cases} . \quad (7)$$

By the definition of u in (3) and by (1), we have $\dot{u} = C_y Ax + C_y B_q\mathrm{dz}(y) + C_y B_w w + D_{yq}\phi(x, w)$. Then conditions (7) in Fact 3.3 impose that, for all ξ and for all x, q, $\phi = dq/dt$, solutions to (1), one has

$$\begin{aligned} \Phi_3(\Pi_3(\xi)) = \phi^T\Pi_3(\xi)\{&C_y Ax + C_y B_q q + C_y B_w w \\ &+ (D_{yq} - I_d)\phi\} \equiv 0 \\ \Phi_4(\Pi_4(\xi)) = \theta^T\Pi_4(\xi)\{&C_y Ax + C_y B_q q + C_y B_w w \\ &+ (D_{yq} - I_d)\phi\} \equiv 0 \end{aligned} \quad (8)$$

where $\Pi_3(\xi), \Pi_4(\xi) \in \mathbb{P}^{d\times d}_{diag}$.

According to the non decreasing property of saturations and deadzones, the following fact can also be proven.

Fact 3.4: Given a vector ξ and any polynomial diagonal matrix $\xi \mapsto \Pi(\xi) \in \mathbb{P}^{d\times d}_{diag}$ such that $\Pi(\xi) \geq 0$ for all ξ, for all $\psi_1, \psi_2 \in \mathbb{R}^d$:

$$\{\mathrm{dz}(\psi_1) - \mathrm{dz}(\psi_2)\}^T\Pi(\xi)\{\mathrm{sat}(\psi_1) - \mathrm{sat}(\psi_2)\} \geq 0. \quad (9)$$

By means of Fact 3.4, the following polynomial constraint holds for all ξ, q, θ and w:

$$\begin{aligned} \Phi_5(\Pi_5(\xi)) := \{\theta - q\}^T\Pi_5(\xi)\{&(D_{yq} - I_d)\theta + (I_d - D_{yq})q \\ &- D_{yw}w\} \geq 0, \end{aligned} \quad (10)$$

where $\Pi_5(\xi) \in \mathbb{P}^{d\times d}_{diag}$ satisfies $\Pi_5(\xi) \geq 0$ for all ξ.

B. Global stability and $\mathcal{L}_{2m}$ gain analysis

In the previous section, polynomial constraints have been introduced to describe the nonlinearities and their dynamics. Thanks to their polynomial nature, these constraints can be exploited to obtain sufficient condition for the stability of system (1) by way of a piecewise polynomial Lyapunov function.

Theorem 3.1: Consider system (1) satisfying Assumption 3.1 and the set of polynomial inequalities

$$\Pi_i(\xi) \geq 0, i = 1, 2, 5 \quad (11a)$$

$$V(x, \theta) - \epsilon|x|^{k_1} \geq 0 \quad (11b)$$

$$-\dot{V}(x, \theta, q, \phi, w) - \textstyle\sum_{i=1}^{5}\Phi_i(\Pi_i(\xi)) - \Psi - \epsilon|x|^{k_2} \geq 0 \quad (11c)$$

where $\Phi_i(\cdot)$, $i = 1, \ldots, 5$, are given in (5), (8), (10), and $\dot{V}(x, \theta, q, \phi, w)$ is a shortcut notation for

$$\left\langle \nabla V(x, \theta), \begin{bmatrix} Ax + B_q q + B_w w \\ \phi \end{bmatrix} \right\rangle.$$

If there exist a polynomial function $V(x,\theta) \in \mathbb{P}$, polynomial matrices $\Pi_i(\xi) \in \mathbb{P}^{d\times d}_{diag}, i = 1,\dots,5$, two reals $k_1, k_2 \geq 1$, a scalar $\epsilon > 0$ and a positive integer $m \in \mathbb{N}$ such that (11) is satisfied, respectively, with

1) $w = 0$, $\theta = q$ (yielding $\Phi_1(\cdot) = \Phi_2(\cdot)$, $\Phi_5(\Pi_5(\xi)) \equiv 0$) and $\Psi \equiv 0$;
2) $\Psi \equiv -(w^T w)^m$;
3) $\Psi \equiv \gamma^{-2m}(z^T z)^m - (w^T w)^m$ with $\gamma \in \mathbb{R},\ \gamma > 0$,

then the following, respectively, holds:

1) (asymptotic stability) the origin of system (1) is globally asymptotically stable;
2) (reachable set) $x(0) = 0$ and $\|w\|_{2m} \leq \rho$ imply $x(t) \in \{x:\ V(x, \mathrm{dz}(u(x))) \leq \rho^{2m}\}$ where $u(x)$ is the unique solution to (3);
3) (global $\mathcal{L}_{2m}$ gain) $x(0) = 0$ implies $\|z\|_{2m} \leq \gamma\|w\|_{2m}$ i.e. the global finite $\mathcal{L}_{2m}$-gain of (1) from w to z is bounded by γ.

The proofs are omitted due to space constraints.

