2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Lyapunov-based second-order sliding mode control for a class of uncertain reaction-diffusion processes

Yury Orlov
CICESE Research Center, Ensenada (Mexico)
Email: yorlov@cicese.mx

Alessandro Pisano, Stefano Scodina, Elio Usai
Dept. of Electrical and Electronic Engineering (DIEE)
University of Cagliari (Italy)
Email:{pisano,stefano.scodina,eusai}@diee.unica.it

Abstract— **This paper addresses the design of a distributed, second-order sliding-mode based, tracking controller for a class of uncertain diffusion-reaction processes. Spatially varying uncertain parameters and mixed boundary conditions, along with the presence of an uncertain distributed disturbance, characterize the considered class of processes. The paper presents a constructive Lyapunov-based stability analysis which leads to simple tuning conditions for the controller parameters, The good performance of the proposed control systems are verified by means of computer simulations.**
Keywords: ***Distributed parameter systems, Reaction-Diffusion equation, Second-order sliding mode control, Lyapunov analysis.***

I. INTRODUCTION

Sliding-mode control has long been recognized as a powerful control method to counteract non-vanishing external disturbances and unmodelled dynamics when controlling dynamical systems of finite and infinite dimension (see [20]).

Presently, the discontinuous control synthesis in the infinite-dimensional setting is well documented (see [10], [13], [15], [16], [14]) and it is generally shown to retain the main robustness features as those possessed by its finite-dimensional counterpart. Other robust control paradigms have been fruitfully applied in the infinite dimensional setting such as adaptive and model-reference control (see [9], [4]), geometric and Lyapunov-based design (see [2]), H_∞ and LMI-based design (see [6]).

In the present paper we consider a generalized uncertain form of the heat equation, under the effect of a persistent external smooth disturbance. Even in the finite-dimensional setting, and with a perfectly known linear and time-invariant controlled plant, the asymptotic rejection of persistent disturbances is difficult to achieve when the unique information is the existence of *a priori* known upper bounds to the magnitude of the disturbance and of its time derivative. Despite structured perturbations (e.g., constant signals or sinusoids with known frequency) can be easily rejected by linear control techniques, the problem remains open when arbitrarily shaped disturbances need to be taken into account. The only known solution appears to be the discontinuous sliding-mode control feedback (see [20]), which is effectively implemented in the infinite dimensional setup via the so-called (distributed) "unit-vector" control (see [13], [15]).

In some recent authors' publications (see [18], [19], [17])) two finite dimensional robust control algorithms, namely, the "Super-Twisting" and "Twisting" second-order sliding-mode (2-SM) controllers (see [5], [11] for details) have been generalized to the infinite-dimensional setting and applied for controlling heat and wave processes, respectively. The mentioned 2-SM controllers are of special interest because in the finite dimensional setting they significantly improve the performance of sliding-mode control systems, in terms of accuracy and chattering avoidance, as compared to the standard "first-order" sliding mode control techniques (see [1]).

In this paper we enlarge the class of controlled distributed-parameter dynamics as compared to the publications [18], [19], [17] by considering more general diffusion-reaction dynamics. More precisely, we consider the presence of an additional dispersion term in the plant equation and, furthermore, we let all the system parameters (diffusivity and dispersion coefficients) to be uncertain and possibly spatially-varying. We additionally put the constraint that the distributed control input must be a continuous (although possibly non-smooth) function of the space and time variables.

The rest of the paper is structured as follows. Some notations are introduced in the remainder of the Introduction. Section 2 presents the tracking control problem formulation for the considered reaction-diffusion process, and section 3 describes the associated solution, based on a proper combination between (distributed forms of) PI and "Super-Twisting" 2-SM control. Section 4 illustrates some relevant numerical simulation results. Finally, Section 5 gives some concluding remarks and draws possible direction of improvement of the proposed result.

Notation. The notation used throughout is fairly standard (see [3] for details). $L_2(a, b)$, with $a \leq b$, stands for the Hilbert space of square integrable functions $z(\zeta)$, $\zeta \in [a, b]$, equipped with the L_2 norm

$$\|z(\cdot)\|_2 = \sqrt[2]{\int_a^b z^2(\zeta)d\zeta} \tag{1}$$

$W^{l,2}(a,b)$ denotes the Sobolev space of absolutely continuous scalar functions $z(\zeta),\ \zeta \in [a,b]$ with square integrable derivatives $z^{(i)}(\zeta)$ up to the order $l \geq 1$

