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More About Almost Controllability Subspaces

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***Abstract*— In this paper, we show on a particular example that any *Almost Controllability Subspace* can also be interpreted as a subspace that can be made unobservable by means of a P.D. feedback. This is done using singularly perturbed techniques for analyzing the high gain feedbacks related to the *Almost Controllability Subspace*.**

Notation

Script capitals $\mathscr{V}$, $\mathscr{W}$, ..., denote linear spaces with elements v, w, ...; $\{0\}$ is the zero subspace. The dimension of a space $\mathscr{V}$ is denoted $\dim(\mathscr{V})$. The direct sum of independent spaces is written as $\oplus$. Capital letters X, denote both, matrices $X \in \mathbb{R}^{\varphi \times \rho}$, and linear maps $X \in \mathscr{V} \to \mathscr{W}$. Given a linear map $X : \mathscr{V} \to \mathscr{W}$, $\operatorname{Im} X = X\mathscr{V}$ denotes its image, and $\operatorname{Ker} X$ denotes its kernel. For the special map, $B : \mathscr{U} \to \mathscr{X}$, its image is denoted by $\mathscr{B}$. We write $X^{-1}\mathscr{T}$ for the inverse image of the subspace $\mathscr{T}$ by the linear map X. I stands for the identity operator. e_i stands for the vector with a 1 in its i-th component and 0 in its other components. $\{\mathscr{B}_i\}_{i=1}^{k}$ denotes a chain in $\mathscr{B}$, namely $\mathscr{B} \supset \mathscr{B}_1 \supset \mathscr{B}_2 \supset \cdots \supset \mathscr{B}_k$. A_F denotes $A + BF$. $\|v\|$ stands for the Euclidean norm, $\sqrt{v^T v}$. $\mathrm{d}\,(x, \mathscr{S})$ denotes the distance of a vector $x \in \mathscr{X}$ to the subspace $\mathscr{S} \subset \mathscr{X}$, namely $\inf_{x' \in \mathscr{S}} \|x - x'\|$ and $\mathrm{d}_\infty\,(x(t), \mathscr{S}) = \sup_{t \in \mathbb{R}^+} \{\mathrm{d}\,(x(t), \mathscr{S})\}$. I_n denotes the identity matrix of size $n \times n$. $\mathrm{BDM}\,\{H_1, \ldots, H_n\}$ denotes a block diagonal matrix whose block diagonal matrices are $\{H_1, \ldots, H_n\}$. $\mathbb{R}^+ = \{r \in \mathbb{R} : r \geq 0\}$ (similar for $\mathbb{Z}$), $\mathbb{R}^{*+} = \mathbb{R}^+ \setminus \{0\}$ and $\mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$. $\mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^q)$ is the set of infinitely differentiable functions mapping from $\mathbb{R}^+$ to $\mathbb{R}^q$. We write: $f(\varepsilon) = \mathcal{O}\,(\varphi(\varepsilon))$ when there exist $\varepsilon^* > 0$ and $K > 0$ such that $|f(\varepsilon)| \leq K\varphi(\varepsilon)$ for $\varepsilon \in (0, \varepsilon^*)$ and $\varphi(\varepsilon) > 0$. $g + \mathcal{O}\,(\varphi(\varepsilon))$ means: $g + f(\varepsilon)$ with $f(\varepsilon) = \mathcal{O}\,(\varphi(\varepsilon))$ [8].

I. Introduction

With the seminal papers of Brunovsky [7] and Morse [10] began the structural study of linear systems. They made it possible to tackle control problems from a very formal point of view, and to understand how systems structures play a deep role in the solvability of such control problems.

In particular [10] is one of the key papers about structure and geometric approach. More precisely, some important structural properties can be interpreted in terms of the (A, B)–*Invariant* and *Controllability Subspaces*, which are related with the maps of the state space representations of the systems. In a very simplistic way, these subspaces tell us which are the parts of the system, which can be made unobservable (made invariant inside the kernel of the output map) by state feedback, and for some part with assignable dynamics. This was the starting point for a systematic study of the structure of linear systems. In the important works of Wonham [13] and Basile and Marro [2] the principal results of the geometric approach are summarized.

