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Observer design for systems with an energy preserving nonlinearity, with application to fluid flow

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***Abstract*— In this paper observer design is considered for a class of non-linear systems whose non-linear part is energy preserving. Examples of such systems arise from considering finite dimensional approximations of fluid flows. A strategy to construct convergent observers for this class of non-linear system is presented. The approach has the advantage that it is possible, via convex programming, to prove whether the constructed observer converges, in contrast to several existing approaches to observer design for non-linear systems. Finally the method is used to produce a globally convergent observer for the Lorenz attractor.**

I. INTRODUCTION

Observer design for non-linear systems is an important and difficult problem. In this paper, observer design is considered for a class of non-linear dynamical systems which are closely related to fluid flows. In particular,

$$\begin{array}{rcll} \dot{x}(t) &=& Ax(t)+N(x(t))x(t), & t\geq 0, \\ y(t) &=& Cx(t), & t\geq 0, \\ x(0) &=& x_0\in\mathbb{R}^n. \end{array} \qquad (1)$$

where $A\in\mathbb{R}^{n\times n}$, $C\in\mathbb{R}^{p\times n}$ and $N:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ is a linear operator. Furthermore, it is assumed that the non-linearity $N(x)x$ has the *energy preserving* property

$$x^TN(x)x=0, \qquad x\in\mathbb{R}^n. \qquad (2)$$

The importance of dynamical systems of the form (1) is that they arise in relation to fluid flows, when the Navier-Stokes equations are approximated by a finite dimensional system [4]. In experimental practice, such an approximation is referred to as a 'low order model' and can be created directly from experimental data by using, for example, the method of Proper Orthogonal Decomposition [6], [15]. From a theoretical perspective, it is therefore of great interest to study the control theoretic properties of such systems, with a view to guiding experimental implementation. The link between the Navier-Stokes equations and (1) is presented in Section III.

Observer design for non-linear systems has received much attention, with approaches falling into two main categories. One approach, first considered in [12] and generalized in [7], [8], [10], [11], is to apply a change of co-ordinates to linearize the system, up to an additional term involving the output $y(t)$. Subsequently, linear design methods can be applied to create an observer for the transformed system, then the co-ordinate transformation is inverted to form an observer for the original, non-linear, system. The main drawback of this approach is that it is usually impossible to prove that the chosen co-ordinate transformation is invertible. Hence, while this is a powerful technique for observer design, it is difficult to prove in practice that the constructed observer will actually converge.

The second approach is to assume a Lipschitz-type bound on the nonlinear part of the system. For example, in addition to the standard Lipschitz assumption [1], [14], one-sided Lipschitz conditions [5], [16] and a 'less conservative' Lipschitz condition [13] have been studied. These techniques apply a Luenberger-type observer and require that the non-linearity is 'small enough' with respect to the linear part of the dynamics. A major drawback of this approach is that systems with a dominant non-linear term often have a large Lipschitz bound and if this is the case, it is unlikely to be possible to prove that a given observer converges.

The difficulties of the above techniques arise either from excessive generality or overly restrictive assumptions. The co-ordinate transformation technique may theoretically be applied to *any* non-linear system, and is therefore unlikely to succeed in every case. For the Lipschitz approaches, a small global Lipschitz bound restricts the class of systems to which the results may be applied. For this reason, we aim for an approach to observer design that sits between these two extremes by only considering the particular class of non-linear system (1).

The following notation will be used. A matrix $P\in\mathbb{R}^{n\times n}$ is said to be *positive definite* (written $P\succ 0$) if its symmetric part satisfies $x^T(P+P^T)x>0$, for any $x\in\mathbb{R}^n$, and *negative definite* if $-P$ is positive definite (written $P\prec 0$). The set of symmetric matrices of dimension n is denoted $\mathbb{S}^n$. For matrices A,B and C of appropriate sizes, the shorthand

$$\left[\begin{array}{cc} A & B \\ (*) & C \end{array}\right] := \left[\begin{array}{cc} A & B \\ B^T & C \end{array}\right]$$

is used to simplify the block matrix. For $r>0$ and $d\in\mathbb{R}^n$, the $\|\cdot\|_2$-norm ball centered at d of radius r is denoted $B_r(d)$. For $i=1,\dots,n$,

$$\boldsymbol{e}_i=\underbrace{(0,\dots 0,1,0,\dots,0)^T}_{i^{\text{th}}\text{ entry}}$$

denotes the i^{th} element of the standard basis of $\mathbb{R}^n$. For sets

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$S, T \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

$$S + T := \{s + t : s \in S, t \in T\}, \quad \alpha S := \{\alpha s : s \in S\}.$$