II. Problem formulation

Consider the space- and time-varying scalar field $Q(\xi,t)$ evolving in a Hilbert space $L_2(0,1)$, where $\xi \in [0,1]$ is the mono-dimensional spatial variable and $t \geq 0$ is time. Let it be governed by the following perturbed reaction-diffusion equation with spatially-varying parameters

$$Q_t(\xi,t) = [\theta_1(\xi)Q_\xi(\xi,t)]_\xi + \theta_2(\xi)Q(\xi,t) + u(\xi,t) + \psi(\xi,t), \tag{2}$$

where the subindexes t and ξ denote the temporal and spatial partial derivative, respectively, $\theta_1(\cdot) \in C^1(0,1)$ is a positive-definite spatially-varying parameter called *thermal conductivity* (or, more generally, *diffusivity*), $\theta_2(\cdot) \in C(0,1)$ is another spatially-varying parameter called *dispersion* (or *reaction constant*), $u(\xi,t)$ is the modifiable source term (the distributed control input), and $\psi(\xi,t)$ represents a distributed **uncertain** disturbance source term. This uncertain term is supposed to satisfy the following conditions

$$\psi(\xi,t) \in L_2(0,1), \qquad \psi_t(\xi,t) \in W^{1,2}(0,1). \tag{3}$$

The spatially-varying diffusivity and dispersion coefficients $\theta_1(\xi)$ and $\theta_2(\xi)$ are supposed to be uncertain, too. We consider non-homogeneous mixed boundary conditions (BCs)

$$Q(0,t) - \alpha_0 Q_\xi(0,t) = Q_0(t) \in W^{1,2}(0,\infty), \tag{4}$$
$$Q(1,t) + \alpha_1 Q_\xi(1,t) = Q_1(t) \in W^{1,2}(0,\infty), \tag{5}$$

with some positive **uncertain** constants α_0, α_1. The initial conditions (ICs)

$$Q(\xi,0) = \omega_0(\xi) \in W^{2,2}(0,1) \tag{6}$$

are assumed to meet the same BCs (4)-(5). Since non-homogeneous BCs are in force, a solution of the above boundary-value problem is defined in the mild sense (see [3]) as that of the corresponding integral equation, written in terms of the strongly continuous semigroup, generated by the infinitesimal plant operator.

The control task is to make the scalar field $Q(\xi,t)$ to track a given reference $Q^r(\xi,t) \in W^{2,2}(0,1)$ which should be selected in accordance with the BCs (4)-(5) and which should also satisfy the following condition

$$Q_t^r \in W^{3,2}(0,1). \tag{7}$$

III. Robust Control of an Uncertain Reaction-Diffusion Process

Consider the deviation variable

$$x(\xi,t) = Q(\xi,t) - Q^r(\xi,t) \tag{8}$$

whose L_2 norm will be driven to zero by the designed feedback control. The dynamics of the error variable (8) are easily derived as

$$\begin{aligned} x_t(\xi,t) =& [\theta_1(\xi)x_\xi(\xi,t)]_\xi + \theta_2(\xi)x(\xi,t) \\ &+ u(\xi,t) - Q_t^r(\xi,t) + \eta(\xi,t), \end{aligned} \tag{9}$$

with the "augmented" disturbance

$$\eta(\xi,t) = [\theta_1(\xi)Q_\xi^r(\xi,t)]_\xi + \theta_2(\xi)Q^r(\xi,t) + \psi(\xi,t), \tag{10}$$

and the next ICs and homogeneous mixed BCs

$$x(\xi,0) = \omega_0(\xi) - Q^r(\xi,0) \in W^{2,2}(0,1) \tag{11}$$

$$x(0,t) - \alpha_0 x_\xi(0,t) = x(1,t) + \alpha_1 x_\xi(1,t) = 0. \tag{12}$$

Assume what follows:

Assumption 1: There exist *a priori* known constants Θ_{1m}, Θ_{1M} and Θ_{2M} such that

$$\begin{aligned} &0 < \Theta_{1m} \leq \theta_1(\xi) \leq \Theta_{1M}, \\ &|\theta_2(\xi)| \leq \Theta_{2M} \quad \forall \quad \xi \in [0,1]. \end{aligned} \tag{13}$$

Assumption 2: There exist *a priori* known constants H_0, ..., H_3, Ψ_0 and Ψ_1 such that the following inequalities hold for all $t \geq 0$

$$\|\theta_2(\cdot)Q_t^r(\cdot,t)\|_2 \leq H_0,\ \|[\theta_2(\xi)Q_t^r(\cdot,t)]_\xi\|_2 \leq H_1, \tag{14}$$

$$\|[\theta_1(\xi)Q_\xi^r(\cdot,t)]_{\xi\ t}\|_2 \leq H_2,\ \ \|[\theta_1(\xi)Q_\xi^r(\cdot,t)]_{\xi\xi\ t}\|_2 \leq H_3, \tag{15}$$

$$\|\psi_t(\cdot,t)\|_2 \leq \Psi_0,\ \ \|\psi_{t\xi}(\cdot,t)\|_2 \leq \Psi_1 \tag{16}$$

By the Assumption 2, it follows that the L_2 norm of the augmented disturbance time derivative $\eta_t(\xi,t)$, and that of its spatial derivative, fulfill the next conditions

$$\|\eta_t(\cdot,t)\|_2 \leq M,\ \ \|\eta_{t\xi}(\cdot,t)\|_2 \leq M_\xi,\ \ \forall t \geq 0, \tag{17}$$

with

$$M = H_2 + H_0 + \Psi_0,\ \ M_\xi = H_3 + H_1 + \Psi_1. \tag{18}$$

The class of admissible "augmented" disturbances is further specified by the following additional restriction, being introduced in [19]:

Assumption 3: There exist *a priori* known constant M_x such that the following inequality holds uniformly beyond the origin $\|x(\cdot,t)\|_2 = 0$ in the state space $L_2(0,1)$:

$$|\eta_t(\xi,t)| \leq M_x \frac{|x(\xi,t)|}{\|x(\cdot,t)\|_2},\ \ \forall t \geq 0, \forall \xi \in [0,1] \tag{19}$$

It is worth noticing that according to the Assumption 3 an admissible disturbance has a time derivative which is not necessarily vanishing as $\|x(\cdot,t)\|_2 \to 0$ because the norm of the right-hand side of the disturbance restriction (19) remains unit according to relation $\left\|\frac{|x(\cdot,t)|}{\|x(\cdot,t)\|_2}\right\|_2 = 1$. Particularly, with $M_x \geq M$ a finite-dimensional counterpart of (19) would not impose any further restrictions on admissible disturbances in addition to the first relation of (17).

It should also be noted that the assumptions on the ICs and BCs, made above, allow us to deal with strong, sufficiently smooth solutions of the uncertain error dynamics (9)-(12) in the open-loop when no control input is applied.

In order to stabilize the error dynamics it is proposed a dynamical distributed controller defined as follows

$$u(\xi,t) = Q_t^r(\xi,t) - \lambda_1\sqrt{|x(\xi,t)|}\ \mathtt{sign}(x(\xi,t)) - \lambda_2 x(\xi,t) + v(\xi,t) \tag{20}$$

$$v_t(\xi,t) = -W_1\frac{x(\xi,t)}{\|x(\cdot,t)\|_2} - W_2 x(\xi,t), \qquad v(\xi,0)=0 \tag{21}$$

which can be seen as a distributed version of the finite-dimensional "Super-Twisting" second-order sliding-mode controller (see [5], [11]) complemented by a feed-forward term $Q_t^r(\xi,t)$ and by the two additional proportional and integral linear terms $-\lambda_2 x(\xi,t)$ and $-W_2 x(\xi,t)$. For ease of reference, the combined Distributed Super-Twisting/PI controller (20)-(21) will be abbreviated as DSTPI.

The non-smooth nature of the DSTPI controller (20)-(21), that undergoes discontinuities on the manifold $x=0$ due to the discontinous term $\frac{x(\xi,t)}{\|x(\cdot,t)\|_2}$, requires appropriate analysis about the meaning of the corresponding solutions for the resulting discontinuous feedback system. Throughout, the precise meaning of the solutions of (9), (11), (12) with the piece-wise continuously differentiable control input (20)-(21) is defined in a generalized sense according to [14] as a limiting result obtained through a certain regularization procedure, similar to that proposed for finite-dimensional systems (see [7], [20]). The existence of generalized solutions, thus defined, has been established within the abstract framework of Hilbert space-valued dynamic systems (cf., e.g., [14, Theorem 2.4]) whereas the uniqueness and well-posedness appear to follow from the fact that in the system in question no sliding mode can occur but in the origin $x=0$.

The performance of the closed-loop system is analyzed in the next theorem.

Theorem 1: Consider the perturbed diffusion/dispersion equation (2) along with the boundary conditions (4) and with the system parameters, reference trajectory and uncertain disturbance satisfying the Assumptions 1-3. Then, the DSTPI control strategy (20)-(21) with the parameters λ_1, λ_2, W_1 and W_2 selected according to

$$\begin{aligned}&\lambda_2 \geq \Theta_{2M},\\ &W_1 \geq \mathtt{max}\left\{M + \frac{\Theta_{1M}M_\xi}{2(\lambda_2-\Theta_{2M})},\ \frac{1}{2}\frac{\Theta_{1M}}{\Theta_{1m}}M_\xi, 2M_x\right\},\\ &\lambda_1 \geq \mathtt{max}\left\{2M, \frac{2M_x}{W_1}\right\},\ \ W_2 \geq 0,\end{aligned} \tag{22}$$

guarantees that the L_2-norm $\|x(\cdot,t)\|_2$ of the tracking error tends to zero as t tends to infinity.

Proof of Theorem 1. Let us define the auxiliary variable

$$\delta(\xi,t) = v(\xi,t) + \eta(\xi,t) \tag{23}$$

System (9) with the control law (20)-(21) yields the following closed-loop dynamics in the new $x-\delta$ coordinates

$$\begin{aligned}x_t(\xi,t) &= [\theta_1(\xi)x_\xi(\xi,t)]_\xi - \lambda_1\sqrt{|x(\xi,t)|}\ \mathtt{sign}(x(\xi,t))\\ &- (\lambda_2-\theta_2(\xi))x(\xi,t) + \delta(\xi,t)\end{aligned} \tag{24}$$

$$\delta_t(\xi,t) = -W_1\frac{x(\xi,t)}{\|x(\cdot,t)\|_2} - W_2 x(\xi,t) + \eta_t(\xi,t) \tag{25}$$

