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Using discrete-time controller to globally stabilize a class of feedforward nonlinear systems

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Abstract— **This paper considers the problem of using a sampled-data controller to globally stabilize a class of feedforward nonlinear systems. Based on the continuous-time controller proposed in [3], a nested saturation sampled-data control law is first designed to drive states of the feedforward system into a small region around the origin in a finite time. Inside this small region, the nested saturation sampled-data control law is then reduced to a linear sampled-data control law. An explicit formula for the maximum allowable sampling period is computed to guarantee global stability of feedforward systems under the proposed sampled-data controller with appropriate gains.**

I. INTRODUCTION

This paper considers the global stabilization problem for a class of feedforward systems under sampled-data control. The feedforward system is described by

$$\begin{aligned}\dot{x}_i(t) &= x_{i+1}(t) + f_i(t, x_{i+1}(t), \cdots, x_n(t)),\\ &\qquad\qquad\qquad i = 1, \cdots, n-1,\\ \dot{x}_n(t) &= u(t), \end{aligned} \tag{1}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^T \in \mathbb{R}^n$ is system state, $u(t) \in \mathbb{R}$ is control input, and $f_i(t, x_{i+1}, \cdots, x_n)$ is an unknown continuous function with $f_i(t, 0, \cdots, 0) = 0$ for $i = 1, \cdots, n-1$. The control law is implemented in discrete-time under a sampler and zero-order hold device, i.e.,

$$\begin{aligned} u(t) = u(t_k), \quad \forall t \in [t_k, t_{k+1}), \quad t_{k+1} = t_k + T,\\ k \in N = \{0, 1, 2, \cdots\}, \end{aligned} \tag{2}$$

where the time instants t_k, t_{k+1} are the sampling points and T is the sampling period. Our objective is to design a sampled-data controller $u(t_k)$ which globally stabilizes the feedforward system (1).

The problem of stabilizing feedforward systems has attracted a great deal of attention due to its practical and theoretical importance [2], [5], [6], [8]. In the literature, there has been developed some methods to solve the stabilizing problem, such as the nested saturation design method [9], [10], forwarding design method [6], [8], and the method integrated the nested saturation with the Lyapunov design together [11], [3]. Nevertheless, the aforementioned results on global stabilization of feedforward systems are based on continuous-time feedback. In practice, most of the controllers are being implemented using digital computers [1]. Hence, how to design a digital controller to stabilize the feedforward system (1) becomes imperative. Due to the presence of the unknown functions $f_i(t, x_{i+1}, \cdots, x_n)$'s, it is challenging to find a discrete-time controller to globally stabilize the feedforward system (1).

Usually, one approach for designing a discrete-time controller is based on the discrete-time approximation of the nonlinear plant. However, the results using this approach will only guarantee local or semi-global stabilization due to the existence of approximation errors which are inevitable for nonlinear systems. To achieve global stabilization result, it was shown that by carefully choosing the sampling period, the emulation method by discretizing continuous-time controllers can guarantee the global stability for some nonlinear systems [7], [4]. However, the feedforward systems (1) do not satisfy the assumptions imposed in [7], [4].

In this paper, we focus on solving the problem of global stabilization for a class of feedforward systems under sampled-data control. Here, the design of sampled-data controller is carried out by using the emulation approach. Specifically, first, a continuous-time controller will be designed to globally stabilize the feedforward system, which has been achieved in [3]. Then the controller is discretized and implemented digitally where the key issue is to compute the *maximum allowable sampling period* (MASP) that guarantees global asymptotic stability. With the help of the combined method of the nested saturation and Lyapunov design, an explicit formula for the maximum allowable sampling period is computed to guarantee global stability of the feedforward systems (1) under sampled-data control.

II. MAIN RESULTS

In this section, we show that the problem of global stabilization for system (1) under sampled-data control is solvable under the following assumption. For the sake of statement, throughout of this paper, let x denote $x(t)$ when there is no confusion.

Assumption 2.1: In a neighborhood of the origin, there exists a positive constant ρ such that

$$|f_i(t, x_{i+1}, \cdots, x_n)| \le \rho(|x_{i+1}|^{p_{i,i+1}} + \cdots + |x_n|^{p_{i,n}}), \tag{3}$$

where the constants $p_{i,j} > 1, j = i+1, \cdots, n, i = 1, \cdots, n-1$.

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For simplicity of statement, denote $X_i = (x_1, \cdots, x_i), i = 1, 2, \cdots, n$. The sampled-data control law is constructed as following form:

$$\begin{aligned} u(t_k) &= u_n(X_n(t_k)) \\ &= -b_n \left[\varepsilon \sigma \left(\frac{x_n(t_k) - u_{n-1}(X_{n-1}(t_k))}{\varepsilon} \right) \right], \\ &\qquad \forall t \in [t_k, t_{k+1}), \end{aligned} \tag{4}$$

with $u_i(X_i(t_k)) = -b_i \left[\varepsilon \sigma \left(\frac{x_i(t_k) - u_{i-1}(X_{i-1}(t_k))}{\varepsilon} \right) \right], i = 1, \cdots, n$, $u_0 = 0$, and $\sigma(s) = \begin{cases} \text{sign}(s), |s| > 1 \\ s, \quad\quad |s| \le 1 \end{cases}$ where $\varepsilon > 0$ is a small constant to be determined later and the constants b_i's are chosen as

$$\begin{aligned} & b_1 > \max\{2, n\}, \\ & b_i > \max\{\alpha_{i-1}(b_1, \cdots, b_{i-1}) + 2, \\ & \qquad \beta_{i-1}(b_1, \cdots, b_{i-1}) + n - i + 1\}, i = 2, \cdots, n, \end{aligned} \tag{5}$$

with

$$\begin{aligned} & \alpha_1(b_1) = b_1(2 + b_1), \ \beta_1(b_1) = \frac{1}{2} + b_1 + \frac{1}{2} b_1^4, \\ & \alpha_j(b_1, \cdots, b_j) = b_j (2 + b_j + \alpha_{j-1}(b_1, \cdots, b_{j-1})), \\ & \beta_j(b_1, \cdots, b_j) = \frac{1}{2} + b_j + \frac{b_j^2 (b_j - b_{j-1})^2}{2} \\ & \qquad + \sum_{k=1}^{j-1} \frac{(b_j^2 \cdots b_k^2)(b_k - b_{k-1})^2}{4}, \quad b_0 = 0, \\ & \qquad j = 2, \cdots, n-1. \end{aligned} \tag{6}$$