II. Observer Design

The approach taken in this paper is to exploit the bilinear and energy preserving properties of the non-linearity in (1) to obtain a method for constructing a convergent observer. In particular, for a given gain matrix $L \in \mathbb{R}^{n\times p}$, the observer $(\hat{x}(t))_{t\geq 0}$ is assumed to have dynamics

$$\dot{\hat{x}} = A\hat{x} + N(\hat{x})\hat{x} - L(y - C\hat{x}), \qquad \hat{x}(0) = \hat{x}_0 \in \mathbb{R}^n. \quad (3)$$

Therefore, the observer error $e := x - \hat{x}$ satisfies

$$\dot{e} = (A + LC)e + N(x)x - N(\hat{x})\hat{x}. \quad (4)$$

The aim of this paper is to find a constructive method of calculating L such that

$$e(t) \to 0, \qquad t \to \infty.$$

The main results, Algorithm 2.4, Theorem 2.5 and Theorem 2.7, provide methods of constructing such a gain L by solving a series of convex optimization problems.

The property of the non-linear system (1) that is advantageous for observer design is that the energy preserving property (2) implies the existence of an *invariant set* for the system dynamics. A set $S \subset \mathbb{R}^n$ is said to be invariant for the dynamical system (1) if $x(t_0) \in S$ at time $t_0 \geq 0$ implies that $x(t) \in S$ for every subsequent time $t \geq t_0$. Invariant sets can be described in terms of perturbations of the linear part A of the system. In the following, given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $d \in \mathbb{R}^n$ define a perturbed matrix $A_d \in \mathbb{R}^{n\times n}$ by

$$A_d x := Ax + N(x)d + N(d)x, \qquad x \in \mathbb{R}^n.$$

Subsequently, we make the following assumption.

(A1) There exists $d \in \mathbb{R}^n$ such that $A_d \prec 0$.

Clearly, assumption (A1) holds if $A \prec 0$. Furthermore, it is shown in Proposition 3.1 that (A1) holds for the class of systems representing finite dimensional approximations of fluid flows.

Lemma 2.1: Suppose that there exist $d \in \mathbb{R}^n$ and $\alpha > 0$ such that $A_d + \alpha I \prec 0$. Then $B_r(d)$ is invariant for (1), with

$$r = \frac{1}{\alpha}\|Ad + N(d)d\|_2. \quad (5)$$

Proof: Using the linearity of $N : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, the system's dynamics can be written

$$\begin{aligned}\dot{x} &= Ax + N(x-d)(x-d) + N(x)d + N(d)x - N(d)d \\ &= A_d(x-d) + N(x-d)(x-d) + A_d d - N(d)d.\end{aligned}$$

For $D(x) := \frac{1}{2}\|x - d\|_2^2$, the energy preserving property (2) implies that

$$\begin{aligned}\dot{D}(x) &= (x-d)^T A_d(x-d) + (x-d)^T(Ad + N(d)d). \\ &\leq -\alpha\|x-d\|_2^2 + \|x-d\|_2\|Ad + N(d)d\|_2.\end{aligned}$$

Therefore, $\dot{D}(x) < 0$ whenever $\|x - d\|_2 > r$, with r given by (5). Hence, $B_r(d)$ is invariant for (1). ∎

If (1) represents a fluid system, an invariant set may be calculated more explicitly, as described in Proposition 3.1 and Corollary 3.2. In general, since A_d is affine in d, the condition $\{d : A_d + A_d^T \prec 0\} \neq \emptyset$ can be checked by solving a semidefinite program [2].

Ideally, one would like to calculate an invariant ball with the smallest possible radius. However, due to the non-linear dependence of (5) upon $(\alpha, d) \in \mathbb{R} \times \mathbb{R}^n$, it is difficult to minimize (5) by convex optimization methods. In order to remove the non-linear dependence upon d from (5), the search can be restricted to vectors d such that $N(d)d = 0$. To do this, select a matrix $Q \in \mathbb{R}^{n\times n}$ with the property

$$d \in \ker(Q) \Rightarrow N(d)d = 0. \quad (6)$$

For example, let $Q^{(i)}_{jk} := N(\boldsymbol{e}_j)_{ik}$, for each $i, j, k \in \{1, \dots, n\}$. Then

$$N(d)d = \left(d^T Q^{(1)} d, \dots, d^T Q^{(n)} d\right)^T$$

and hence, if Q is chosen such that $\ker(Q)$ is any of the spaces

$$\bigcap_{i=1}^{n} \ker(\Phi_j^{(i)}), \qquad \Phi_j^{(i)} \in \left\{Q^{(i)}, Q^{(i)^T}\right\},$$

then Q satisfies (6).