The family of candidate Lyapunov functions that we introduce to perform the computation of the $\mathcal{L}_{2m}$ gain has the form

$$V(x,\theta) = \left(\begin{bmatrix} x \\ \theta \end{bmatrix}^{\{m\}}\right)^T P \left(\begin{bmatrix} x \\ \theta \end{bmatrix}^{\{m\}}\right), \qquad (12)$$

that is, a form of degree $2m$ in the variables x and θ. This class of piecewise polynomial functions is a generalization of the piecewise quadratic Lyapunov functions proposed in [3].

With this polynomial version, $\dot{V}(x,\theta)$ becomes a form of degree $2m$ in the variables x, θ, q, w, ϕ, whose monomials are the elements of vector $\left(\begin{smallmatrix} x \\ \theta \end{smallmatrix}\right)^{\{2m-1\}} \otimes \left(\begin{smallmatrix} x \\ q \\ w \\ \phi \end{smallmatrix}\right)$. The polynomial constraint (11c) can then be described as a form of degree $2m$ if the multipliers $\Pi_i(\xi)$ are chosen to match the degree of $\dot{V}(x,\theta)$. In particular, if the elements of the multipliers $\Pi_i(\xi)$ $i = 1,\dots,5$ are homogeneous polynomials of degree $2(m-1)$ in the variables x, θ, q, w, ϕ, then the polynomials $\Phi_i(\cdot)$, $i = 1,\dots,5$ become forms of degree $2m$. A possible choice for the multipliers $\Pi_i(\xi)$ $i = 1,\dots,5$ is then given by:

$$\begin{array}{rcl} \Pi_i(\xi) & = & \mathrm{diag}(\pi_{i,1}(\xi),\dots,\pi_{i,d}(\xi)), \\ \xi & = & \begin{bmatrix} x^T & \theta^T & q^T & w^T & \phi^T \end{bmatrix}^T \end{array} \qquad (13)$$

where

$$\pi_{i,j}(\xi) = q_{i,j}\xi^{\{2m-2\}},\ j = 1,\dots,d \qquad (14)$$

with $q_{i,j}$ real vectors of compatible dimensions describing the coefficients of the polynomial $\pi_{i,j}(\xi)$.

Finally, with the selection (12)-(14) and with $k_1 = k_2 = 2m$ the constraints (11b) and (11c) become forms of degree $2m$.

The following proposition arises from a sum-of-squares relaxation of the constraints in Theorem 3.1:

Proposition 3.1: Consider $V(x,\theta)$ given by (12) and $\Pi_i(\xi)$, $i = 1,\dots 5$, as in (13)-(14), and the relaxation of constraints (11) given by:

$$\Pi_i(\xi) \in \Sigma^{d\times d}_{diag}, i = 1,2,5, \qquad (15a)$$

$$V(x,\theta) - \epsilon|x|^{2m} \in \Sigma\ , \qquad (15b)$$

$$-\dot{V}(x,\theta) - \textstyle\sum_{i=1}^{5} \Phi_i(\Pi_i(\xi)) - \Psi - \epsilon|x|^{2m} \in \Sigma\ , \qquad (15c)$$

which are SOS constraints in the unknown variables P, vectors $q_{i,j}$, $i = 1,\dots,5$, $j = 1,\dots,d$, and $\epsilon > 0$. If (15) holds with

1) $w = 0$, $q = \theta$ and $\Psi \equiv 0$, then the origin of system (1) is globally asymptotically stable.
2) $\Psi \equiv -(w^T w)^m$, then if $x(0) = 0$, $\|w\|_{2m} \leq \rho$ we have $x \in \{x;\ \ V(x, \mathrm{dz}(u(x))) \leq \rho^{2m}\}$.
3) $\Psi = \gamma^{-2m}(z^T z)^m - (w^T w)^m$, then if $x(0) = 0$ then $\|z\|_{2m} \leq \gamma\|w\|_{2m}$ i.e. the global $\mathcal{L}_{2m}$-gain of (1) is bounded by γ.

Note that, (15a) is a homogeneous SOS constraint of degree $2m-2$, while (15b)-(15c) are homogeneous SOS constraints of degree $2m$, both in the variable ξ (see (14)).

C. Regional Stability and $\mathcal{L}_{2m}$ gain analysis

In this section, paralleling the results in [3], the global results of Section III-B are generalized to regional results, with the goal of estimating the region of attraction (we refer to this as "regional stability") and the nonlinear $\mathcal{L}_{2m}$ gain in terms of the $\mathcal{L}_{2m}$ norm of the input. When focusing on regional properties, Fact 3.2 provides the following polynomial constraints

$$\begin{aligned} \Phi_{R1}(\Pi_1(\xi)) &= q^T\Pi_1(\xi)\{C_y x + (D_{yq} - I_d)q + D_{yw}w \\ &\qquad -h_1(x)\} \geq 0 \\ \Phi_{R2}(\Pi_2(\xi)) &= \theta^T\Pi_2(\xi)\{C_y x + (D_{yq} - I_d)\theta - h_2(x)\} \geq 0, \end{aligned} \qquad (16)$$

where $\Pi_1(\xi), \Pi_2(\xi) \in \mathbb{P}^{d\times d}_{diag}$ satisfy $\Pi_1(\xi) \geq 0$, $\Pi_2(\xi) \geq 0$.