A second milestone occurred with Willems' introduction of the *Almost* (A, B)–*Invariant* and *Almost Controllability Subspaces*, which are related with the maps of the state space representations of the systems [14], [15], [16]. These subspaces are useful when non exact solutions are looked for to some control problems. *Almost Invariance* and *Almost Controllability* have been connected with the use of high gain state feedback, as approximations of distributional state feedbacks.

In this paper we continue the study of [5] with respect to the comparison of the high gain control laws, based on some *Almost Controllability Subspace*, with the P.D. control laws. In Section II, we recall the principal characteristics of the *Almost Controllability Subspaces*. In Section III, we recall the characterization of the high gains of [12]. In Section IV, we analyze the closed loop system using the singularly perturbed techniques [9]. For the sake of shortness this is done by means of an illustrative example. In Section V, we show that the high gain feedback proposed by Trentelman also tends to a P.D. feedback. In Section VI, we show that the *Supremal Almost Controllability Subspace* contained in $\mathscr{K} = \operatorname{Ker} C$, $\mathscr{S}_{\mathscr{K}}^{\infty}$, is indeed the *(supremal) differential redundant space* of the system obtained after appling a P.D. feedback, $\widehat{\mathscr{R}}_a^*$. And in Section VII, we conclude.

II. Almost Controllability Subspaces

In this paper we consider the *input/state system* [11], $\Sigma_{i/s} = \left(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}, \mathfrak{B}_{[A,B]}\right)$, with behavior, $\mathfrak{B}_{[A,B]}$:

$$\mathfrak{B}_{[A,B]} = \left\{ (u, x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}) \Big| \begin{bmatrix} (\mathrm{Id}/\mathrm{dt} - A) & -B \end{bmatrix} \begin{bmatrix} x^T & u^T \end{bmatrix}^T = 0 \right\} \tag{1}$$

where $u \in \mathscr{U} \approx \mathbb{R}^m$ is the input variable and $x \in \mathscr{X} \approx \mathbb{R}^n$ is the state variable; we assume that B is monic.

Let us write the definition and some geometric characterizations of the *Almost Controllability Subspaces*:

Definition 1 ([14]): A subspace $\mathscr{R}_a \subset \mathscr{X}$ is said to be an *Almost Controllability Subspace* if $\forall\, x_0, x_1 \in \mathscr{R}_a$, $\exists\, T > 0$ such that $\forall\, \rho > 0\ \exists\, (u, x) \in \mathfrak{B}_{[A,B]}$ with the properties that $x(0) = x_0$, $x(T) = x_1$ and $\sup_{t \in \mathbb{R}^+} \inf_{x' \in \mathscr{R}_a} \|x(t) - x'\| \leq \rho$.

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Let $\mathscr{K}$ be a subspace of $\mathscr{X}$, then the subspace $\mathscr{S}^\infty_{\mathscr{K}}$ is the limit of the non decreasing algorithm:

$$\mathscr{S}^0 = \{0\};\ \mathscr{S}^{\mu+1} = \mathscr{K} \cap (A\mathscr{S}^\mu + \mathscr{B}),\ \mu \in \mathbb{Z}^+ \qquad (2)$$

Corollary 2 ([14], Corollary 1.23–[12]): A subspace $\mathscr{R}_a$ of $\mathscr{X}$ is an *Almost Controllability Subspace* if and only if there is a linear map $F : \mathscr{X} \to \mathscr{U}$ and a chain $\{\mathscr{B}_i\}_{i=1}^k$ in $\mathscr{B}$ such that $\mathscr{R}_a = \mathscr{B}_1 + A_F\mathscr{B}_2 + \cdots + A_F^{k-1}\mathscr{B}_k$. Moreover, there exist a $k \in \mathbb{Z}^+$, $k \le \dim \mathscr{R}_a$, a chain $\{\mathscr{B}_i\}_{i=1}^k$ in $\mathscr{B}$ and a linear map $F^* : \mathscr{X} \to \mathscr{U}$ such that ($i \in \{1, \ldots, k\}$):

$$\begin{array}{c} \mathscr{R}_a = \mathscr{B}_1 \oplus A_{F^*}\mathscr{B}_2 \oplus \cdots \oplus A_{F^*}^{k-1}\mathscr{B}_k, \quad \mathscr{B}_1 = \mathscr{R}_a \cap \mathscr{B} \\ \dim \mathscr{B}_i = \dim A_F^{i-1}\mathscr{B}_i = \dim \mathscr{S}^i - \dim \mathscr{S}^{i-1} \end{array} \qquad (3)$$

where the $\mathscr{S}^i$ are the steps of algorithm (2) with $\mathscr{K} = \mathscr{R}_a$.