In order to simplify the notation, the dependence of the system coordinates from the space and time variables (ξ,t) is omitted from this point on. Consider the following Lyapunov functional

$$V_1(t) = 2W_1\|x\|_2 + W_2\|x\|_2^2 + \frac{1}{2}\|\delta\|_2^2 + \frac{1}{2}\|s\|_2^2 \tag{26}$$

inspired from the finite-dimensional treatment in [12], where

$$\begin{aligned}&s = x_t =\\ &[\theta_1(\xi)x_\xi]_\xi - \lambda_1\sqrt{|x|}\ \mathtt{sign}(x) - [\lambda_2-\theta_2(\xi)]x + \delta.\end{aligned} \tag{27}$$

The time derivative of $V_1(t)$ is given by

$$\begin{aligned}\dot V_1(t) &= \frac{2W_1}{\|x\|_2}\int_0^1 xs d\xi + 2W_2\int_0^1 xs d\xi\\ &+ \int_0^1 \delta\delta_t d\xi + \int_0^1 ss_t d\xi\end{aligned} \tag{28}$$

Let us evaluate the time derivative of the auxiliary signal s in (27) along the strong solutions of (24)-(25)

$$\begin{aligned}&s_t = x_{tt} = [\theta_1(\xi)s_\xi]_\xi - \frac{1}{2}\lambda_1\frac{s}{\sqrt{|x|}} - [\lambda_2-\theta_2(\xi)]s\\ &- W_1\frac{x}{\|x\|_2} - W_2 x + \eta_t\end{aligned} \tag{29}$$

Substituting (25) and (29) into (28) and rearranging it yields

$$\begin{aligned}\dot V_1(t) &= \frac{2W_1}{\|x\|_2}\int_0^1 xs d\xi + 2W_2\int_0^1 xs d\xi - \frac{W_1}{\|x\|_2}\int_0^1 \delta x d\xi\\ &- W_2\int_0^1 \delta x d\xi + \int_0^1 \delta\eta_t d\xi + \int_0^1 s[\theta_1(\xi)s_\xi]_\xi d\xi\\ &- \frac{1}{2}\lambda_1\int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}} - \int_0^1 [\lambda_2-\theta_2(\xi)]s^2 d\xi\\ &- \frac{W_1}{\|x\|_2}\int_0^1 xs d\xi - W_2\int_0^1 xs d\xi + \int_0^1 s\eta_t d\xi\end{aligned} \tag{30}$$

which can be manipulated as follows by virtue of Assumption 1

$$\begin{aligned}\dot V_1(t) &\leq -\frac{W_1}{\|x\|_2}\int_0^1 x(\delta-s)d\xi - W_2\int_0^1 x(\delta-s)d\xi\\ &+ \int_0^1 s[\theta_1(\xi)s_\xi]_\xi d\xi - -\frac{1}{2}\lambda_1\int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}}\\ &- [\lambda_2-\Theta_{2M}]\int_0^1 s^2 d\xi + \int_0^1 (\delta+s)\eta_t d\xi\end{aligned}$$

By (27), one has

$$\delta - s = \lambda_1\sqrt{|x|}\ \mathtt{sign}(x) + [\lambda_2-\theta_2(\xi)]x - [\theta_1(\xi)x_\xi]_\xi \tag{31}$$

$$\delta + s = 2s + \lambda_1\sqrt{|x|}\ \mathtt{sign}(x) + [\lambda_2-\theta_2(\xi)]x - [\theta_1(\xi)x_\xi]_\xi. \tag{32}$$

Due to this, and considering once more the Assumption 1, (31) can further be manipulated as

$$
\begin{aligned}
\dot V_1(t) \le & -\frac{W_1\lambda_1}{\|x\|_2}\int_0^1 x\sqrt{|x|}\ \mathtt{sign}(x)d\xi \\
& -\frac{W_1[\lambda_2-\Theta_{2M}]}{\|x\|_2}\int_0^1 x^2 d\xi + \frac{W_1}{\|x\|_2}\int_0^1 x[\theta_1(\xi)x_\xi]_\xi d\xi \\
& - W_2\lambda_1\int_0^1 x\sqrt{|x|}\ \mathtt{sign}(x)d\xi - W_2[\lambda_2-\Theta_{2M}]\int_0^1 x^2 d\xi \\
& + W_2\int_0^1 x[\theta_1(\xi)x_\xi]_\xi d\xi + \int_0^1 s[\theta_1(\xi)s_\xi]_\xi d\xi \\
& - \frac{1}{2}\lambda_1\int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}} - [\lambda_2-\Theta_{2M}]\int_0^1 s^2 d\xi + 2\int_0^1 s\eta_t d\xi \\
& + \lambda_1\int_0^1 \sqrt{|x|}\ \mathtt{sign}(x)\eta_t d\xi \\
& + [\lambda_2-\Theta_{2M}]\int_0^1 x\eta_t d\xi - \int_0^1 [\theta_1(\xi)x_\xi]_\xi \eta_t d\xi.
\end{aligned} \tag{33}
$$