The above inequalities then hold $\forall\xi$, $\forall q,\theta \in \mathbb{R}^d$, $\forall w \in \mathbb{R}^r$ and $\forall x \in \mathbb{R}^n$ in the set where $\mathrm{sat}(h_j(x)) = h_j(x)$, $j = 1,2$ respectively, namely the sets

$$\mathcal{L}(h_j(x)) = \left\{x \in \mathbb{R}^n : |\bar{U}^{-1}h_j(x)|_\infty \leq 1\right\}, j = 1,2 \qquad (17)$$

where $\bar{U} = diag(\bar{u}_1,\dots,\bar{u}_d)$.

Moreover, denoting by $\eta_i(x)$ the i-th element of a polynomial vector function $\eta(x)$ establishes sufficient conditions for a sublevel set

$$\mathcal{E}(W(x),\rho^{2m}) = \left\{x : W(x) \leq \rho^{2m}\right\}$$

of a function $W(x)$ to be contained in $\mathcal{L}(\eta(x))$:

Lemma 1: Given $s \in \mathbb{R}$, $s > 0$, $m \in \mathbb{N}$, a polynomial function $\eta(x) : \mathbb{R}^n \mapsto \mathbb{R}^d$, and a positive definite function $W(x)$, if there exist a polynomial $p(x) \geq 0$ such that

$$\rho^{2m} - \rho^{2m}p(x) - 2\nu\eta_i(x) + \left(\frac{\bar{u}_i}{\rho^m}\right)^2\nu^2 + p(x)W(x) \geq 0 \qquad i = 1,\dots,d \qquad (18)$$

then $\mathcal{E}(W(x),\rho^{2m}) \subseteq \mathcal{L}(\eta(x))$.

Lemma 1 allows us to consider constraints (16) in the estimate of the regional $\mathcal{L}_{2m}$-gain of system (1) as established in the following theorem:

Theorem 3.2: Consider system (1) satisfying Assumption 3.1 and the set of polynomial inequalities

$$\Pi_i(\xi) \geq 0, i = 1,2,5 \qquad (19a)$$

$$V(x,\theta) - \epsilon|x|^{k_1} \geq 0 \qquad (19b)$$

$$-\dot{V}(x,\theta,q,\phi,w) - \textstyle\sum_{i=1}^{5} \Phi_{Ri}(\Pi_i(\xi)) - \Psi - \epsilon|x|^{k_2} \geq 0 \qquad (19c)$$

$$\rho^{2m} - \rho^{2m}p_j(x) - 2\nu h_{ji}(x) + \frac{\bar{u}_i^2}{\rho^{2m}}\nu^2 + p_j(x)V(x,\theta) \geq 0 \qquad i = 1,\dots,d\ j = 1,2, \qquad (19d)$$

where $h_{ji}(x)$, $i = 1,\dots,d$, denotes the i-th entry of the function $h_j(x)$, $j = 1,2$ introduced in (16), $\Phi_{R1}(\cdot)$ and $\Phi_{R2}(\cdot)$ are given in (16) and $\Phi_{Ri}(\cdot) = \Phi_i(\cdot)$ for $i = 3,4,5$, are given in (8), (10). If there exist a polynomial function $V(x,\theta) \in \mathbb{P}$ with $m \in \mathbb{N}$, polynomial matrices $\Pi_i(\xi) \in \mathbb{P}^{d\times d}_{diag}, i = 1,\dots,5$, two vector of polynomials $h_j(x)$ $j = 1,2$, two polynomial functions $p_j(x) \geq 0$, $j = 1,2$, two reals $k_1, k_2 \geq 1$ and a scalar $\epsilon > 0$ such that (19) is satisfied with

1) $w = 0$, $\theta = q$ (yielding $\Phi_{R1}(\cdot) = \Phi_{R2}(\cdot)$, $h_1(\cdot) = h_2(\cdot)$, $\Phi_5(\Pi_5(\xi)) \equiv 0$) and $\Psi \equiv 0$;

2) $\Psi \equiv -(w^T w)^m$;
3) $\Psi \equiv \gamma^{-2m}(z^T z)^m - (w^T w)^m$ with $\gamma \in \mathbb{R},\ \gamma > 0$,

then the following, respectively, holds:

1) (regional asymptotic stability) the origin of system (1) is locally asymptotically stable. Moreover, denoting by $u(x)$ the unique solution to (3) if $x(0) \in \mathcal{E}(V(x, \mathrm{dz}(u(x))), \rho^{2m})$ and $w = 0$, then $x(t) \in \mathcal{E}(V(x, \mathrm{dz}(u(x))), \rho^{2m})$ and $\lim_{t\to\infty} x(t) = 0$;
2) (reachable set) $x(0) = 0$ and $\|w\|_{2m} \le \rho$ imply $x(t) \in \{x : \ V(x, \mathrm{dz}(u(x))) \le \rho^{2m}\}$ for all $t \ge 0$;
3) (regional $\mathcal{L}_{2m}$ gain) $x(0) = 0$ and $\|w\|_{2m} \le \rho$ imply $\|z\|_{2m} \le \gamma\|w\|_{2m}$ i.e. the regional finite $\mathcal{L}_{2m}$-gain of system (1) from w to z is bounded by γ.