We will show that there exist a small enough constant ε and a maximum allowable sampling period (MASP) $T^* > 0$ ($\sup_{k \in N} \{t_{k+1} - t_k\} \le T^*$) such that the sampled-data control system (1)-(4) is globally asymptotically stable.

Remark 2.1: Based on the definition of $\alpha_i(\cdot)$, the following relations can be obtained:

$$b_1 \le b_2 \le \cdots \le b_{n-1} \le b_n \tag{7}$$

$$2 \le b_1, \quad 2 + \alpha_{i-1}(\cdot) \le b_i, \quad i = 2, \cdots, n. \tag{8}$$

To show how controller (4) globally stabilizes system (1), we first introduce three lemmas. The first lemma studies the local dynamic behavior of closed-loop system (1)-(4) around the origin, where the saturated sampled-data control law (4) is equivalent to the linear sampled-data control law

$$\begin{aligned} u(t) = & - b_n(x_n(t_k) + b_{n-1}[x_{n-1}(t_k) + \cdots \\ & + b_2(x_2(t_k) + b_1 x_1(t_k))]), \quad \forall t \in [t_k, t_{k+1}). \end{aligned} \tag{9}$$

Lemma 2.1: Under Assumption 2.1, the following inequality holds for the closed-loop system (1)-(9)

$$\begin{aligned} \dot{V}(X_n)|_{(1)-(9)} \le & - (\xi_1^2 + \cdots + \xi_n^2) + b_n \xi_n (\xi_n(t) - \xi_n(t_k)) \\ & + \omega_1(X_n) f_1(\cdot) + \cdots + \omega_{n-1}(X_n) f_{n-1}(\cdot), \\ & \forall t \in [t_k, t_{k+1}) \end{aligned} \tag{10}$$

where $\xi_1 = x_1$, $\xi_{i+1} = x_{i+1} + b_i \xi_i$, $\omega_i(X_n) = \partial V(X_n)/\partial x_i$, $i = 1, \cdots, n-1$, and $V(X_n) = \frac{1}{2}(\xi_1^2 + \cdots + \xi_n^2)$.

Proof: We first consider the nominal system of (1)

$$\dot{x}_i = x_{i+1}, \quad i = 1, \cdots, n-1, \quad \dot{x}_n = u. \tag{11}$$

Next, we will show that the following inequality holds:

$$\begin{aligned} \dot{V}(X_n)|_{(11)-(9)} \le & - (\xi_1^2 + \cdots + \xi_n^2) \\ & + b_n \xi_n(t) (\xi_n(t) - \xi_n(t_k)). \end{aligned} \tag{12}$$

Step 1. For the Lyapunov function $V_1(x_1) = \frac{1}{2} x_1^2$, the derivative of V_1 along system (11) is

$$\dot{V}_1(x_1) = x_1 x_2^* + x_1(x_2 - x_2^*) \tag{13}$$

where x_2^* is a virtual control law. Select the virtual controller x_2^* as $x_2^* = -b_1 \xi_1$ with $\xi_1 = x_1$ and b_1 satisfying (5). Substituting this virtual controller into (13) results in

$$\dot{V}_1 \le -n \xi_1^2 + \xi_1(x_2 - x_2^*). \tag{14}$$

Inductive Step. Assume that at step $i-1$, under the virtual controller $x_i^* = -b_{i-1} \xi_{i-1}$, we have the following inequality

$$\dot{V}_{i-1}(X_{i-1}) \le -(n-(i-2))(\xi_1^2 + \cdots + \xi_{i-1}^2) + \xi_{i-1}(x_i - x_i^*) \tag{15}$$

where $\xi_1 = x_1$, $\xi_j = x_j - x_j^*$, $x_j^* = -b_{j-1} \xi_{j-1}, j = 2, \cdots, i-1$, and $V_{i-1}(X_{i-1}) = \frac{1}{2} \sum_{k=1}^{i-1} \xi_k^2$.

Next we show that (15) also holds at step i. To this end, define $V_i(X_i) = V_{i-1}(X_{i-1}) + \frac{1}{2}(x_i - x_i^*)^2 = V_{i-1}(X_{i-1}) + \frac{1}{2} \xi_i^2$ where $\xi_i = x_i - x_i^*$. The derivative of $V_i(X_i)$ along system (11) is

$$\begin{aligned} \dot{V}_i(X_i) \le & - (n - (i-2))(\xi_1^2 + \cdots + \xi_{i-1}^2) + \frac{1}{2} \xi_{i-1}^2 + \frac{1}{2} \xi_i^2 \\ & + \xi_i x_{i+1} + |\xi_i \dot{x}_i^*|. \end{aligned} \tag{16}$$

By the definition of x_i^*, it can be verified that

$$\begin{aligned} \dot{x}_i^* = & - b_{i-1} \xi_i + b_{i-1}(b_{i-1} - b_{i-2}) \xi_{i-1} + \cdots \\ & + (b_{i-1} b_{i-2} \cdots b_2)(b_2 - b_1) \xi_2 + (b_{i-1} b_{i-2} \cdots b_1) b_1 \xi_1. \end{aligned}$$