By taking into account the BCs (12) and their time derivatives, standard integration by parts yields

$$
\begin{aligned}
&\int_0^1 x[\theta_1(\xi)x_\xi]_\xi d\xi = \\
&-\int_0^1 \theta_1(\xi)x_\xi^2 d\xi + \theta_1(1)x(1,t)x_\xi(1,t) - \theta_1(0)x(0,t)x_\xi(0,t) \\
\le \quad & -\Theta_{1m}\|x_\xi\|_2^2 - \theta_1(1)\frac{x^2(1,t)}{\alpha_1} - \theta_1(0)\frac{x^2(0,t)}{\alpha_0}
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&\int_0^1 s[\theta_1(\xi)s_\xi]_\xi d\xi \\
\le \quad & -\Theta_{1m}\|s_\xi\|_2^2 - \theta_1(1)\frac{s^2(1,t)}{\alpha_1} - \theta_1(0)\frac{s^2(0,t)}{\alpha_0}
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&\int_0^1 [\theta_1(\xi)x_\xi]_\xi \eta_t d\xi = \\
&-\int_0^1 \theta_1(\xi)x_\xi\eta_{t\xi} d\xi + \theta_1(1)\eta_t(1,t)x_\xi(1,t) \\
&- \theta_1(0)\eta_t(0,t)x_\xi(0,t) = -\int_0^1 \theta_1(\xi)x_\xi\eta_{t\xi} d\xi \\
&- \theta_1(1)\eta_t(1,t)\frac{x(1,t)}{\alpha_1} - \theta_1(0)\eta_t(0,t)\frac{x(0,t)}{\alpha_0}
\end{aligned} \tag{36}
$$

Additional straightforward manipulations of (33) taking into account (34) and (35) yield

$$
\begin{aligned}
\dot V_1(t) \le & -W_1[\lambda_2-\Theta_{2M}]\|x\|_2 - W_2[\lambda_2-\Theta_{2M}]\|x\|_2^2 \\
& - W_1\Theta_{1m}\frac{\|x_\xi\|_2^2}{\|x\|_2} - \frac{W_1}{\|x\|_2}\theta_1(1)\frac{x^2(1,t)}{\alpha_1} \\
& - \frac{W_1}{\|x\|_2}\theta_1(0)\frac{x^2(0,t)}{\alpha_0} - W_2\Theta_{1m}\|x_\xi\|_2^2 \\
& - W_2\theta_1(1)\frac{x^2(1,t)}{\alpha_1} - W_2\theta_1(0)\frac{x^2(0,t)}{\alpha_0} \\
& - [\lambda_2-\Theta_{2M}]\|s\|_2^2 - \Theta_{1m}\|s_\xi\|_2^2 - \theta_1(1)\frac{s^2(1,t)}{\alpha_1} \\
& - \theta_1(0)\frac{s^2(0,t)}{\alpha_0} - W_2\lambda_1\int_0^1 |x|^{3/2} d\xi \\
& - \frac{1}{2}\lambda_1\int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}} - \frac{W_1\lambda_1}{\|x\|_2}\int_0^1 \sqrt{|x|}|x| d\xi \\
& + 2\int_0^1 s\eta_t d\xi + \lambda_1\int_0^1 \sqrt{|x|}\ \mathtt{sign}(x)\eta_t d\xi \\
& + [\lambda_2-\Theta_{2M}]\int_0^1 x\eta_t d\xi + \int_0^1 \theta_1(\xi)x_\xi\eta_{t\xi} d\xi \\
& + \theta_1(1)\eta_t(1,t)\frac{x(1,t)}{\alpha_1} + \theta_1(0)\eta_t(0,t)\frac{x(0,t)}{\alpha_0}.
\end{aligned} \tag{37}
$$

It is worth noting that by virtue of the tuning inequality $\lambda_2 > \Theta_{2M}$ in (22) all terms appearing in the right hand side of (37) are negative definite except those depending on the augmented disturbance term η_t and its spatial derivative. Some estimations involving those sign-indefinite terms are now derived by simple application of the Cauchy-Schwartz and Young's inequalities and by considering the Assumptions 1 and 2, the BCs (12) and the derived conditions (17)-(18):

$$
\begin{aligned}
2\left|\int_0^1 s\eta_t d\xi\right| &\le 2\int_0^1 |s||\eta_t| d\xi = 2\int_0^1 \frac{|s|\sqrt{|\eta_t|}\sqrt{|\eta_t|}\sqrt{|x|}}{\sqrt{|x|}} d\xi \\
&\le \int_0^1 \frac{|\eta_t|s^2 + |\eta_t||x|}{\sqrt{|x|}} d\xi \\
&\le M\int_0^1 \frac{s^2}{\sqrt{|x|}} d\xi + \int_0^1 \eta_t\sqrt{|x|} d\xi.
\end{aligned} \tag{38}
$$