Theorem 3 ([14], Theorem 1.24–[12]): Let $\mathscr{K}$ be a subspace of $\mathscr{X}$ and $\mathscr{R}^*_{a,\mathscr{K}}$ be the *Supremal Almost Controllability Subspace* contained in $\mathscr{K}$. Then:

$$\mathscr{R}^*_{a,\mathscr{K}} = \Big\{ x_0 \in \mathscr{K} \,\Big|\, \forall\, \rho > 0\ \exists\ (u, x) \in \mathfrak{B}_{[A,B]},\ x(0) = x_0, \text{ such that } x(T) = 0 \text{ and } \mathrm{d}_\infty(x, \mathscr{K}) \le \rho \Big\} \qquad (4)$$

Moreover, $\mathscr{R}^*_{a,\mathscr{K}} = \mathscr{S}^\infty_{\mathscr{K}}$.

The following Lemma gives a nice space decomposition, in terms of a suitable feedback:

Lemma 4 (Lemma 1.15–[12]): Let $\mathscr{K}$ be a subspace of $\mathscr{X}$. There are subspaces $\mathscr{X}_1$, $\mathscr{X}_2$ and $\mathscr{X}_3$ of $\mathscr{X}$ and $\mathscr{U}_1$, $\mathscr{U}_2$ and $\mathscr{U}_3$ of $\mathscr{U}$, a linear map $F^* : \mathscr{X} \to \mathscr{U}$, an integer $k \le \dim \mathscr{K}$ and integers r_i, such that: *1)* $\mathscr{S}^\infty_{\mathscr{K}} = \mathscr{X}_1 \oplus \mathscr{X}_2$, *2)* $\mathscr{X} = \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus \mathscr{X}_3$, *3)* $A_{F^*}\mathscr{X}_1 \subset \mathscr{X}_1 \oplus \mathscr{X}_2$, *4)* $B\mathscr{U}_i \subset \mathscr{X}_i$, $i \in \{1, 2, 3\}$, *5)* When applying the state feedback $u = F^*x + \bar{u}$ to (1), then under the decompositions $\mathscr{X} = \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus \mathscr{X}_3$ and $\mathscr{U} = \mathscr{U}_1 \oplus \mathscr{U}_2 \oplus \mathscr{U}_3$, the state space representation is:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}}_{A_{F^*}} x + \underbrace{\begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{bmatrix}}_{B} \bar{u} \qquad (5)$$

where:[1] *(a)* $\mathscr{X}_2 = \mathrm{Im}\, A_{21} \oplus \mathrm{Im}\, B_2$, *(b)* let $\overline{A}_{21} = P_{A_{21}}A_{21}$, where $P_{A_{21}}$ is the natural projection on $\mathrm{Im}\, A_{21}$ along $\mathrm{Im}\, B_2$, then $\mathscr{X}_1 = A_{11}^{-1}\mathrm{Im}\, B_1 \oplus \mathrm{Ker}\, \overline{A}_{21}$ and $\mathrm{Im}\, \overline{A}_{21} \approx \mathrm{Im}\, B_1$, *(c)* the associated pencil, $\left[\begin{array}{c|c} \lambda \mathrm{I} - A_{11} & -B_1 \\ \hline \overline{A}_{21} & \end{array}\right]$, $\lambda \in \mathbb{C}$, only contains *infinite elementary divisors*, namely the standard controllable triple $(\overline{A}_{21}, A_{11}, B_1)$ is prime [10].

Morse [10] introduced the prime systems, which roughly speaking are controllable and observable systems, represented by a (C, A, B) state space form. In his Theorem 3.1 shows that there exist, F and L, such that: $(A + BF - LC) \sim \mathrm{BDM}\{A_1, \ldots, A_m\}$, $B \sim \mathrm{BDM}\{B_1, \ldots, B_m\}$, and $C \sim \mathrm{BDM}\{C_1, \ldots, C_m\}$; where:

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & \cdots & 0 & 1 \\ 0 & \cdot & \cdot & \cdots & 0 \end{bmatrix},\ B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ C_i^T = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

[1] These geometric properties directly follow from the matricial expressions of Trentelman [12]. For example, for item a): $\mathrm{Ker}\, [A_{21}\ B_2]^T = \{0\}$ implies $\mathscr{X}_2 = \mathrm{Im}\, A_{21} + \mathrm{Im}\, B_2$ and $\dim \mathscr{X}_2 = \mathrm{rank}\, A_{21} + \mathrm{rank}\, B_2$ implies $\mathscr{X}_2 = \mathrm{Im}\, A_{21} \oplus \mathrm{Im}\, B_2$.