With this in mind, completing the square for each individual cross term results in

$$|\xi_i \dot{x}_i^*| \le \xi_1^2 + \cdots + \xi_{i-2}^2 + \frac{1}{2} \xi_{i-1}^2 + [\beta_{i-1}(\cdot) - 1/2] \xi_i^2. \tag{17}$$

where β_{i-1} is a function defined in (6). Substituting (17) into (16) leads to

$$\begin{aligned} \dot{V}_i(X_i) \le & - (n - (i-1))(\xi_1^2 + \cdots + \xi_{i-1}^2) \\ & + \beta_{i-1}(\cdot) \xi_i^2 + \xi_i x_{i+1}. \end{aligned} \tag{18}$$

Since $b_i > [\beta_{i-1}(\cdot) + n - i + 1]$ as chosen in (5), the virtual controller $x_{i+1}^* = -b_i \xi_i$ yields

$$\dot{V}_i(X_i) \le -(n - (i-1))(\xi_1^2 + \cdots + \xi_i^2) + \xi_i(x_{i+1} - x_{i+1}^*).$$

This completes the inductive proof.

Following the same line, we can obtain

$$\dot{V}_n(X_n)|_{(11)-(9)} \le -(\xi_1^2 + \cdots + \xi_{n-1}^2 + \xi_n^2) + \xi_n(u(t) - x_{n+1}^*) \tag{19}$$

where $\xi_1 = x_1$, $\xi_i = x_i - x_i^*$, $x_i^* = -b_{i-1}\xi_{i-1}, i = 2, \cdots, n$and $x_{n+1}^* = -b_n\xi_n(t)$. Note that the linear sampled-data controller (9) can be rewritten as

$$\begin{aligned} u(t) &= -b_n\xi_n(t_k) \\ &= x_{n+1}^* + b_n\left(\xi_n(t) - \xi_n(t_k)\right), t \in [t_k, t_{k+1}). \end{aligned} \tag{20}$$

Substituting (20) into (19) yields

$$\begin{aligned} \dot{V}_n(X_n)|_{(11)-(9)} \leq &-(\xi_1^2 + \cdots + \xi_n^2) \\ &+ b_n\xi_n(t)\left(\xi_n(t) - \xi_n(t_k)\right). \end{aligned} \tag{21}$$

Next, let us consider the nonlinear system (1). By (21), taking the derivative of $V(X_n)$ along system (1) under the linear sampled-data control law (9) leads to (10). ∎

Remark 2.2: By its definition, clearly $V(X_n)$ is a quadratic function. Hence $\omega_i(X_n) = \partial V(X_n)/\partial x_i$ is a linear function. Define $W(X_n) = \xi_1^2 + \cdots + \xi_n^2$, then $W(X_n)$ is a positive definition quadratic function. Hence, there exist constants c_i, $h_i > 0, i = 1, 2, \cdots, n-1$, such that

$$|\omega_i(X_n)|(|x_{i+1}| + |x_{i+2}| + \cdots + |x_n|) \leq c_i W(X_n), \tag{22}$$

$$2|x_{i+1}| + |x_{i+2}| + \cdots + |x_n| \leq h_i\sqrt{W(X_n)}. \tag{23}$$

The second lemma was first introduced in [3] where a continuous-time stabilizer was used. However, even though the controller employed in this paper is a sampled-data controller, the proof procedure is similar to the one in [3]. The reason is that Lemma 2.2 mainly discusses the properties of the first $n-1$ equations of system (1) while the different controller is only applied to the n-th equation. The detailed proof of Lemma 2.2 can be found in [3].

Lemma 2.2: Consider the closed-loop system (1)-(4) under Assumption 2.1. For any given constants $b_i, i = 1, \cdots, n-1$, there exists a constant $\varepsilon_0 \in (0, 1)$ satisfying

$$\begin{aligned} \lambda_i(\varepsilon_0) = \rho\Big(&|1 + b_i|^{p_{i,i+1}}\varepsilon_0^{p_{i,i+1}-1} + \cdots \\ &+|1 + b_{n-1}|^{p_{i,n}}\varepsilon_0^{p_{i,n}-1}\Big) \leq 1, i = 1, \cdots, n-1, \end{aligned} \tag{24}$$

for any $\varepsilon \in (0, \varepsilon_0)$ the following inequalities hold for $i = 1, \cdots, n-1$

$$|f_i(t, x_{i+1}(t), \cdots, x_n(t))| < \varepsilon, \forall t \in [\underline{t}, \bar{t}], \tag{25}$$

$$\left|u_i(X_i(\bar{t})) - u_i(X_i(\underline{t}))\right| \leq \alpha_i(b_1, \cdots, b_i)(\bar{t} - \underline{t}), \quad \forall \bar{t} \geq \underline{t}, \tag{26}$$

provided $|x_j| \leq \varepsilon(1 + b_{j-1}), j = i+1, \cdots, n$.

The next lemma shows that under the proposed sampled-data saturated control law (4), by carefully tuning the saturation level and choosing the sampling period, the states of the upper-triangular system will enter into a small region around the origin in a finite time and stay there forever. Inside this small region, the nested saturation sampled-data control law is no longer saturated. The main goal of this lemma is to use the feature of nested saturation to handle the high-order nonlinearities in system (1). This idea has also been used in [11], [3] to design continuous-time feedback controller. However, since the controller proposed here is in the discrete-time form, the proof is significantly different from that in [11], [3].