Depending upon the system under consideration, it may be possible to find Q_j such that $\bigcup_j \ker(Q_j) = \{d : N(d)d = 0\}$, as is shown for the Lorenz attractor in Section IV. Even if this is not the case, a natural choice for Q may be apparent given the system's underlying structure – see Section III. The advantage of this assumption is that if the search for the centre of an invariant set is conducted over $\ker(Q)$, it can be performed by solving a semidefinite program. Subsequently, it will be assumed that:

(A2) A matrix $Q \in \mathbb{R}^{n\times n}$ is chosen such that $N(d)d = 0$ whenever $d \in \ker(Q)$.

Proposition 2.2: Suppose that the semidefinite program

$$\begin{aligned}\text{minimize} \quad & s \\ \text{subject to} \quad & \begin{bmatrix} s & (Ax)^T \\ (*) & sI_n \end{bmatrix} \succeq 0 \\ & tA + (A_x - A) + I \preceq 0 \\ & t \geq 0 \\ & Qx = 0\end{aligned} \quad (7)$$

with variables $s, t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ has optimal values (s^*, t^*, x^*) with $t^* > 0$. Then $B_{s^*}(x^*/t^*)$ is an invariant set for $(x(t))_{t\geq 0}$,

$$\sigma^* := \inf_{\substack{\lambda > 0 \\ d \in \mathbb{R}^n}} \left\{ \frac{\|Ad + N(d)d\|_2}{\lambda} : A_d + \lambda I \prec 0, d \in \ker(Q) \right\} < \infty \quad (8)$$

and $s^* = \sigma^*$.

Proof: Suppose that (s^*, t^*, x^*) are optimal values of

(7) and $t^* > 0$. Then since,

$$A_{(x^*/t^*)} + I/t^* \preceq 0, \qquad \|Ax^*\|_2 \leq s^*, \tag{9}$$

Lemma 2.1 implies that $B_{s^*}(x^*/t^*)$ is invariant for $(x(t))_{t\geq 0}$. Now suppose that $(\lambda, d) \in \mathbb{R}_+ \times \mathbb{R}^n$ satisfy the minimization constraints in (8). Since

$$A/\lambda + (A_{(d/\lambda)} - A) + I \preceq 0, \qquad Qd = 0,$$

it follows that

$$s^* \leq \|A(d/\lambda)\|_2 = \frac{\|Ad + N(d)d\|_2}{\lambda}.$$

Therefore, $s^* \leq \sigma^*$. Conversely, (9) implies that

$$\sigma^* \leq t^* \left\| \frac{Ax^*}{t^*} \right\|_2 = \|Ax^*\|_2 \leq s^*.$$

Hence, $s^* = \sigma^*$. ■

Given an invariant set for the state and an observer gain L, the following result provides a sufficient condition for the existence of an invariant set for the observer $(\hat{x}(t))_{t\geq 0}$.

Lemma 2.3: Suppose that $B_r(d)$ is invariant for (1) and that $x_0 \in B_r(d)$. If there exist $\hat{d} \in \mathbb{R}^n$ and $\alpha > 0$ such that $A_{\hat{d}} + LC + \alpha I \prec 0$, then $B_{\hat{r}}(\hat{d})$ is invariant for $(\hat{x}(t))_{t\geq 0}$ under the dynamics (3), where

$$\hat{r} = \frac{1}{\alpha} \sup_{v \in B_r(d - \hat{d})} \left\| LCv - A\hat{d} - N(\hat{d})\hat{d} \right\|_2. \tag{10}$$

Proof: Suppose that $\hat{d} \in \mathbb{R}^n$ is such that $A_{\hat{d}} + LC + \alpha I \prec 0$. Then,

$$\begin{aligned}\dot{\hat{x}} = (A_{\hat{d}} + LC)(\hat{x} - \hat{d}) + N(\hat{x} - \hat{d})(\hat{x} - \hat{d}) \\ - (LC(x - \hat{d}) - A\hat{d} - N(\hat{d})\hat{d}).\end{aligned}$$

If $D(\hat{x}) := \frac{1}{2}\|\hat{x} - \hat{d}\|_2^2$, then

$$\begin{aligned}\dot{D}(\hat{x}) &= (\hat{x} - \hat{d})^T (A_{\hat{d}} + LC)(\hat{x} - \hat{d}) \\ &\quad - (\hat{x} - \hat{d})^T (LC(x - \hat{d}) - A\hat{d} - N(\hat{d})\hat{d}) \\ &< -\alpha \|\hat{x} - \hat{d}\|_2^2 \\ &\quad - (\hat{x} - \hat{d})^T (LC(x - \hat{d}) - A\hat{d} - N(\hat{d})\hat{d}).\end{aligned}$$