$$
\left|\int_0^1 x\eta_t d\xi\right| \le \left[\int_0^1 x^2 d\xi\right]^{1/2}\left[\int_0^1 \eta_t^2 d\xi\right]^{1/2} \le M\|x\|_2. \tag{39}
$$

$$
\begin{aligned}
\left|\int_0^1 \theta_1(\xi)x_\xi\eta_{t\xi} d\xi\right| &\le \Theta_{1M}\int_0^1 |x_\xi||\eta_{t\xi}| d\xi \\
&= \Theta_{1M}\int_0^1 \frac{|x_\xi|\sqrt{|\eta_{t\xi}|}\sqrt{|\eta_{t\xi}|}\|x\|_2}{\|x\|_2} d\xi \\
&\le \frac{1}{2}\Theta_{1M}\int_0^1 \frac{x_\xi^2|\eta_{t\xi}| + |\eta_{t\xi}|\|x\|_2^2}{\|x\|_2} d\xi \\
&\le \frac{1}{2}\Theta_{1M}M_\xi\frac{\|x_\xi\|_2^2}{\|x\|_2} + \frac{1}{2}\Theta_{1M}M_\xi\|x\|_2.
\end{aligned} \tag{40}
$$

Taking into account (38)-(40), the right-hand side of (37) can be estimated as

$$\begin{aligned}
\dot{V}_1(t) \ \le \ & -(\lambda_2-\Theta_{2M})\left[W_1 - M - \frac{\Theta_{1M}M_\xi}{2(\lambda_2-\Theta_{2M})}\right]\|x\|_2 \\
& - W_2(\lambda_2-\Theta_{2M})\|x\|_2^2 - \left[W_1\Theta_{1m} - \frac{1}{2}\Theta_{1M}M_\xi\right]\frac{\|x_\xi\|_2^2}{\|x\|_2} \\
& - W_2\Theta_{1m}\|x_\xi\|_2^2 - (\lambda_2-\Theta_{2M})\|s\|_2^2 - \Theta_{1m}\|s_\xi\|_2^2 \\
& - W_2\lambda_1\int_0^1 |x|^{3/2}d\xi - \frac{1}{2}(\lambda_1 - 2M)\int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}} \\
& - \int_0^1 \sqrt{|x|}\left[\frac{W_1\lambda_1}{2\|x\|_2}|x| - \eta_t\right]d\xi \\
& - \lambda_1\int_0^1 \sqrt{|x|}\left[\frac{W_1}{2\|x\|_2}|x| - \eta_t\right]d\xi \\
& - \theta_1(1)\frac{|x(1,t)|}{\alpha_1}\left[\frac{W_1}{\|x\|_2}|x(1,t)| - \eta_t(1,t)\right] \\
& - \theta_1(0)\frac{|x(0,t)|}{\alpha_0}\left[\frac{W_1}{\|x\|_2}|x(0,t)| - \eta_t(0,t)\right] \\
& - W_2\theta_1(1)\frac{x^2(1,t)}{\alpha_1} - W_2\theta_1(0)\frac{x^2(0,t)}{\alpha_0} \\
& - \theta_1(1)\frac{s^2(1,t)}{\alpha_1} - \theta_1(0)\frac{s^2(0,t)}{\alpha_0}
\end{aligned} \tag{41}$$

By virtue of Assumption 3, the next inequalities guarantee that all terms in the right hand side of (41) are negative definite

$$\begin{aligned}
& \lambda_2 > \Theta_{2M}, \quad W_1 > M + \frac{\Theta_{1M}M_\xi}{2(\lambda_2-\Theta_{2M})}, \\
& W_2 > 0, \quad W_1 > \frac{1}{2}\frac{\Theta_{1M}}{\Theta_{1m}}M_\xi, \\
& \lambda_1 > 2M, \quad W_1\lambda_1 > 2M_x, \quad W_1 > 2M_x, \quad W_1 > M_x
\end{aligned} \tag{42}$$

The above inequalities collected together form the tuning conditions (22). It remains to demonstrate that

$$\|x(\cdot,t)\|_2 \to 0 \text{ as } t \to \infty. \tag{43}$$

For this purpose, let us integrate the relation

$$\dot{\bar{V}}(t) \le -(\lambda_2-\Theta_{2M})\left[W_1 - M - \frac{\Theta_{1M}M_\xi}{2(\lambda_2-\Theta_{2M})}\right]\|x\|_2, \tag{44}$$

straightforwardly resulting from the negative definiteness of all terms in the right hand side of (41), to conclude that

$$\int_0^\infty \|x(\cdot,t)\|_2 dt < \infty \tag{45}$$

The inequality $\dot{V}_1(t) \le 0$, which is readily concluded from (41) and (42) in light of the Assumption 3, guarantees that $V_1(t) \le V_1(0)$ for any $t \ge 0$. From this, and considering (26), one can conclude that the L_2 norm of $s = x_t$ fulfills the estimation

$$\|x_t\|_2^2 \le 2V_1(0), \qquad \forall t \ge 0 \tag{46}$$

Thus, the integrand $\omega(t) = \|x(\cdot,t)\|_2$ of (45) possesses a uniformly bounded time derivative

$$\dot{\omega}(t) = \frac{\int_0^1 x x_t d\xi}{\|x\|_2} \le \|x_t\|_2 \le \sqrt{2R} \tag{47}$$

on the semi-infinite time interval $t \in [0,\infty)$, where R is any positive constant such that $R \ge V_1(0)$. Convergence (43) is then verified by applying the Barbalat lemma (see [8]). Since the Lyapunov functional (26) is radially unbounded the global asymptotic stability of the closed-loop system (9)-(12) is thus established in the L_2 space. Theorem 1 is proved. □