Before giving this lemma, define function $\lambda(\cdot)$ and constant h as follows:

$$\begin{aligned} \lambda(\varepsilon) &= c_1\lambda_1(\varepsilon) + \cdots + c_{n-1}\lambda_{n-1}(\varepsilon), \\ h &= b_n + \sum_{j=1}^{n-1}(\prod_{i=j}^{n-1} b_i)h_j, \end{aligned} \tag{27}$$

where the function $\lambda_i(\cdot)$ is defined as (24), and the positive constants c_i's and h_i's are defined in (22)-(23). Clearly, the function $\lambda(\varepsilon) \in \mathcal{K}_\infty$[1], and hence there exist small constants $\varepsilon_1, T_1 \in (0, 1)$ such that the following inequalities hold

$$\lambda(\varepsilon_1) < \frac{1}{2}, \qquad T_1 h b_n < \frac{1}{2}. \tag{28}$$

With the help of above selections of small constants ε_1, T_1, we are ready to prove the following Lemma.

Lemma 2.3: Consider the closed-loop system (1)-(4) under Assumption 2.1. If the constant ε and sampling period T are chosen as

$$0 < \varepsilon \leq \varepsilon^* = \min\{\varepsilon_0, \varepsilon_1\}, \quad 0 < T \leq T^* = \min\{\frac{1}{b_n}, T_1\}, \tag{29}$$

then there exists a sampling time point $K_n \in N$ such that for any $k \geq K_n$,

$$|x_i(t) - u_{i-1}(X_{i-1}(t_k))| \leq \varepsilon, \forall t \in [t_k, t_{k+1}), i = 1, \cdots, n. \tag{30}$$

Proof. An inductive method will be used to show that inequality (30) holds. For the sake of brevity, denote

$$\begin{aligned} e_i(t, t_k) = x_i(t) - u_{i-1}(X_{i-1}(t_k)), \forall t \in [t_k, t_{k+1}), \\ i = 1, \cdots, n. \end{aligned}$$

Initial Step. In this step, we will prove that inequality (30) holds for $i = n$, i.e., there exists a sampling point $K_1 \in N$ such that for any $k \geq K_1$,

$$|e_n(t, t_k)| = |x_n(t) - u_{n-1}(X_{n-1}(t_k))| \leq \varepsilon, \forall t \in [t_k, t_{k+1}).$$

Since the proof can be easily obtained by following the proof procedure of Inductive Step, we omit it here.

Inductive step. We suppose that at step $i-1$, there exist $0 \leq K_1 \leq \cdots \leq K_{i-1}$, such that for any $k \geq K_{i-1}$,

$$\begin{aligned} |e_j(t, t_k)| = |x_j(t) - u_{j-1}(X_{j-1}(t_k))| \leq \varepsilon, \\ \forall t \in [t_k, t_{k+1}), \quad j = n-i+2, \cdots, n. \end{aligned} \tag{31}$$

In what follows, we will prove that the above relation will also hold at step i.

We first show that there exists a sampling point $K_i \geq K_{i-1}$ such that $|e_{n-i+1}(t_{K_i}, t_{K_i})| \leq \varepsilon$. If there is no such K_i, it can be assumed that for any $k \geq K_{i-1}$, $|e_{n-i+1}(t_k, t_k)| > \varepsilon$.

[1]A continuous function $g : [0, \infty) \to [0, \infty)$ is said to be class $\mathcal{K}_\infty$ if it is strictly increasing and $g(0) = 0, g(r) \to \infty$ as $r \to \infty$.

Case 1: $e_{n-i+1}(t_k) = x_{n-i+1}(t_k) - u_{n-i}(X_{n-i}(t_k)) < -\varepsilon$, $\forall k \geq K_{i-1}$. In this case, we have

$$\begin{aligned}\dot{x}_{n-i+1}(t) =& u_{n-i+1}(X_{n-i+1}(t_k)) + e_{n-i+2}(t, t_k) \\ &+ f_{n-i+1}(t, x_{n-i+2}, \cdots, x_n) \\ =& b_{n-i+1}\varepsilon + e_{n-i+2}(t, t_k) + f_{n-i+1}(\cdot), \\ & \forall t \in [t_k, t_{k+1}). \end{aligned} \tag{32}$$

Using (31), for any $k \geq K_{i-1}$, we know $|x_j(t)| \leq \varepsilon(1 + b_{j-1}), \forall t \in [t_k, t_{k+1}), j = n-i+2, \cdots, n$, which leads to

$$|f_{n-i+1}(t, x_{n-i+2}, \cdots, x_n)| < \varepsilon, \forall t \in [t_k, t_{k+1}), \tag{33}$$

from Lemma 2.2. As a result, according to (32) and inductive assumption condition (31), we get

$$\dot{x}_{n-i+1}(t) > (b_{n-i+1} - 2)\varepsilon. \tag{34}$$

Noticing that $b_{n-i+1} - 2 > 0$ from (8), it can be concluded that $x_{n-i+1}(t) \to +\infty$ as $t \to \infty$, which leads to a contradiction disavowing $e_{n-i+1}(t_k, t_k) = x_{n-i+1}(t_k) - u_{n-i}(X_{n-i}(t_k)) < -\varepsilon, \forall k \geq K_{i-1}$.

Case 2: $e_{n-i+1}(t_k, t_k) = x_{n-i+1}(t_k) - u_{n-i}(X_{n-i}(t_k)) > \varepsilon, \forall k \geq K_{i-1}$. Using a proof similar to that of Case 1, we can prove that this case is also impossible.

Case 3: There exists a $K \in N$ such that $e_{n-i+1}(t_K, t_K) < -\varepsilon$, $e_{n-i+1}(t_{K+1}, t_{K+1}) > \varepsilon$, i.e., $x_{n-i+1}(t_K) - u_{n-i}(X_{n-i}(t_K)) < -\varepsilon$, $x_{n-i+1}(t_{K+1}) - u_{n-i}(X_{n-i}(t_{K+1})) > \varepsilon$. In this case, by continuity of $x_{n-i+1}(t) - u_{n-i}(X_{n-i}(t))$, it can be concluded that there exists a time $\bar{t} \in [t_K, t_{K+1})$ such that $x_{n-i+1}(\bar{t}) - u_{n-i}(X_{n-i}(\bar{t})) = -\varepsilon$.