In order to consider sum-of-squares relaxation for the positivity of polynomials in (19) we fix the degrees of the multipliers as in (13)-(14) and make the particular choice for $h_j(x)$ and $p_j(x)$, $j = 1, 2$ given by $h_j(x) = H_j x$ and $p_j(x) = p_{0j}$, $p_{0j} \in \mathbb{R}^+$. We obtain the following proposition:

Proposition 3.2: Consider $V(x,\theta)$ given by (12) and $\Pi_i(\xi)$, $i = 1, \dots 5$, as in (13)-(14), and constraints

$$\begin{array}{l} \Pi_i(\xi) \in \Sigma^{d\times d}_{diag}, i = 1, 2, 5 \\ V(x,\theta) - \epsilon|x|^{2m} \in \Sigma \\ -\dot{V}(x,\theta,q,\phi,w) - \sum_{i=1}^{5} \Phi_{Ri}(\Pi_i(\xi)) - \Psi - \epsilon|x|^{2m} \in \Sigma \\ \rho^{2m} - \rho^{2m} p_{0j} - 2\nu H_{ji} x + \dfrac{\bar{u}_i^2}{\rho^{2m}}\nu^2 + p_{0j} V(x,\theta) \in \Sigma \\ \qquad\qquad\qquad i = 1, \dots, d \ j = 1, 2. \end{array} \tag{20}$$

where H_{ji}, $i = 1, \dots, p$ denotes the i-th row of matrix H_j, $j = 1, 2$. If there exist matrix P, vectors $q_{i,j}$, $i = 1, \dots, 5$, $j = 1, \dots, d$, and a scalar $\epsilon > 0$ such that (20) holds

1) with $w = 0$, $q = \theta$ and $\Psi \equiv 0$, then the origin of system (1) is locally asymptotically stable. If $x(0) \in \mathcal{E}(W(x), \rho^{2m})$ then $x(t) \in \mathcal{E}(W(x), \rho^{2m})$ and $\lim_{t\to\infty} x(t) = 0$.
2) with $\Psi \equiv -(w^T w)^m$, then if $x(0) = 0$, $\|w\|_{2m} \le \rho$ we have $x \in \{x; \ V(x, \mathrm{dz}(u(x))) \le \rho^{2m}\}$.
3) with $\Psi = \gamma^{-2m}(z^T z)^m - (w^T w)^m$, then if $x(0) = 0$ and $\|w\|_{2m} \le \rho$ then $\|z\|_{2m} \le \gamma\|w\|_{2m}$ i.e. the regional $\mathcal{L}_{2m}$-gain of (1) is bounded by γ.

The first three constraints of (20) are forms of degree $2m$ in variables x, θ, q, ϕ and w. The last constraint is a polynomial of degree $2m$ (containing also quadratic and constant terms) in variables x, θ and ν.

IV. Extension to systems with parametric uncertainty

In this section the results of Section III are extended to a class of uncertain systems. A system with deadzones and parametric uncertainties can be described by the following compact form:

$$\dot{x} = Ax + B_q q + B_w w + B_v v \tag{21a}$$
$$y = C_y x + D_{yq} q + D_{yw} w + D_{yv} v \tag{21b}$$
$$z = C_z x + D_{zq} q + D_{zw} w + D_{zv} v \tag{21c}$$
$$p = C_p x + D_{pq} q + D_{pw} w + D_{pv} v \tag{21d}$$
$$q = \mathrm{dz}(y) \tag{21e}$$
$$v = \Delta(\delta) p \tag{21f}$$

where the uncertainty matrix $\Delta(\delta)$ corresponds to

$$\Delta(\delta) = \mathrm{diag}(\delta_1 I_{s_1}, \dots, \delta_{n_\delta} I_{s_{n_\delta}}), \tag{22}$$

and where $\delta_i \in \mathbb{R}^{s_i}$ denotes the i-th component of the uncertainty vector δ which is assumed to be unknown but constant. Moreover, n_δ is the number of uncertain parameters, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^d$, $p \in \mathbb{R}^l$, with $l = \sum_{i=1}^{n_\delta} s_i$ and all the matrices are real matrices of appropriate dimensions. Let us define the

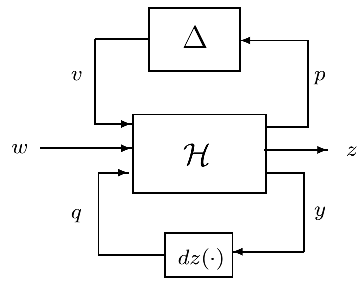

Fig. 2. Representation of a system with deadzones and uncertainties.

uncertainty operator domain as $\mathbf{\Delta} = \{\Delta(\delta) : \delta \in \mathcal{Q}\}$ where $\mathcal{Q}$ defines a polyhedral set in $\mathbb{R}^{n_\delta}$:

$$\mathcal{Q} = \left\{\delta : \underline{\delta}_{ek} \le c_k \delta \le \overline{\delta}_{ek}, \text{ for } k = 1, \dots, n_e\right\}.$$

To generalize Assumption 3.1 to the uncertain case addressed here, it is necessary to assume that the nonlinear algebraic loop introduced by the uncertainty is well defined (namely, it admits a unique explicit solution) for all values of $\delta \in \mathcal{Q}$. This is formalized in the next assumption.