By assumption, $x(t) \in B_r(d)$, for any time $t \geq 0$. Therefore,

$$x(t) - \hat{d} \in B_r(d - \hat{d}), \qquad t \geq 0.$$

The result follows since $\dot{D}(\hat{x}) < 0$ whenever $\hat{x} \in B_{\hat{r}}(\hat{d})$, with $\hat{r}$ given by (10). ■

To study observer convergence, it is useful to rewrite the nonlinear part of the observer error dynamics (4). For $x, \hat{x} \in \mathbb{R}^n$,

$$\begin{aligned}N(x)x - N(\hat{x})\hat{x} &= N(x - \hat{x})x - N(\hat{x})\hat{x} + N(\hat{x})x \\ &= N(x - \hat{x})x + N(\hat{x})(x - \hat{x}) \\ &= N(e)x + N(x)e - N(e)e \\ &= N(e)(x - e/2) + N(x - e/2)e \\ &= N(e)((x + \hat{x})/2) + N((x + \hat{x})/2)(e) \\ &= (A_{\frac{x+\hat{x}}{2}} - A)e.\end{aligned} \tag{11}$$

Hence, the error dynamics (4) may be written

$$\dot{e} = \left(A_{\frac{x+\hat{x}}{2}} + LC\right)e. \tag{12}$$

The observer error dynamics can therefore be considered as a linear time varying system, and the problem of observer design is to find a gain L which stabilizes (12). The proposed strategy for constructing a convergent observer is split into two stages: observer design; and convergence certificate. This strategy, stated formally as Algorithm 2.4 and Theorem 2.5 below, can be described as follows.

Observer design: Use Proposition 2.2 to calculate an invariant set $B_r(d)$ for the state $(x(t))_{t\geq 0}$. For the observer $(\hat{x}(t))_{t\geq 0}$ to converge to the state, it is sensible to assume that its invariant set contains $B_r(d)$. Hence, we search for a gain L which stabilizes (12) under the assumption

$$\frac{1}{2}(x(t) + \hat{x}(t)) \in S, \qquad t \geq 0, \tag{13}$$

for a pre-defined set $S \supseteq B_r(d)$.

To ensure that Lemma 2.3 can be used to construct an invariant set for the observer, check that $\{\alpha > 0 : \exists x \in \mathbb{R}^n \text{ s.t. } A_x + LC + \alpha I \prec 0\} \neq \emptyset$. The tuning parameters α_1, α_2 are included in this stage, to provide control over $\|LC\|_2$. By Lemma 2.3, $\|LC\|_2$ influences the radius of the invariant set calculated for $(\hat{x}(t))_{t\geq 0}$ in the second stage.

Convergence certificate: Once L has been calculated, we wish to check whether the observer error converges to zero. Use Lemma 2.3 to calculate an invariant set $B_{\hat{d}}(\hat{r})$ for $(\hat{x}(t))_{t\geq 0}$. Consequently,

$$\frac{1}{2}(x(t) + \hat{x}(t)) \in B_{\frac{r+\hat{r}}{2}}\left((d + \hat{d})/2\right), \qquad t \geq 0. \tag{14}$$

Hence, $e(t) \to 0$ if it can be verified that (12) is stable under the assumption (14).

Algorithm 2.4 (Observer design):

(i) Use Proposition 2.2 to select $d \in \mathbb{R}^n, r > 0$ such that $B_r(d)$ is invariant for (1).

(ii) Select $\alpha_1, \alpha_2 \geq 0$ and pick $(y_i)_{i=1}^N \subset \mathbb{R}^n$ such that $B_r(d) \subset \text{conv}\{y_i : i = 1 \dots N\}$. Calculate positive definite $P \in \mathbb{S}^n$ and $R \in \mathbb{R}^{n\times p}$ such that:

$$P - \alpha_1 I \succ 0; \tag{15}$$

$$\begin{bmatrix} \alpha_2 I_n & RC \\ (*) & \alpha_2 I_p \end{bmatrix} \succeq 0; \tag{16}$$

$$PA_{y_i} + A_{y_i}^T P + RC + C^T R^T \prec 0, \tag{17}$$

for each $i = 1 \dots N$. Define $L := P^{-1}R$.

(iii) Solve the semidefinite program

$$\alpha^* = \max\{\alpha \in \mathbb{R} : \exists x \in \mathbb{R}^n \text{ s.t. } A_x + LC + \alpha I \prec 0\}. \tag{18}$$

Step (iii) of Algorithm 2.4 is a check to determine whether it is possible to use Lemma 2.3 to calculate an invariant set for the observer $(\hat{x}(t))_{t\geq 0}$. If a prospective candidate for L is found for which $\alpha^* > 0$, the following result may be used to verify observer convergence.