In this case, since $e_{n-i+1}(t_K, t_K) < -\varepsilon$ which is also assumed in Case 1, the relations (32) and (34) hold as well. By continuity of $x_n(t)$, for any $t \in [\bar{t}, t_{K+1}]$,

$$x_{n-i+1}(t) \geq x_{n-i+1}(\bar{t}) = -\varepsilon + u_{n-i}(X_{n-i}(\bar{t})). \tag{35}$$

Also, by (32), we get $\dot{x}_{n-i+1}(t) \leq (b_{n-i+1} + 2)\varepsilon$, which leads to

$$\begin{aligned} x_{n-i+1}(t) &\leq x_{n-i+1}(\bar{t}) + (b_{n-i+1} + 2)\varepsilon(t - \bar{t}) \\ &= -\varepsilon + u_{n-i}(X_{n-i}(\bar{t})) + (b_{n-i+1} + 2)\varepsilon(t - \bar{t}), \\ & \qquad \forall t \in [\bar{t}, t_{K+1}]. \end{aligned} \tag{36}$$

In addition, noticing that for any $t \in [\bar{t}, t_{K+1}]$,

$$(b_n + 2)(t - \bar{t}) \leq (b_n + 2)T^* \leq (b_n + b_1)T^* \leq 2b_nT^* \leq 2$$

and $|u_{n-i}(X_{n-i}(\bar{t}))| \leq b_{n-i}\varepsilon$, it follows from (35) and (36) that $|x_{n-i+1}(t)| \leq (1 + b_{n-i})\varepsilon, \forall t \in [\bar{t}, t_{K+1}]$. Under this condition and using the inductive assumption condition (31), we have $|x_j(t)| \leq (1 + b_{j-1})\varepsilon, \forall t \in [\bar{t}, t_{K+1}], j = n-i+1, \cdots, n$. Then, by Lemma 2.2, it can be concluded that

$$\begin{aligned} &\left|u_{n-i}(X_{n-i}(\bar{t})) - u_{n-i}(X_{n-i}(t_{K+1}))\right| \\ &\qquad \leq \alpha_{n-i}(\cdot)\varepsilon(t_{K+1} - \bar{t}). \end{aligned} \tag{37}$$

By the definition of $e_{n-i+1}(t, t_k)$ and (36), we have

$$\begin{aligned} e_{n-i+1}(t_{K+1}, t_{K+1}) &= x_{n-i+1}(t_{K+1}) - u_{n-i}(X_{n-i}(t_{K+1})) \\ &\leq -\varepsilon + (b_{n-i+1} + 2)\varepsilon(t_{K+1} - \bar{t}) \\ &\quad + u_{n-i}(X_{n-i}(\bar{t})) - u_{n-i}(X_{n-i}(t_{K+1})). \end{aligned} \tag{38}$$

Since $b_{n-i+1} - \alpha_{n-i}(\cdot) - 2 \geq 0$ from (8) and $b_{n-i+1}T^* \leq b_nT^* \leq 1$, then it follows from (38) and (37) that

$$\begin{aligned} e_{n-i+1}(t_{K+1}, t_{K+1}) &\leq -\varepsilon + (b_{n-i+1} + 2)\varepsilon(t_{K+1} - \bar{t}) \\ &\quad + \alpha_{n-i}(\cdot)\varepsilon(t_{K+1} - \bar{t}) \\ &= -\varepsilon + b_{n-i+1}\varepsilon(t_{K+1} - \bar{t}) + b_{n-i+1}\varepsilon(t_{K+1} - \bar{t}) \\ &\quad - (b_{n-i+1} - \alpha_{n-i}(\cdot) - 2)\varepsilon(t_{K+1} - \bar{t}) \\ &\leq -\varepsilon + \varepsilon b_{n-i+1}T^* + \varepsilon b_{n-i+1}T^* \leq \varepsilon, \end{aligned} \tag{39}$$

which contradicts to the assumption $e_{n-i+1}(t_{K+1}, t_{K+1}) > \varepsilon$.

Case 4: There exists a $K \in N$ such that $e_{n-i+1}(t_K, t_K) > \varepsilon$, $e_{n-i+1}(t_{K+1}, t_{K+1}) < -\varepsilon$. By a similar proof, we can show this case is also impossible.

To summarize Cases 1-4, we can conclude that there exists a sampling point K_i such that

$$|e_{n-i+1}(t_{K_i}, t_{K_i})| = |x_{n-i+1}(t_{K_i}) - u_{n-i}(X_{n-i}(t_{K_i}))| \leq \varepsilon.$$

Next, we will prove that $|e_{n-i+1}(t, t_k)| = |x_{n-i+1}(t) - u_{n-i}(X_{n-i}(t_k))| \leq \varepsilon, \forall t \in [t_k, t_{k+1}), \forall k \geq K_i$.

First of all, we consider the case $e_{n-i+1}(t_{K_i}, t_{K_i}) = -\gamma\varepsilon, 0 \leq \gamma \leq 1$. By the system dynamic, we have

$$\begin{aligned} \dot{e}_{n-i+1}(t, t_{K_i}) = \dot{x}_{n-i+1}(t) &= b_{n-i+1}\gamma\varepsilon + e_{n-i+2}(t, t_{K_i}) \\ &\quad + f_{n-i+1}(\cdot), \quad \forall t \in [t_{K_i}, t_{K_i+1}). \end{aligned} \tag{40}$$

Note that for any $t \in [t_{K_i}, t_{K_i+1})$, $|e_{n-i+2}(t, t_{K_i})| \leq \varepsilon$ from the inductive assumption condition (31) and $|f_{n-i+1}(t, x_{n-i+2}, \cdots, x_n)| < \varepsilon, \forall t \in [t_{K_i}, t_{K_i+1})$ from (33). As a result, it follows from (40) that

$$\begin{aligned} b_{n-i+1}\gamma\varepsilon - \varepsilon - \varepsilon < \dot{e}_{n-i+1}(t, t_{K_i}) < b_{n-i+1}\gamma\varepsilon + \varepsilon + \varepsilon, \\ \forall t \in [t_{K_i}, t_{K_i+1}). \end{aligned} \tag{41}$$