III. High Gain Feedback

High gain feedback for *Almost Controllability Subspaces* is characterized in the next Theorem:

Theorem 5 (Th. 2.32–[12], 2.35–[12], & L. 2.34–[12]): Let $\mathscr{S}^\infty_{\mathscr{K}}$ be the *Supremal Almost Controllability Subspace* contained in $\mathscr{K}$ and $\{\mathscr{B}_i\}_{i=1}^k$ be a chain in $\mathscr{S}^\infty_{\mathscr{K}} \cap \mathscr{B}$.
Let $F^* : \mathscr{X} \to \mathscr{U}$ be a linear map satisfying (3).
Let, $\mathfrak{L}_1, \cdots, \mathfrak{L}_k$, be subspaces of $\mathscr{S}^\infty_{\mathscr{K}}$ such that:

$$\mathfrak{L}_i = b_i \oplus A_{F^*}b_i + \cdots + \oplus A_{F^*}{}^{n_i - 1}b_i, \quad i \in \{1, \ldots, k\} \qquad (6)$$

where $\mathrm{span}\{b_1, \cdots, b_k\} = \mathscr{B}$. Let $N \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^{*+}$ such that:[2] $N \ge \max\{|\lambda| \mid \lambda \in \sigma(A_{F^*})\}$ and $\varepsilon < 1/N$.
Then, for each $i \in \{1, \ldots, k\}$, there exist a sequence of (A, B)–*Invariant Subspaces*, $\{\mathfrak{L}_i(\varepsilon)\}$, such that[3] $\lim_{\varepsilon \to 0} \mathfrak{L}_i(\varepsilon) = \mathfrak{L}_i$, respectively, and they are generated by the sequences of vectors, $\{x_{i,1}(\varepsilon, \bar{u}_i), \cdots, x_{i,k_i}(\varepsilon, \bar{u}_i)\}$, defined recursively by ($i \in \{1, \ldots, k_i\}$ and $B\bar{u}_j \in \{\mathscr{B}_i\}_{i=1}^{k_i}$): [4]

$$\begin{array}{rcl} x^b_{i,j}(\varepsilon, \bar{u}_i) & = & \left(\mathrm{I} - \varepsilon A_{F^*}\right)^{-1} B\bar{u}_i, \\ x^b_{i+1,j}(\varepsilon, \bar{u}_{i+1}) & = & \left(\mathrm{I} - \varepsilon A_{F^*}\right)^{-1} A_{F^*} x^b_{i,j}(\varepsilon, \bar{u}_i). \end{array} \qquad (7)$$

Moreover, for a given sequence of *friend feedbacks* [13], $\{F_i :\ \mathcal{L}_i(\varepsilon) \to \mathscr{U}\}$, such that:

$$F_i x_{i,j}(\varepsilon, \bar{u}_j) = -\left(1/\varepsilon\right)^i \bar{u}_j, \qquad (8)$$

let $(Fx, x) \in \mathfrak{B}_{[A_{F^*}, B]}$, where $F|\mathcal{L}_i(\varepsilon) = F_i$ and with $x(0) = x_0 \in \mathscr{S}^\infty_{\mathscr{K}}$, then for all $\rho \in \mathbb{R}^{*+}$ there exists a $K \in \mathbb{N}$ such that for all $\varepsilon \le 1/K$: $\mathrm{d}_\infty(x, \mathscr{S}^\infty_{\mathscr{K}}) \le \rho$