Theorem 2.5 (Convergence certificate): Suppose that Algorithm 2.4 has been completed, providing an invariant set

$B_r(d)$ for $(x(t))_{t\geq 0}$, a candidate $L = P^{-1}R$ for the observer gain and $\alpha^* > 0$. Assume that the observer $(\hat{x}(t))_{t\geq 0}$ satisfies the dynamics (3) and that $x_0 \in B_r(d)$. Suppose that $(s^*, t^*, \xi^*) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$ is an optimal solution to

$$\begin{aligned} &\text{minimize} \quad s \\ &\text{subject to} \quad \begin{bmatrix} s - tr\|LC\|_2 & (LC(td-\xi) - A\xi)^T \\ (*) & (s - tr\|LC\|_2)I_n \end{bmatrix} \succeq 0 \qquad (19) \\ &\qquad t(A+LC) + (A_\xi - A) + I \preceq 0 \qquad (20) \\ &\qquad Q\xi = 0, \qquad i = 1, \ldots, n, \qquad (21) \end{aligned}$$

with variables $(s, t, \xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$. Then

(i) If $t^* > 0$, the set $B_{\hat{r}}(\hat{d})$ is invariant for $(\hat{x}(t))_{t\geq 0}$, for $\hat{r} := s^*$, $\hat{d} := \xi^*/t^*$.

(ii) If $(\hat{y}_i)_{i=1}^N \subset \mathbb{R}^n$ are such that

$$B_{\frac{r+\hat{r}}{2}}((d+\hat{d})/2) \subset \operatorname{conv}\{\hat{y}_i : i = 1, \ldots, N\}$$

and there exists positive definite $\hat{P} \in \mathbb{S}^n$ such that

$$\hat{P}(A_{\hat{y}_i} + LC) \prec 0, \qquad i = 1, \ldots, N, \qquad (22)$$

the observer error satisfies $e(t) \to 0, t \to \infty$, for any initial condition $\hat{x}_0 \in B_{\hat{r}}(\hat{d})$.

Proof: (i) By (20), (21) and Lemma 2.3, $B_{\tilde{r}}(\xi^*/t^*)$ is invariant for $(\hat{x}(t))_{t\geq 0}$ with

$$\tilde{r} = \sup_{v \in B_r(d - \xi^*/t^*)} t^* \left\| LCv - A(\xi^*/t^*) \right\|_2 .$$

By (19),

$$\begin{aligned} \tilde{r} &= \sup_{v \in B_r(0)} t^* \|LCv + LC(d - \xi^*/t^*) - A(\xi^*/t^*)\|_2 \\ &\leq t^* r \|LC\|_2 + \|LCt^*d - (A+LC)\xi^*\|_2 \\ &\leq s^*. \end{aligned}$$

Hence, $B_{\hat{r}}(\hat{d})$ is invariant for $(\hat{x}(t))_{t\geq 0}$, for $\hat{r} = s^*$ and $\hat{d} = \xi^*/t^*$.

(ii) Suppose that $x_0 \in B_r(d)$ and $\hat{x}_0 \in B_{\hat{r}}(\hat{d})$. Then by invariance,

$$\begin{aligned} \frac{1}{2}(x(t) + \hat{x}(t)) &\in \frac{1}{2}\left(B_r(d) + B_{\hat{r}}(\hat{d})\right) \\ &\subset B_{\frac{r+\hat{r}}{2}}\left((d+\hat{d})/2\right) \\ &\subset \operatorname{conv}\{\hat{y}_i : i = 1, \ldots, N\}, \qquad t \geq 0. \end{aligned}$$

It follows from (12) and (22) that $e(t) \to 0, t \to \infty$. ■

Remark 2.6: The choice of vectors $(y_i)_{i=1}^N$ and $(\hat{y}_i)_{i=1}^N$ may be influenced by the dimension of the system under consideration. For example, one possible choice is a cube in $\mathbb{R}^n$ of smallest side-length containing the ball $B_r(d) \subset \mathbb{R}^n$, which requires $N = 2^n$. However, for large dimensions n, solving the associated semidefinite program may be computationally intractable. In this case, a more conservative bounding set may be used. For example, the $\|\cdot\|_1$-norm ball

$$B_r(d) \subset \operatorname{conv}\{d \pm \sqrt{n} r \boldsymbol{e}_i : i = 1, \ldots, n\}$$

containing $B_r(d)$ requires $N = 2n$ points. A simplex, which can be represented as the convex hull of $N = n+1$ points in $\mathbb{R}^n$, can also be used.