Integrating (41) yields that for any $t \in [t_{K_i}, t_{K_i+1})$,

$$\begin{aligned} e_{n-i+1}(t, t_{K_i}) \leq & e_{n-i+1}(t_{K_i}, t_{K_i}) \\ & + (b_{n-i+1}\gamma + 2)\varepsilon(t - t_{K_i}) \\ \leq & -\gamma\varepsilon + \gamma\varepsilon b_{n-i+1}T^* + 2\varepsilon T^*. \end{aligned} \tag{42}$$

Based on the fact that $2T^* \leq b_{n-i+1}T^* \leq b_nT^* \leq 1$, (42) leads to

$$e_{n-i+1}(t, t_{K_i}) \leq \varepsilon. \tag{43}$$

Similarly, from (41) we have

$$\begin{aligned} e_{n-i+1}(t, t_{K_i}) &\geq -\gamma\varepsilon + (b_{n-i+1}\gamma - 2)\varepsilon(t - t_{K_i}) \\ &= -\gamma\varepsilon + (\gamma - 1)b_{n-i+1}\varepsilon(t - t_{K_i}) \\ &\quad + (b_{n-i+1} - 2)\varepsilon(t - t_{K_i}) \end{aligned} \tag{44}$$

From the definition of b_{n-i+1} we know that $b_{n-i+1} - 2 \geq 0$. With this in mind, (44) leads to

$$e_{n-i+1}(t, t_{K_i}) \geq -\gamma\varepsilon - (1 - \gamma)b_{n-i+1}\varepsilon T^* \geq -\varepsilon. \tag{45}$$

Hence,

$$|e_{n-i+1}(t, t_{K_i})| \leq \varepsilon, \forall t \in [t_{K_i}, t_{K_i+1}). \tag{46}$$

Now, we discuss $e_{n-i+1}(t_{K_i+1}, t_{K_i+1})$. Note that

$$e_{n-i+1}(t_{K_i+1}, t_{K_i+1}) = e_{n-i+1}(t^-_{K_i+1}, t_{K_i}) + u_{n-i}(X_{n-i}(t_{K_i})) - u_{n-i}(X_{n-i}(t_{K_i+1})) \quad (47)$$

where $e_{n-i+1}(t^-_{K_i+1}, t_{K_i}) = \lim_{t \to t_{K_i+1}} e_{n-i+1}(t, t_{K_i})$.

By (46), we know that

$$|x_{n-i+1}(t)| = \left|e_{n-i+1}(t, t_{K_i}) + u_{n-i}(X_{n-i}(t))\right| \leq (1 + b_{n-i})\varepsilon, \forall t \in [t_{K_i}, t_{K_i+1}). \quad (48)$$

By continuity of $x_{n-i+1}(t)$, we know (48) holds for all $t \in [t_{K_i}, t_{K_i+1}]$. Under this condition and using the inductive assumption condition (31), we have $|x_j(t)| \leq (1+b_{j-1})\varepsilon, \forall t \in [t_{K_i}, t_{K_i+1}], j = n-i+1, \cdots, n$. Applying Lemma 2.2 results in

$$|u_{n-i}(X_{n-i}(t_{K_i})) - u_{n-i}(X_{n-i}(t_{K_i+1}))| \leq \alpha_{n-i}(\cdot)\varepsilon(t_{K_i+1} - t_{K_i}). \quad (49)$$

Meanwhile, from (42) and (44), we get

$$-\gamma\varepsilon + (b_{n-i+1}\gamma - 2)\varepsilon(t_{K_i+1} - t_{K_i}) \leq e_{n-i+1}(t^-_{K_i+1}, t_{K_i}) \leq -\gamma\varepsilon + (b_{n-i+1}\gamma + 2)\varepsilon(t_{K_i+1} - t_{K_i}). \quad (50)$$

Substituting (49) and (50) into (47) yields

$$\begin{aligned} e_{n-i+1}(t_{K_i+1}, t_{K_i+1}) &\leq -\gamma\varepsilon \\ &\quad + (b_{n-i+1}\gamma + 2 + \alpha_{n-i}(\cdot))\varepsilon(t_{K_i+1} - t_{K_i}) \\ &= -\gamma\varepsilon + (b_{n-i+1}\gamma + b_{n-i+1})\varepsilon(t_{K_i+1} - t_{K_i}) \\ &\quad - (b_{n-i+1} - \alpha_{n-i}(\cdot) - 2)\varepsilon(t_{K_i+1} - t_{K_i}). \end{aligned} \quad (51)$$

Noticing that $b_{n-i+1}T^* \leq b_{n-i+1}T^* \leq b_n T^* \leq 1$ and $b_{n-i+1} - \alpha_{n-i}(\cdot) - 2 > 0$ from (8), (51) becomes

$$\begin{aligned} e_{n-i+1}(t_{K_i+1}, t_{K_i+1}) &\leq -\gamma\varepsilon + \gamma\varepsilon b_{n-i+1}T^* + \varepsilon b_{n-i+1}T^* \\ &\leq \varepsilon b_{n-i+1}T^* \leq \varepsilon. \end{aligned} \quad (52)$$

On the other hand, similarly we have

$$\begin{aligned} e_{n-i+1}(t_{K_i+1}, t_{K_i+1}) &\geq -\gamma\varepsilon \\ &\quad + (b_{n-i+1}\gamma - 2 - \alpha_{n-i}(\cdot))\varepsilon(t_{K_i+1} - t_{K_i}) \\ &= -\gamma\varepsilon + (b_{n-i+1}\gamma - b_{n-i+1})\varepsilon(t_{K_i+1} - t_{K_i}) \\ &\quad + (b_{n-i+1} - \alpha_{n-i}(\cdot) - 2)\varepsilon(t_{K_i+1} - t_{K_i}). \end{aligned} \quad (53)$$