IV. NUMERICAL SIMULATIONS

Consider the perturbed heat equation (2) with the following spatially-varying diffusion and reaction parameters:

$$\theta_1(\xi) = 0.1(1 + 0.2sin(\pi\xi)), \tag{48}$$

$$\theta_2(\xi) = 1 + 0.1sin(4\pi\xi). \tag{49}$$

According to (4)-(5), the boundary conditions are of Robin's type:

$$Q(0,t) - Q_\xi(0,t) = Q(1,t) + Q_\xi(1,t) = 20, \tag{50}$$

with the parameters $\alpha_0 = \alpha_1 = 1$ being uncertain to the designer, and the initial condition is:

$$Q(\xi,0) = 20 + sin(6\pi\xi). \tag{51}$$

A spatially variant set-point $Q^r(\xi,t) = 20 + sin(t)sin(2\pi\xi)$ is considered, which meets the actual BCs. A space- and time-varying disturbance term is considered in the form:

$$\psi(\xi,t) = 5sin(2\pi\xi)cos(t) \tag{52}$$

such that the augmented disturbance (10) now specializes to the form:

$$\eta(\xi,t) = 20 + 2sin(4\pi\xi) + 5sin(2\pi\xi)cos(t) \tag{53}$$

The L_2 norm bounds M and M_ξ of the disturbance derivatives η_t and $\eta_{t\xi}$, which are involved in the controller tuning inequalities (22) can be easily estimated as $M = 132$ and $M_\xi = 2500$, and the constant M_x is selected as $M_x = 132$. Then, the DSTPI controller gains are set in accordance with (22) to the values

$$W_1 = 1520, \quad \lambda_1 = 264, \quad W_2 = 2, \quad \lambda_2 = 2 \tag{54}$$

For solving the PDEs governing the closed-loop system, standard finite-difference approximation method is used by discretizing the spatial solution domain $\xi \in [0,1]$ into a finite number of N uniformly spaced solution nodes $\xi_i = ih$, $h = 1/(N+1)$, $i = 1, 2, ..., N$. The value $N = 40$ has been used in the present simulations. The resulting 40-th order discretized system is implemented in Matlab-Simulink and solved by fixed-step Euler integration method with constant step $T_s = 10^{-4}$.

Figure 1 depicts the solution $Q(\xi,t)$, which converges to the given set-point, and Figure 2 shows the time profile of the tracking error L_2 norm $\|x(\cdot,t)\|_2$. Figure 3 depicts the control input $u(\xi,t)$ which, as expected, appears to be a smooth function of both time and space. The attained results confirm the validity of the presented analysis.

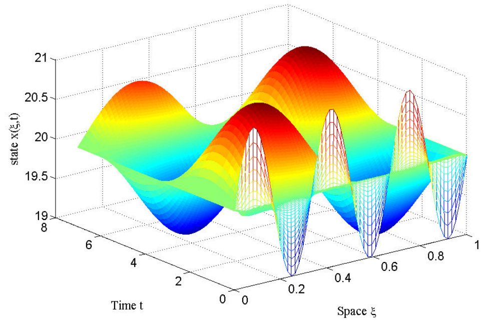

Fig. 1. The solution $Q(\xi,t)$.

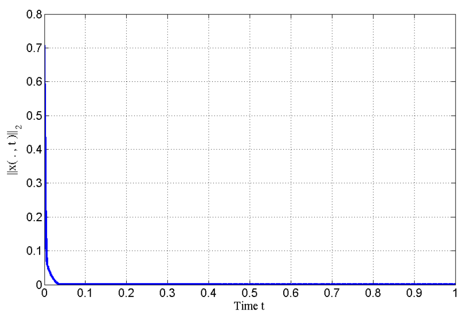

Fig. 2. The tracking error L_2 norm $\|x(\cdot,t)\|_2$.

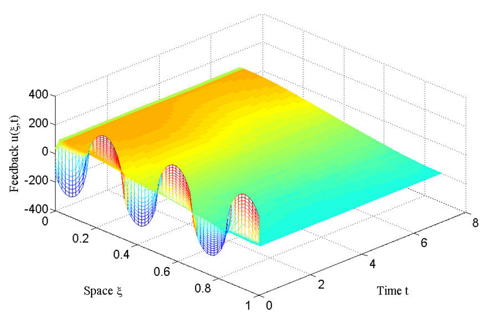

Fig. 3. The distributed control $u(\xi,t)$.