Ideally, the search for an observer gain L and invariant set $B_{\hat{r}}(\hat{d})$ would be performed simultaneously. The reason for this is that the radius of the ball in (14) depends upon $\hat{r}$, which in turn depends upon LC via (10). Due to the complexity introduced by the system's non-linearity, this is difficult to achieve if the search for P in Algorithm 2.4 is conducted over $\mathbb{S}^n$. However, if the search is taken over a particular subset of $\mathbb{S}^n$, defined in terms of the non-linearity N, it is possible to remove the need to find $B_{\hat{r}}(\hat{d})$. Define

$$\mathbb{S}^n_N := \{P \in \mathbb{S}^n : e^T P N(e) e = 0, \qquad e \in \mathbb{R}^n\}.$$

Since the energy preserving property (2) holds, it is the case that $\mathbb{S}^n_N \neq \emptyset$. Notice also that, since $e^T P N(e) e$ is linear in P, it is easy to calculate $\mathbb{S}^n_N$ for a given non-linearity N. The following result provides conditions for global observer convergence.

Theorem 2.7: Suppose that $B_r(d)$, calculated by Proposition 2.2, is invariant for (1) and let $x_0 \in B_r(d)$. Pick $(y_i)_{i=1}^N \subset \mathbb{R}^n$ such that $B_r(d) \subset \operatorname{conv}\{y_i, i = 1, \ldots, N\}$ and suppose that there exist positive definite $P \in S^n_N$ and $R \in \mathbb{R}^{n \times p}$ such that

$$PA_{y_i} + RC \prec 0, \qquad i = 1, \ldots, N. \qquad (23)$$

Then if $L := P^{-1}R$, the observer $(\hat{x}(t))_{t\geq 0}$ defined by (3) satisfies $e(t) \to 0$, $t \to \infty$, for any initial condition $\hat{x}_0 \in \mathbb{R}^n$.

Proof: Let $V(e) = e^T P e$. Then by (11) and (12),

$$\begin{aligned} \nabla V \cdot \dot{e} &= e^T P(A_{\frac{x+\hat{x}}{2}} + LC)e \\ &= e^T P(A + LC)e \\ &\qquad + e^T P(N(x)e + N(e)x - N(e)e) \\ (P \in \mathbb{S}^n_N) &= e^T P(A_x + LC)e. \\ (\text{by (23)}) &< 0, \qquad e \in \mathbb{R}^n. \end{aligned}$$

Hence, $e(t) \to 0, t \to \infty$ and since $\hat{x}$ does not appear in the expression for $\nabla V \cdot \dot{e}$, the observer error converges to zero for any initial condition $\hat{x}_0 \in \mathbb{R}^n$. ■

III. Observation for fluid systems

Finite dimensional approximations of fluid flows satisfying the Navier-Stokes equations can be represented in the form (1), and consequently observers for fluid systems may designed by the methods in Section II. For $\boldsymbol{u} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$, the incompressible Navier-Stokes equations are

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -\nabla p + \frac{1}{R}\nabla^2 \boldsymbol{u} + \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned}$$

where $p : \Omega \to \mathbb{R}$ represents the pressure, $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ is an external force and R the Reynold's number of the flow. A common assumption [6], [15] is that the flow field can be decomposed in the form

$$\boldsymbol{u}(x,t) = \sum_{i=1}^{\infty} a_i(t) \boldsymbol{u}_i(x),$$

and a finite dimensional approximation of the flow obtained by considering the truncation $\boldsymbol{u} = \sum_{i=1}^{n} a_i \boldsymbol{u}_i$.

A set of ordinary differential equations for the time-dependent coefficients a_i can be obtained via the method of Galerkin projection (see e.g. [3]), leading to

$$\dot{a}_i = \frac{\langle \boldsymbol{f}, \boldsymbol{u}_i \rangle}{\|\boldsymbol{u}_i\|^2} - \frac{\lambda_i}{R} a_i + \sum_{j,k} \frac{a_j a_k}{\|\boldsymbol{u}_i\|^2} \langle (\boldsymbol{u}_j \cdot \nabla)\boldsymbol{u}_k, \boldsymbol{u}_i \rangle, \quad (24)$$

for each $i = 1, \ldots, n$, where $\lambda_i > 0$ are fixed constants. To remove the constant term from (24) it is assumed that (24) has a known stationary point $a = c$. Making the transformation $x = a - c$, the perturbations about c have dynamics

$$\dot{x} = Ax + N(x)x, \quad (25)$$

where

$$Ax := \frac{1}{R}\Lambda x + N(c)x + N(x)c, \qquad x \in \mathbb{R}^n,$$

for $\Lambda \prec 0$ and $N : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ has the form

$$N(x) = \sum_{i=1}^{n} x_i Q^{(i)}, \qquad x \in \mathbb{R}^n.$$

Here, the matrices $Q^{(i)} \in \mathbb{R}^{n \times n}$ are anti-symmetric and satisfy the additional property

$$Q_{ij}^{(i)} = 0, \qquad i, j = 1, \ldots, n. \quad (26)$$

As a consequence, N has the energy preserving property (2). This additional structure allows the calculation of an invariant set for the fluid system.