Using the same gain relations to obtain (52), (53) can be estimated as

$$\begin{aligned} e_{n-i+1}(t_{K_i+1}, t_{K_i+1}) &\geq -\gamma\varepsilon + b_{n-i+1}\gamma\varepsilon(t_{K_i+1} - t_{K_i}) \\ &\quad - b_{n-i+1}\varepsilon(t_{K_i+1} - t_{K_i}) \\ &\geq -\varepsilon. \end{aligned} \quad (54)$$

Thus $|e_{n-i+1}(t_{K_i+1}, t_{K_i+1})| \leq \varepsilon, \quad \forall t \in [t_{K_i}, t_{K_i+1}]$.

A similar proof will show that the assumption of $e_{n-i+1}(t_{K_i}, t_{K_i}) = \gamma\varepsilon, 0 \leq \gamma \leq 1$, will also lead to the conclusion that $|e_{n-i+1}(t_{K_i+1}, t_{K_i+1})| \leq \varepsilon, \forall t \in [t_{K_i}, t_{K_i+1}]$. Then, using an inductive method, it can be concluded that $|e_{n-i+1}(t, t_k)| \leq \varepsilon, \forall t \in [t_k, t_{k+1}), \forall k \geq K_i$.

According to the results of Initial step and Inductive step, it can be concluded that there exists a K_n such that for any $k \geq K_n$

$$\begin{aligned} &|x_1(t)| \leq \varepsilon, |x_2(t) - u_1(x_1(t_k))| \leq \varepsilon, \cdots, \\ &|x_n(t) - u_{n-1}(X_{n-1}(t_k))| \leq \varepsilon, \forall t \in [t_k, t_{k+1}). \end{aligned} \quad (55)$$

This completes the proof. ∎

With the help of Lemmas 2.1-2.3, we are ready to prove the main result of this paper.

Theorem 2.1: Under Assumption 2.1, system (1) can be globally stabilized by the sampled-data control law (4) if the constant ε and sampling period T satisfy condition (29).

Proof. First of all, according to Lemma 2.3, we know that after the sampling time t_{K_n}, the states $x(t)$ will stay in the following region

$$|x_1(t)| \leq \varepsilon, \quad |x_i(t)| \leq (1 + b_{i-1})\varepsilon, \quad i = 2, \cdots, n, \quad (56)$$

and hence the sampled-data control law (4) is equivalent to the linear sampled-data control law (9), i.e.,

$$\begin{aligned} u(t) = &- b_n(x_n(t_k) + b_{n-1}[x_{n-1}(t_k) + \cdots + b_2(x_2(t_k) \\ &+ b_1 x_1(t_k))]]), \quad \forall t \in [t_k, t_{k+1}), k \geq K_n. \end{aligned} \quad (57)$$

Next, we only need to prove that system (1) can be stabilized by the linear sampled-data control law (57) under the condition (56).

By Lemma 2.1, the derivative of $V(X_n)$ along system (1) under the control law (57) is

$$\begin{aligned} \dot{V}(X_n)_{(1)-(57)} \leq &-W(X_n) + b_n\xi_n(t)\left(\xi_n(t) - \xi_n(t_k)\right) \\ &+ \omega_1(X_n)f_1(\cdot) + \cdots + \omega_{n-1}(X_n)f_{n-1}(\cdot). \end{aligned} \quad (58)$$

By Assumption 2.1, we get

$$\begin{aligned} |f_i(t, x_{i+1}, \cdots, x_n)| \leq &(|x_{i+1}| + \cdots + |x_n|) \\ &\times \rho(|x_{i+1}|^{p_{i,i+1}-1} + \cdots + |x_n|^{p_{i,n}-1}). \end{aligned} \quad (59)$$

By condition (56) and the definition of $\lambda_i(\cdot)$ in (24), we have

$$|f_i(t, x_{i+1}, \cdots, x_n)| \leq (|x_{i+1}| + \cdots + |x_n|)\lambda_i(\varepsilon). \quad (60)$$

With the help of (22), it follows from (60) that

$$|\omega_i(X_n)f_i(\cdot)| \leq c_i\lambda_i(\varepsilon)W(X_n), \quad i = 1, \cdots, n-1. \quad (61)$$

Substituting (61) into (58) results in

$$\begin{aligned} \dot{V}(X_n)_{(1)-(57)} \leq &- W(X_n) + \lambda(\varepsilon)W(X_n) \\ &+ b_n\xi_n(t)\left(\xi_n(t) - \xi_n(t_k)\right) \end{aligned} \quad (62)$$

where $\lambda(\varepsilon)$ is defined in (27).

To handle the term of $\xi_n(t) - \xi_n(t_k)$, we estimate

$$|\xi_n(t) - \xi_n(t_k)| \leq \int_{t_k}^{t} |\dot{\xi}_n(\tau)| d\tau, \quad t \in [t_k, t_{k+1}). \quad (63)$$

Since $\xi_n(\tau) = x_n(\tau) + \sum_{j=1}^{n-1}(\prod_{i=j}^{n-1} b_i)x_j(\tau)$,

$$\begin{aligned}|\dot{\xi}_n(\tau)| \leq& |u(\tau)| \\ &+ \sum_{j=1}^{n-1}(\prod_{i=j}^{n-1} b_i)|x_{j+1}(\tau) + f_i(\tau, x_{j+1}(\tau), \cdots, x_n(\tau))|.\end{aligned} \tag{64}$$