Assumption 4.1: For each value of $\delta \in \mathcal{Q}$, the following holds:

- the matrix $I - D_{pv}\Delta(\delta)$ is nonsingular;
- the nonlinear algebraic loop defined by the implicit equation

$$y - \left(D_{yq} + D_{yv}\Delta(\delta)(I - D_{pv}\Delta(\delta))^{-1} D_{pq}\right) \mathrm{dz}(y) = \zeta,$$

is well posed. Namely, for any $\zeta \in \mathbb{R}^d$, there exists a unique value y satisfying the corresponding nonlinear equation.

A. Conditions for deadzones and uncertainties

In Section III-A, conditions have been derived to describe the nonlinearities in the feedback loop and their dynamics. Here the aim is to devise polynomial constraints which enable us to describe the uncertainty loop. Because of the presence of uncertainty, the conditions describing the deadzones and corresponding to equations (16), (8), and (10) generalize respectively to the following constraints which hold for all ξ and all $x,\ q,\ v,\ \phi = dq/dt$, solutions to (21),

$$\begin{array}{rcl} \Omega_1(\Pi_1(\xi)) &=& q^T \Pi_1(\xi)\{C_y x + (D_{yq} - I_d) q + D_{yw} w \\ && +D_{yv} v - h_1(x)\} \ge 0; \\ \Omega_2(\Pi_2(\xi)) &=& \theta^T \Pi_2(\xi)\{C_y x + (D_{yq} - I_d)\theta \\ && +D_{yv} v - h_2(x)\} \ge 0; \\ \Omega_3(\Pi_3(\xi)) &=& \phi^T \Pi_3(\xi)\{C_y(Ax + B_q q + B_w w + B_v v) \\ && +(D_{yq} - I_d)\phi\} \equiv 0; \\ \Omega_4(\Pi_4(\xi)) &=& \theta^T \Pi_4(\xi)\{C_y(Ax + B_q q + B_w w + B_v v) \\ && +(D_{yq} - I_d)\phi\} \equiv 0; \\ \Omega_5(\Pi_5(\xi)) &=& \{\theta - q\}^T \Pi_5(\xi)\{(D_{yq} - I_d)\theta \\ && +(I_d - D_{yq}) q - D_{yw} w\} \ge 0 \end{array} \tag{23}$$

where $\Pi_i(\xi) \in \mathbb{P}^{d\times d}_{diag}$ $i = 1, \dots, 5$ and $\Pi_i(\xi) \ge 0$, $i = 1, 2, 5$. Additional conditions pertaining the uncertain parameter δ can also be exploited. Consider the algebraic loop imposed by (21d) and (21f). Premultiplying (21d) by $\Delta(\delta)$ and using (21f), we get

$$0 = \Delta(\delta)(C_p x + D_{pq} q + D_{pw} w) + (\Delta(\delta) D_{pv} - I_l) v, \tag{24}$$

which is a collection of polynomial equality constraints yielding the following constraint

$$\begin{array}{r} \Omega_6(\Pi_6(\xi)) = \Pi_6(\xi)\{\Delta(\delta)(C_p x + D_{pq} q + D_{pw} w) \\ +(\Delta(\delta) D_{pv} - I_l) v\} \equiv 0 \end{array} \tag{25}$$

which, given any $\Pi_6(\xi) \in \mathbb{P}^{1\times l}$, is satisfied for all ξ, for all $\delta \in \mathcal{Q}$ and for all x, q, v, w solution of (21). For the allowable domain $\mathcal{Q}$ for δ, we have that

$$\Omega_7(\Pi_7(\xi)) = -\left(C_e\delta - \underline{\delta}_e\right)^T \Pi_7(\xi)\left(C_e\delta - \overline{\delta}_e\right) \geq 0, \qquad (26)$$

with, $C_e = \left[c_1^T \dots c_{n_e}^T\right]^T$, $\underline{\delta}_e = \left[\underline{\delta}_{e1} \dots \underline{\delta}_{en_e}\right]^T$, $\overline{\delta}_e = \left[\overline{\delta}_{e1} \dots \overline{\delta}_{en_e}\right]^T$ and $\Pi_7(\xi) \in \mathbb{P}^{n_e\times n_e}_{diag}$ satisfying $\Pi_7(\xi) \geq 0$ for all ξ.