Proposition 3.1: For the dynamics (25) with stationary point $c \in \mathbb{R}^n$, there exists $r > 0$ such that $B_r(-c)$ is invariant for (25).

Proof: Note that $A_{-c} = \frac{1}{R}\Lambda \prec 0$. By Proposition 2.1 it follows that $\|x + c\|_2^2$ is decreasing if

$$\frac{1}{R}(x+c)^T \Lambda (x+c)^T + (x+c)^T \left(-\frac{1}{R}\Lambda c + N(c)c \right) < 0.$$

Standard algebraic manipulation shows that the set of $x \in \mathbb{R}^n$ for which the above inequality holds is equal to $\mathbb{R}^n \backslash E$, where E is the ellipsoid

$$E := \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i \left(x_i - \frac{R}{2\lambda_i}(\Lambda c + N(c)c)_i \right)^2 \right.$$
$$\left. \leq \sum_{i=1}^{n} \frac{R^2}{4\lambda_i}(\Lambda c + N(c)c)_i^2 \right\}$$

The result follows if $r > 0$ is chosen such that $E \subset B_r(-c)$. ■

In order to more easily calculate the invariant set for a fluid flow, it is convenient to select the basis functions $\boldsymbol{u}_i$ appropriately. For example, $\boldsymbol{u}_1$ may be chosen [3] to coincide with the laminar solution to the flow. In this case $c = \boldsymbol{e}_1$.

Corollary 3.2: Suppose that $c = \boldsymbol{e}_1$. Then the ball $B_{\sqrt{\lambda_1/\lambda_{\min}}}(-c)$ is invariant for (25).

Proof: By (26), $N(c)c = 0$. Applying Proposition 3.1, the associated ellipsoid is

$$E := \left\{ x \in \mathbb{R}^n : \lambda_1 \left(x_1 + \frac{1}{2} \right)^2 + \sum_{i=1}^{n} \lambda_i x_i^2 \leq \frac{\lambda_1}{4} \right\}$$
$$\subset B_{\sqrt{\lambda_1/\lambda_{\min}}}(-c).$$

■

Consequently, Algorithm 2.4, Theorem 2.5 or Theorem 2.7 can be applied with state invariant set $B_r(d) = B_{\lambda_1/\lambda_{\min}}(-\boldsymbol{e}_1)$. Even if it is not possible to prove global observer convergence, the structure of the non-linearity N for fluid systems implies that if Algorithm 2.4 can be completed, the resulting observer is locally convergent.

Theorem 3.3: Suppose that $B_r(d)$ is invariant for $(x(t))_{t \geq 0}$, let $(y_i)_{i=1}^{N}$ be such that $B_r(d) \subset \operatorname{conv}\{y_i : i = 1, \ldots, N\}$ and assume there exist $P \in \mathbb{S}^n$ and $R \in \mathbb{R}^{n \times p}$ such that

$$PA_{y_i} + RC \prec 0, \qquad i = 1, \ldots, n.$$

Then if $(\hat{x}(t))_{t \geq 0}$ has dynamics (3) for $L := P^{-1}R$, there exists $\rho > 0$ such that $\|x_0 - \hat{x}_0\|_2 < \rho$ implies $e(t) \to 0, t \to \infty$.

Proof: Consider the error dynamics as a time varying system by writing

$$\dot{e} = f(t, e) := (A_{x(t)} + LC)e - N(e)e, \qquad t \geq 0.$$

Then the linearized error dynamics are

$$\left. \frac{\partial f}{\partial e} \right|_{e=0} = A_{x(t)} + LC.$$

The result follows from [9, Theorem 3.11], if it can be shown the Jacobian $J(t, e) := \partial f / \partial e$ satisfies

$$\|J(t, e) - J(t, \tilde{e})\|_2 \leq L\|e - \tilde{e}\|_2, \qquad e, \tilde{e} \in \mathbb{R}^n, t \geq 0,$$

for some constant $L > 0$. By property (26),

$$\left(\frac{\partial N(e)e}{\partial e} \right)_{ij} = \sum_{k=1}^{n} e_k Q_{ij}^{(k)}, \qquad i, j = 1, \ldots, n.$$