A. Illustrative Example (Part 1)

Let us consider the state space representation (5) with the following matrices:[5]

$$\begin{array}{l} A_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A_{12} = A_{13} = A_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \\ A_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_2 = B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ A_{23} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A_{33} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \end{array} \qquad (9)$$

with: $x = \begin{bmatrix} x_1^T & x_2^T & x_a^T \end{bmatrix}^T$, $x_1 = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \end{bmatrix}^T$, $x_2 = \begin{bmatrix} x_{2,1} & x_{2,2} & x_{2,3} \end{bmatrix}^T$ and $x_a = \begin{bmatrix} x_{a,1} & x_{a,2} & x_{a,3} \end{bmatrix}^T$; the subspace $\mathscr{K}$ is defined as follows:

$$\mathscr{K} = \mathrm{Ker} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & d \end{bmatrix}}_{C}, \qquad (10)$$

where: $ad \ne cb$. Note that: $\sigma(A_{F^*}) = \{2.15, -0.57 \pm 0.37\imath, -1, 0, 0, 0, -1, 0\}$.

Following Theorem 5, hereafter we synthesize the high gain feedback.

[2] This condition guarantees the invertibility of $(\mathrm{I} - \varepsilon A_{F^*})$.

[3] See Appendix II.

[4] See Section 2.4–[12]

[5] Let us note that (3.a) implies that: $\mathscr{S}^\infty_{\mathscr{K}} \subset \langle A_{F^*} \,|\, \mathscr{K} \cap \mathscr{B} \rangle$, then the couple (A_{22}, B_2) is also controllable, and it can be carried into its Brunovsky canonical form.

1) Subspaces $\mathfrak{L}_i$ and $\mathfrak{L}_i(\varepsilon)$: From (5), (9) and (6), we get ($e_i \in \mathbb{R}^9$): $\mathscr{B}_1 = \text{span}\{e_1, e_3, e_6\}$, $\mathscr{B}_2 = \text{span}\{e_1, e_3\}$, $\mathscr{B}_3 = \text{span}\{e_3\}$, $A_{F^*}\mathscr{B}_2 = \text{span}\{e_4, e_2\}$, $A^2_{F^*}\mathscr{B}_3 = \text{span}\{e_5\}$. Then, $\mathfrak{L}_1 = \text{span}\{b_1, A_{F^*}b_1\} = \text{span}\{e_1, e_4\}$, $\mathfrak{L}_2 = \text{span}\{b_2, A_{F^*}b_2, A^2_{F^*}b_2\} = \text{span}\{e_3, e_2, e_5\}$, $\mathfrak{L}_3 = \text{span}\{b_3\} = \text{span}\{e_6\}$. Thus, the matrix, $\overline{X}_0 \in \mathbb{R}^{9\times 9}$, associated to the subspaces $\mathfrak{L}_i$ is ($\mathbb{R}^9 = \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \mathfrak{L}_3 \oplus \text{span}\{e_7, e_8, e_9\}$):

$$\overline{X}_0 = \left[\begin{array}{cc||ccc||c||ccc} e_1 & e_4 & e_3 & e_2 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array}\right] \tag{11}$$

From (5), (9) and (7), we get ($x^b_{i,j} \in \mathbb{R}^9$):[6]

$$\begin{aligned}
x^b_{1,1} &= \left[\begin{array}{ccccccccc} 1 & \frac{\varepsilon^2}{\alpha_1} & 0 & -\frac{\varepsilon\alpha_3}{\alpha_1} & -\frac{2\varepsilon^3\eta_5}{\alpha_1\alpha_5} & 0 & 0 & \frac{\varepsilon^2}{\alpha_1\alpha_5} & 0 \end{array}\right]^T \\
x^b_{1,2} &= \\
&\left[\begin{array}{ccccccccc} 0 & -\frac{2\varepsilon\alpha_2}{\alpha_1^2} & 0 & \frac{\alpha_4\alpha_5}{\alpha_1^2} & \frac{6\varepsilon^2\alpha_6}{(\alpha_1\alpha_5)^2} & 0 & 0 & -\frac{2\varepsilon\alpha_7}{(\alpha_1\alpha_5)^2} & 0 \end{array}\right]^T \\
x^b_{2,1} &= \left[\begin{array}{ccccccccc} 0 & -\frac{\varepsilon\alpha_3}{\alpha_1} & 1 & -\frac{\varepsilon^3}{\alpha_1} & \frac{\varepsilon^2\beta_5}{\alpha_1} & 0 & 0 & -\frac{\varepsilon^3}{\alpha_1} & 0 \end{array}\right]^T \\
x^b_{2,2} &= \left[\begin{array}{ccccccccc} 0 & \frac{\alpha_4\alpha_5}{\alpha_1^2} & 0 & \frac{3\varepsilon^2\alpha_8}{\alpha_1^2} & -\frac{2\varepsilon\alpha_9}{\alpha_1^2} & 0 & 0 & \frac{3\varepsilon^2\alpha_8}{\alpha_1^2} & 0 \end{array}\right]^T \\
x^b_{2,3} &= \left[\begin{array}{ccccccccc} 0 & -\frac{3\varepsilon\beta_1}{\alpha_1^3} & 0 & -\frac{3\varepsilon\beta_2}{\alpha_1^3} & \frac{\beta_3}{\alpha_1^3} & 0 & 0 & -\frac{3\varepsilon\beta_2}{\alpha_1^3} & 0 \end{array}\right]^T \\
x^b_{3,1} &= \\
&\left[\begin{array}{ccccccccc} 0 & -\frac{\varepsilon\beta_4}{\alpha_1} & 0 & \frac{2\varepsilon^2\eta_5}{\alpha_1} & -\frac{\varepsilon\beta_6}{\alpha_1\alpha_5} & 1 & 0 & -\frac{\varepsilon\beta_4}{\alpha_1\alpha_5} & 0 \end{array}\right]^T
\end{aligned} \tag{12}$$