B. Robust Stability and $\mathcal{L}_{2m}$ Gain Analysis

Robust stability with respect to uncertainties can be assessed by using a candidate Lyapunov function independent of the uncertain parameter δ. This kind of analysis however results to be conservative whenever the uncertainty is either constant or slowly time-varying. In the case under consideration, the uncertainty is constant, therefore it is useful to choose a candidate Lyapunov function $V(x,\theta,\delta)$ depending on the uncertain parameter δ and on $\theta = \mathrm{dz}(u(x,\delta))$, where $u(x,\delta)$ generalizes the solution of (3) to the uncertain case, and corresponds to the unique solution to the implicit equation

$$u(x,\delta) - M_1(\delta)\mathrm{dz}(u(x,\delta)) = M_2(\delta)x, \qquad (27)$$

with $M_1(\delta) = \left(D_{yq} + D_{yv}\Delta(\delta)(I - D_{pv}\Delta(\delta))^{-1}D_{pq}\right)$ and $M_2(\delta) = (D_{yv}\Delta(\delta)(I - D_{pv}\Delta(\delta))^{-1}C_p + C_y)$, which is always well defined under Assumption 4.1. With this Lyapunov function, we can formulate the next theorem stating sufficient conditions for robust stability of (21).

Theorem 4.1: Consider system (21) satisfying Assumption 4.1. If there exist a polynomial function $V(x,\theta,\delta) \in \mathbb{P}$, polynomial matrices $\Pi_i(\xi) \in \mathbb{P}^{d\times d}_{diag}, i = 1,\dots,5$, a vector $\Pi_6(\xi) \in \mathbb{P}^{1\times l}$, a matrix $\Pi_7(\xi) \in \mathbb{P}^{n_\delta\times n_\delta}_{diag}$, polynomials $h_j(x)$, $p_j(x)$, $j = 1,2$, two reals $k_1, k_2 \geq 1$ and a scalar $\epsilon > 0$ such that

$$\Pi_i(\xi) \geq 0,\ i = 1,2,5,7 \qquad (28a)$$

$$V(x,\theta,\delta) - \epsilon|x|^{k_1} \geq 0 \qquad (28b)$$

$$-\dot{V}(x,\theta,q,\phi,w,\delta,v) - \textstyle\sum_{i=1}^{7}\Omega_i(\Pi_i(\xi)) - \Psi - \epsilon|x|^{k_2} \geq 0 \qquad (28c)$$

$$\rho^{2m} - \rho^{2m}p_j(x) - 2\nu h_{ji}(x) + \frac{\bar{u}_i^2}{\rho^{2m}}\nu^2 + p_j(x)V(x,\theta,\delta) \geq 0 \quad i = 1,\dots,d\ j = 1,2, \qquad (28d)$$

is satisfied with

1) $w = 0$, $\theta = q$ (yielding $\Omega_1(\cdot) = \Omega_2(\cdot)$, $h_1(\cdot) = h_2(\cdot)$ and $\Omega_5(\Pi_5(\xi)) \equiv 0$) and $\Psi \equiv 0$;
2) $\Psi \equiv -(w^Tw)^m$;
3) $\Psi \equiv \gamma^{-2m}(z^Tz)^m - (w^Tw)^m$ with $\gamma \in \mathbb{R}$, $\gamma > 0$.

then the following, respectively, holds:

1) (robust regional asymptotic stability) the origin of system (21) is locally asymptotically stable. If $x(0) \in \mathcal{E}(V(x,\mathrm{dz}(u(x,\delta)),\delta),\rho^{2m})$ then $x(t) \in \mathcal{E}(V(x,\mathrm{dz}(u(x,\delta)),\delta),\rho^{2m})$ and $\lim_{t\to\infty} x(t) = 0$;
2) (robust reachable set) $x(0) = 0$ and $\|w\|_{2m} \leq \rho$ imply $x(t) \in \{x:\ V(x,\mathrm{dz}(u(x,\delta),\delta) \leq \rho^{2m}\}$ where $u(x,\delta)$ is the unique solution to (27);
3) (robust regional $\mathcal{L}_{2m}$ gain) $x(0) = 0$ and $\|w\|_{2m} \leq \rho$ imply $\|z\|_{2m} \leq \gamma\|w\|_{2m}$ i.e. the regional finite $\mathcal{L}_{2m}$ gain of system (21) from w to z is bounded by γ.

In order to check for global properties of system (21) we may consider $h_1(\cdot) = h_2(\cdot) = 0$ and enforce constraints (28a)-(28c).

A possible choice for the parameter-dependent Lyapunov function $V(x,\theta,\delta)$ is the extension of (12) to the uncertain case as

$$V(x,\theta,\delta) = \left(\begin{bmatrix}1\\ \delta\end{bmatrix}^{\{s\}} \otimes \begin{bmatrix}x\\ \theta\end{bmatrix}^{\{m\}}\right)^T P \left(\begin{bmatrix}1\\ \delta\end{bmatrix}^{\{s\}} \otimes \begin{bmatrix}x\\ \theta\end{bmatrix}^{\{m\}}\right), \qquad (29)$$