Hence, for any $e, \tilde{e} \in \mathbb{R}^n$ and $t \geq 0$,

$$\begin{aligned} &\|J(t, e) - J(t, \tilde{e})\|_2 \\ &\quad = \left\| \sum_{k=1}^{n} (e_k - \tilde{e}_k) Q_{ij}^{(k)} \right\|_2 \\ &\quad \leq \sqrt{n} \left\| \sum_{k=1}^{n} (e_k - \tilde{e}_k) Q_{ij}^{(k)} \right\|_1 \\ &\quad = \sqrt{n} \max_{1 \leq j \leq n} \sum_{i=1}^{n} \sum_{k=1}^{n} |e_k - \tilde{e}_k| |Q_{ij}^{(k)}|, \\ &\quad \leq \sqrt{n} \max_{1 \leq j \leq n} \sum_{i=1}^{n} \left(\sum_{k=1}^{n} |Q_{ij}^{(k)}|^2 \right)^{\frac{1}{2}} \cdot \|e - \tilde{e}\|_2. \end{aligned}$$

■

IV. Example

We consider the Lorenz attractor, whose dynamics originate as a simplified form of the Navier-Stokes equations. Consequently, the non-linear part of the systems has the energy preserving property (2). In particular, the dynamics have the form (1) with

$$A := \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix},$$

$$N(x) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix}, \quad x \in \mathbb{R}^3.$$

Suppose that it is possible to observe only to first state:

$$y = Cx, \qquad C := \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

The first step towards constructing a convergent observer is to select Q which satisfies (6). Notice that

$$\{d \in \mathbb{R}^3 : N(d)d = 0\} = \text{span}\{\boldsymbol{e}_2, \boldsymbol{e}_3\} \cup \text{span}\{\boldsymbol{e}_1\}.$$

We choose the larger space $\text{span}\{\boldsymbol{e}_2, \boldsymbol{e}_3\} = \ker Q$, for

$$Q := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Proposition 2.2 implies that $B_r(d)$ is invariant for the state $(x(t))_{t\geq 0}$ with $r = 100.70$ and $d = 37.76$. In this case,

$$\mathbb{S}_N^n = \text{span}\left\{\text{diag}\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \text{diag}\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}\right\}.$$

Solving Algorithm 2.4 (ii) with the restriction $P \in \mathbb{S}_N^n$, tuning parameters $\alpha_1 = 0.1$, $\alpha_2 = 10^3$ and the condition $P < 10 \cdot I$ gives

$$P(A_{y_i} + LC) \prec 0, \qquad i = 1, \ldots, 8,$$

where

$$P = \text{diag}\begin{pmatrix} 10 & 0.1642 & 0.1642 \end{pmatrix} \in \mathbb{S}_N^n,$$
$$L = \begin{pmatrix} -99.51 & -599.43 & 0 \end{pmatrix}^T$$

and $(y_i)_{i=1}^8$ are the vertices of a cube of smallest side length containing $B_r(d)$. Hence, Theorem 2.7 implies that the resulting observer is globally convergent.

Observer design for the Lorenz attractor is considered in [12]. Here, the co-ordinate transformation approach is used and it is also assumed that $C = (1\ 0\ 0)$. This approach creates an observer which appears to converge experimentally, but the complexity of the co-ordinate transformation means that it is not possible to *prove* convergence.

For the Lipschitz approach, suppose there exists $\gamma > 0$, symmetric $P \succ 0$ and $R \in \mathbb{R}^{p\times n}$ such that

$$\begin{bmatrix} PA + A^TP + RC + C^TR^T & P \\ (*) & -I/\gamma^2 \end{bmatrix} \prec 0. \qquad (27)$$

It is easy to deduce (see e.g. [5], [13]) that if S is an invariant set for the state and the non-linearity satisfies the Lipschitz condition

$$\|N(x)x - N(y)y\|_2 \leq \gamma\|x - y\|_2, \qquad x, y \in S,$$

then (3), for $L = P^{-1}R$, is a convergent observer. With respect to the Lorenz dynamics, the largest $\gamma > 0$ satisfying (27) is $\gamma = 2.67$. However, letting $x = (x_1, x_2, x_3), y = (y_1, x_2, x_3)$ implies that

$$\frac{\|N(x)x - N(y)y\|_2}{\|x - y\|_2} = \sqrt{x_2^2 + x_3^2}.$$

It is known that there exists x in the range of the Lorenz attractor for which $\sqrt{x_2^2 + x_3^2} > \sqrt{1500}$ and hence, the Lipschitz approach [1] cannot be used to construct a convergent observer for the Lorenz attractor.

V. Conclusions

A method of observer design has been presented for a class of non-linear systems whose non-linearity is energy preserving. Sufficient conditions, which can be verified by standard convex optimization methods, are given which imply either local or global observer convergence. The results are applied to show that there exists a globally convergent observer for the Lorenz attractor. Design of globally convergent observers for more complicated systems may be aided by the introduction of a quadratic innovation term in the observer dynamics. This will be the subject of future research.

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