By Assumption 2.1, we know

$$\begin{aligned}&|x_{j+1}(\tau) + f_i(\tau, x_{j+1}(\tau), \cdots, x_n(\tau))| \\ &\leq |x_{j+1}(\tau)| + (|x_{j+1}(\tau)| + \cdots + |x_n(\tau)|) \\ &\quad \times \rho(|x_{j+1}(\tau)|^{p_{j,j+1}-1} + \cdots + |x_n(\tau)|^{p_{j,n}-1}).\end{aligned} \tag{65}$$

Noticing that $\varepsilon \leq \varepsilon_0$ and by (24)-(56), (65) leads to

$$\begin{aligned}&|x_{j+1}(\tau) + f_i(\tau, x_{j+1}(\tau), \cdots, x_n(\tau))| \\ &\leq 2|x_{j+1}(\tau)| + |x_{j+2}(\tau)| + \cdots + |x_n(\tau)|.\end{aligned} \tag{66}$$

With the help of (23), it follows from (66) that

$$\begin{aligned}&|x_{j+1}(\tau) + f_i(\tau, x_{j+1}(\tau), \cdots, x_n(\tau))| \\ &\leq h_j\sqrt{W(X_n(\tau))} \leq h_j\sqrt{W_{max}(t)}\end{aligned} \tag{67}$$

where $W_{max}(t) = \max_{\forall\tau\in[t_k,t]} W(X_n(\tau))$. In addition, note that $|u(\tau)| = b_n|\xi_n(t_k)| \leq b_n\sqrt{W(X_n(t_k))} \leq b_n\sqrt{W_{max}(t)}$. This, together with (63)-(64)-(67) leads to

$$|\xi_n(t) - \xi_n(t_k)| \leq (t - t_k)h\sqrt{W_{max}(t)}. \tag{68}$$

By substituting (68) into (62), we know that for any $t \in [t_k, t_{k+1}), k \geq K_n$, the following inequality holds:

$$\begin{aligned}\dot{V}(X_n(t))_{(1)-(57)} \leq& -W(X_n(t)) + \lambda(\varepsilon)W(X_n(t)) \\ &+ (t - t_k)hb_n\sqrt{W(X_n(t))}\sqrt{W_{max}(t)}.\end{aligned} \tag{69}$$

Note that the constant ε and sampling time T satisfy the condition (29). Hence the following relations hold

$$\lambda(\varepsilon) < 1/2, \quad Thb_n < 1/2, \quad 1 - \lambda(\varepsilon) - Thb_n < 1. \tag{70}$$

In what follows, we will use the relation (69) together the paramater conditions (70) to prove that

$$\max_{\forall\tau\in[t_k,t_{k+1}]} W(X_n(\tau)) = W(X_n(t_k)).$$

Otherwise, it can be assumed that there exists a time instant $t' \in [t_k, t_{k+1}]$ such that $W(X_n(t')) > W(X_n(t_k))$. Using $\lambda(\varepsilon) < \frac{1}{2}$ in (70), clearly we can prove from (69) that for $X_n(t_k) \neq 0, \dot{W}(X_n(t_k)) = 2\dot{V}(X_n(t_k)) < 0$, which implies $\dot{W}(X_n(t))$ will decrease in a short time starting from t_k. Hence, there is a time instant $t'' \in [t_k, t']$ such that

$$\begin{aligned}&(i)\ W(X_n(t'')) = W(X_n(t_k)), \quad (ii)\ \dot{W}(X_n(t'')) > 0, \\ &\text{and} \quad (iii) W(X_n(t)) \leq W(X_n(t_k)), \ \forall t \in [t_k, t''].\end{aligned} \tag{71}$$

Based on relations (71), it follows from (69) that

$$\begin{aligned}\frac{1}{2}\dot{W}(X_n(t'')) &= \dot{V}(X_n(t'')) \\ &\leq -[1 - \lambda(\varepsilon) - Thb_n]W(X_n(t'')) < 0\end{aligned} \tag{72}$$

which contradicts to the assumption $\dot{W}(X_n(t'')) > 0$. Thus, $\max_{\forall\tau\in[t_k,t_{k+1}]} W(X_n(\tau)) = W(X_n(t_k))$.

With this fact in mind and noticing $V(X_n(t)) = \frac{1}{2}W(X_n(t))$, it follows from (69) that

$$\begin{aligned}\dot{W}(X_n(t))_{(1)-(57)} \leq& -2(1 - \lambda(\varepsilon))W(X_n(t)) \\ &+ 2Thb_n\sqrt{W(X_n(t))}\sqrt{W(X_n(t_k))}.\end{aligned} \tag{73}$$

For simplicity of statement, let $\eta(t) = \sqrt{\frac{W(X_n(t))}{W(X_n(t_k))}}$. A straightforward calculation leads to

$$\dot{\eta}(t) \leq -(1 - \lambda(\varepsilon))\eta(t) + Thb_n. \tag{74}$$

Noticing that $\eta(t_k) = 1$, it follows (74) that

$$\eta(t_{k+1}) \leq e^{-(1-\lambda(\varepsilon))T} + \left(1 - e^{-(1-\lambda(\varepsilon))T}\right)\frac{Thb_n}{1 - \lambda(\varepsilon)} := l. \tag{75}$$

which leads to $W(X_n(t_{k+1})) \leq l^2 W(X_n(t_k))$. By (70), we know $\frac{Thb_n}{1-\lambda(\varepsilon)} < 1$, which implies the constant $l < 1$. Hence, $W(X_n(t_k))$ converges zero as k tends to infinity, i.e., system (1) is globally stabilized by sampled-data controller (4). ∎

III. Conclusion

In this paper, we have designed a sampled-data controller to globally stabilize a class of feedforward systems with unknown nonlinearities. An explicit formula for the maximum allowable sampling period is computed to guarantee global stability of feedforward systems under the proposed sampled-data controller with appropriate gains.

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