representing a function which is homogeneous of degree $2m$ in the variables x and q and polynomial of degree $2s$ with respect to the uncertain parameter δ. The choice in (29) suggests a suitable structure for the multipliers $\Pi_i(\xi)$, $i = 1,\dots,7$ by noticing that the monomials generating Ω_i, $i = 1,\dots,7$, should come from the same base as the one describing the gradient of $V(x,\theta,\delta)$ times the vector field given by (1), $\dot{\theta} = \phi$ and $\dot{\delta} = 0$ which we briefly denote by $\dot{V}(x,\theta,q,\phi,w,\delta,v)$. For this reason, ξ can be chosen as $\xi^T = \begin{bmatrix} x^T & \theta^T & q^T & \phi^T & w^T & \delta^T & v^T \end{bmatrix}$ and $\Pi_i(\xi)$, for $i = 1,\dots,5$, can be chosen to be

$$\Pi_i(\xi) = \mathrm{diag}(\pi_{i,1}(\xi),\dots,\pi_{i,d}(\xi)), \qquad (30)$$

with

$$\pi_{i,j}(\xi) = \varsigma(\xi)^T Q_{i,j}\varsigma(\xi),\ \varsigma(\xi) = \begin{bmatrix}1\\ \delta\end{bmatrix}^{\{s\}} \otimes \vartheta^{\{m-1\}} \qquad (31)$$

with $\vartheta = \begin{bmatrix} x' & \theta' & q' & \phi' & w' & v' \end{bmatrix}'$ and where $Q_{i,j}$ is a real symmetric matrix of compatible dimensions describing the coefficients of the monomials of $\pi_{i,j}(\xi)$. The vector of polynomials $\Pi_6(\xi)$ can be chosen as

$$\Pi_6(\xi) = [\pi_{6,1}(\xi),\dots,\pi_{6,l}(\xi)], \qquad (32)$$

with

$$\pi_{6,j}(\xi) = K_j\left(\begin{bmatrix}1\\ \delta\end{bmatrix}^{\{2s-1\}} \otimes \vartheta^{\{m-1\}}\right), \qquad (33)$$

where K_j, $j = 1,\dots,l$ is a real vector of compatible dimensions describing the coefficients of the monomials of $\pi_{6,j}(\xi)$. Finally, $\Pi_7(\xi)$ can be chosen as

$$\Pi_7(\xi) = \mathrm{diag}(\pi_{7,1}(\xi),\dots,\pi_{7,n_\delta}(\xi)), \qquad (34)$$

with

$$\pi_{7,j}(\xi) = \varsigma(\xi)^T Q_{7,j}\varsigma(\xi), \varsigma(\xi) = \begin{bmatrix}1\\ \delta\end{bmatrix}^{\{s-1\}} \otimes \vartheta^{\{m\}} \qquad (35)$$

where $Q_{7,j}$ is a real symmetric matrix of compatible dimensions describing the coefficients of the monomials of $\pi_{7,j}(\xi)$.

Now, by proceeding like in Section III-B, the positivity constraints (28) in Theorem 4.1 are relaxed to sum-of-squares constraints.

Proposition 4.1: Consider $V(x,\theta,\delta)$ given by (29) $\Pi_i(\xi)$, $i = 1,\dots,7$ as in (30)-(35)

$$\Pi_i(\xi) \in \Sigma^{d\times d}_{diag},\quad i = 1,2,5;\ \Pi_7(\xi) \in \Sigma^{n_\delta\times n_\delta}_{diag} \qquad (36a)$$

$$V(x,\theta,\delta) - \epsilon|x|^{k_1} \geq 0 \qquad (36b)$$

$$-\dot{V}(x,\theta,q,\phi,w,\delta,v) - \textstyle\sum_{i=1}^{7}\Omega_i(\Pi_i(\xi)) - \Psi - \epsilon|x|^{k_2} \in \Sigma \qquad (36c)$$

$$\rho^{2m} - \rho^{2m}p_{0j} - 2\nu H_{ji}x + \frac{\bar{u}_i^2}{\rho^{2m}}\nu^2 + p_{0j}V(x,\theta) \in \Sigma. \qquad (36d)$$

$i = 1,\dots,d$ $j = 1,2$, where H_{ji}, $i = 1,\dots,p$ denotes the i-th row of matrix H_j, $j = 1,2$. If there exist matrix P, matrices $Q_{i,j}$, $i = 1,\dots,5$, $j = 1,\dots,d$; K_j, $j = 1,\dots,l$; $Q_{7,j}$, $j = 1,\dots,n_\delta$, and a scalar $\epsilon > 0$ such that (36) holds

1) with $w = 0$, $\theta = q$ (yielding $\Omega_1(\cdot) = \Omega_2(\cdot)$, $h_1(\cdot) = h_2(\cdot)$ and $\Omega_5(\Pi_5(\xi)) \equiv 0$) and $\Psi \equiv 0$, then the

origin of system (21) is locally asymptotically stable. If $x(0) \in \mathcal{E}(V(x, \mathrm{dz}(u(x,\delta)), \delta), \rho^{2m})$ then $x(t) \in \mathcal{E}(V(x, \mathrm{dz}(u(x,\delta)), \delta), \rho^{2m})$ and $\lim_{t\to\infty} x(t) = 0$;
2) with $\Psi \equiv -(w^T w)^m$ then $x(0) = 0$ and $\|w\|_{2m} \leq \rho$ imply $x(t) \in \{x : V(x, \mathrm{dz}(u(x,\delta), \delta) \leq \rho^{2m}\}$ where $u(x,\delta)$ is the unique solution to (27);
3) with $\Psi \equiv \gamma^{-2m}(z^T z)^m - (w^T w)^m$ with $\gamma \in \mathbb{R}$, $\gamma > 0$, then $x(0) = 0$ and $\|w\|_{2m} \leq \rho$ imply $\|z\|_{2m} \leq \gamma \|w\|_{2m}$ i.e. the regional finite $\mathcal{L}_{2m}$ gain of system (21) from w to z is bounded by γ.