V. Concluding remarks

The "Super-Twisting" 2-SMC algorithm has been used in conjunction with linear PI control in a distributed parameters setting involving a a class of uncertain infinite-dimensional processes. The tracking control problem for a class of diffusion-reaction dynamics with spatially-varying parameters and mixed boundary conditions, subject to a persistent smooth disturbance of arbitrary shape, is tackled. By means of Lyapunov functional analysis, the stability in the L_2 space of the resulting error dynamics is demonstrated. Finite-time convergence of the proposed algorithm, which would be the case whenever confined to a finite dimensional treatment, cannot be proved using the proposed Lyapunov functional, and it remains among other problems to be tackled in the future within the present framework.

References

[1] Bartolini, G., Levant, A., Pisano, A., & Usai, E. (2002) Higher-Order Sliding Modes for Output-Feedback Control of Nonlinear Uncertain Systems. In: Yu X, Xu J.X. (eds), Variable Structure Systems: towards the 21^{st} century, Lecture Notes in Control and Information Science Series. vol. 274, 83-108, Berlin: Springer Verlag.

[2] Christofides, P.D. (2001) *Nonlinear and Robust Control of PDE Systems. Methods and Applications to Transport-Reaction Processes.* Systems & Control: Foundations & Applications series, Boston: Birkh*ä*user.

[3] Curtain, R. & Zwart, H. (1995) *An Introduction To Infinite-dimensional Linear Systems Theory.* Texts In Applied Mathematics. Berlin: Springer Verlag

[4] Demetriou, M.A., Ito, K., & Smith, R.C. (2009) Adaptive techniques for the MRAC, adaptive parameter identification, and on-line fault monitoring and accommodation for a class of positive real infinite dimensional systems, *Int. J. Adaptive Contr. Signal Proc.*, **23**, 2, 193–215.

[5] Fridman, L., & Levant, A. (1996) Higher order sliding modes as a natural phenomenon in control theory. In: Garofalo F, Glielmo L (eds). Robust control via variable structure and Lyapunov techniques. Lecture Notes in Control and Information Science. Berlin: Springer.

[6] Fridman, E. & Orlov, Y (2009) An LMI approach to H_∞ boundary control of semilinear parabolic and hyperbolic systems, *Automatica*, **45**, 9, 2060–2066.

[7] Filippov, A.F. (1988) *Differential Equations with Discontinuous Right-Hand Side.* Dordrecht: Kluwer. t control via variable structure and Lyapunov techniques. Lecture Notes in Control and Information Science Series. Berlin: Springer Verlag.

[8] Khalil, H. (2002) *Nonlinear systems, 3rd ed* Upper Saddle River: Prentice Hall.

[9] Krstic, M., & Smyshlyaev, A. (2008) Adaptive control of PDEs, *Annual Reviews of Control,* **32**, 149–160.

[10] Levaggi, L. (2002) Sliding modes in banach spaces, *Diff. Integr. Equat.,* **15**, 167–189.

[11] Levant,Al. (1993) Sliding order and sliding accuracy in sliding mode control, *Int.J.Contr.,* **58**, 1247-1263.

[12] Moreno, J. & Osorio M., (2009) *A Lyapunov approach to second-order sliding mode controllers and observers.* Proc 47 Conference on Decision and Control (CDC 2008), 2856–2861, December 2008, , Cancun, Mexico.

[13] Orlov, Y., & Utkin, V.I. (1987) Sliding mode control in infinite-dimensional systems, *Automatica,* **23**, 753-757.

[14] Orlov, Y. (2009) *Discontinuous Systems Lyapunov Analysis and Robust Synthesis under Uncertainty Conditions.* Communications and Control Engineering Series, Berlin: Springer Verlag.

[15] Orlov, Y. (2000) Discontinuous unit feedback control of uncertain infinite-dimensional systems, *IEEE Trans. Aut. Contr.,* **45**, 12, 834-843.

[16] Orlov, Y., Liu, Y. & Christofides, P.D. (2004) Robust stabilization of infinite-dimensional systems using sliding-mode output feedback control, *Int. J. Contr.,* **77**, 12, 1115-1136.

[17] Orlov, Y.,Pisano, A. & Usai, E. (2011b) Exponential stabilization of the uncertain wave equation via distributed dynamic input extension, *IEEE Trans. Aut. Contr.*, **56**, 1, 212–217.

[18] Orlov, Y.,Pisano, A. & Usai, E. (2010) Continuous state-feedback tracking of an uncertain heat diffusion process, *Syst. Contr. Lett.*, **59**, 12, 754–759.

[19] Pisano, A., Orlov, Y. & Usai, E. (2011a) Tracking Control of the Uncertain Heat and Wave Equation via Power-Fractional and Sliding-Mode Techniques, *SIAM J. Contr. Optimizat.,* **49**, 363–382.

[20] Utkin, V.I. (1972) *Sliding Modes In Control And Optimization.* Berlin: Springer Verlag