V. Numerical examples

Example 5.1: Consider system (1) with the following matrices:

$$\begin{bmatrix} A & B_q & B_w \\ C_y & D_{yq} & D_{yw} \\ C_z & D_{zq} & D_{zw} \end{bmatrix} = \left[\begin{array}{ccc|cc|c} 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 1 \\ 0 & 1 & -3.96 & 1 & -1 & 1 \\ \hline 1 & 0 & 1 & -3 & -1 & 1 \\ 0 & 1 & 0 & -2 & -4 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & -0.1 \end{array}\right]$$

The system is well-posed and open-loop stable. For this system it is not possible to find a piecewise quadratic Lyapunov function, that is, a function of the form (12) with $m = 1$, satisfying inequalities (15) with $\Psi = \gamma^{-2m}(z^T z)^m - (w^T w)^m$ yielding an estimate of the $\mathcal{L}_2$ gain. However, with $m = 2$ we find the bound for the $\mathcal{L}_4$ gain given by $\gamma_4 = 105$. ⋆

From Theorem 3.2 we have that for each bound ρ on the input $\|w\|_{2m}$ it corresponds a bound for the nonlinear $\mathcal{L}_{2m}$ gain. In order to obtain the curve of the nonlinear $\mathcal{L}_{2m}$ gain estimates, we adopt the following procedure:

Procedure 1 Choose a sequence $\rho_1 < \rho_2 < \ldots < \rho_N$ with N a positive integer.

1) *Initial step.* Minimize γ subject to (15) with $\Psi = \gamma^{-2m}(z^T z)^m - (w^T w)^m$. The resulting γ is an estimate of the global $\mathcal{L}_{2m}$ gain of system (1). Set $i = N$, $\rho = \rho_N$ and go to step 2.
2) *Optimization with fixed multipliers and Lyapunov function.* Using the multipliers Π_k $k = 1, 2$ and the Lyapunov function $V(x, \theta)$ obtained in the previous step, minimize γ subject to inequalities (20). For fixed Π_i $i = 1, 2$ and $V(x, \theta)$, the constraints become LMIs.
3) *Optimization with fixed H_j and p_{0j}, $j = 1, 2$.* Using H_j and p_{0j}, $j = 1, 2$ obtained in the previous step 2, minimize γ subject to inequalities (20). For fixed H_j and p_{0j}, $j = 1, 2$, the constraints are LMIs. If the difference between the minimum γ obtained in this step and that from the previous iteration is greater than the desired accuracy, return to step 2. Otherwise go to step 4.
4) If $i = 1$ finish, otherwise store the pair $\{\rho_i, \gamma_{(i)}\}$, set $\rho = \rho_{i-1}$ and $i = i - 1$ select Π_k $k = 1, 2$ and the Lyapunov function $V(x, \theta)$ obtained in 3) and go to 2).

The above procedure can be used whenever the system is open-loop stable. That is, we know that for any given ρ the output is bounded.

Example 5.2: This example is adapted from Example 2 in [3]. Consider system (1) with the following matrices:

$$\begin{bmatrix} A & B_q & B_w \\ C_y & D_{yq} & D_{yw} \\ C_z & D_{zq} & D_{zw} \end{bmatrix} = \left[\begin{array}{ccc|cc|cc} 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -1 & 1 & 1 \\ \hline 1 & 0 & 1 & -3 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & -4 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right]$$

we apply the Procedure 1 to compute the bounds for the $\mathcal{L}_4$ gain for inputs having different $\mathcal{L}_4$ norms. The bounds for the gain are plotted in Figure 3. The gain tends to a constant value for large and small values of the input norm. This values correspond respectively to the open-loop gain and the closed-loop gain whenever the disturbance is small enough such that the solution for $u(x)$ remains inside the saturation bounds. ⋆

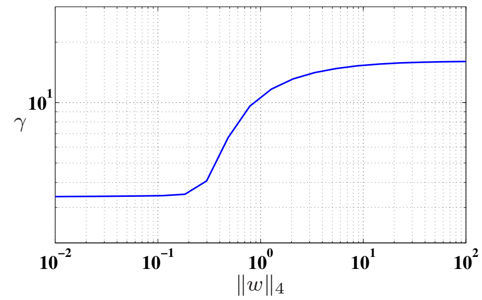

Fig. 3. Estimates of the $\mathcal{L}_4$ gain.

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