Then, $\mathfrak{L}_1(\varepsilon) = \text{span}\left\{x^b_{1,1}, x^b_{1,2}\right\}$, $\mathfrak{L}_2(\varepsilon) = \text{span}\left\{x^b_{2,1}, x^b_{2,2}, x^b_{2,3}\right\}$, $\mathfrak{L}_3(\varepsilon) = \text{span}\left\{x^b_{3,1}\right\}$. Thus, the matrix, $\overline{X}(\varepsilon) \in \mathbb{R}^{9\times 9}$, associated to the subspaces $\mathfrak{L}_i(\varepsilon)$ is ($\mathbb{R}^9 = \mathfrak{L}_1(\varepsilon) \oplus \mathfrak{L}_2(\varepsilon) \oplus \mathfrak{L}_3(\varepsilon) \oplus \text{span}\{e_7, e_8, e_9\}$):

$$\overline{X}(\varepsilon) = \left[\begin{array}{cc||ccc||c||ccc} x^b_{1,1} & x^b_{1,2} & x^b_{2,1} & x^b_{2,2} & x^b_{2,3} & x^b_{3,1} & e_7 & e_8 & e_9 \end{array}\right] \tag{13}$$

2) High gain feedback: From (5), (9), (8), (12) and (13), the high gain feedback, $F_{Tr}(\varepsilon)$, is:[6]

$$\overline{F}_{Tr}(\varepsilon) = \left[\begin{array}{cc||ccc||c||ccc}
\multicolumn{2}{c}{\mathfrak{L}_1(\varepsilon)} & \multicolumn{3}{c}{\mathfrak{L}_2(\varepsilon)} & \mathfrak{L}_3(\varepsilon) & & & \\
-1/\varepsilon & -1/\varepsilon^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\hline
0 & 0 & -1/\varepsilon & -1/\varepsilon^2 & -1/\varepsilon^3 & 0 & 0 & 0 & 0 \\ \hline\hline
0 & 0 & 0 & 0 & 0 & -1/\varepsilon & 0 & 0 & 0 \\ \hline\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\delta_1\gamma_9}{\beta_7} & -\left(1 + \frac{\delta_1\gamma_9}{\beta_7}\right)
\end{array}\right] \tag{14}$$

$$F_{Tr}(\varepsilon) = \overline{F}_{Tr}(\varepsilon)\overline{X}^{-1}(\varepsilon) =$$

$$\left[\begin{array}{ccc|cc|c|ccc}
-\frac{2}{\varepsilon} & -3 & -0 & -\left(\frac{1}{\varepsilon^2} - 3\right) & -\left(\frac{3}{\varepsilon} - 6\right) & -1 & 0 & 0 & 0 \\ \hline
1 & -\left(\frac{3}{\varepsilon^2} - 7\right) & -\frac{3}{\varepsilon} & 5 & -\left(\frac{1}{\varepsilon^3} - \frac{3}{\varepsilon} - 6\right) & -\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} - 6\right) & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & -\frac{1}{\varepsilon} & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2
\end{array}\right] + \mathcal{O}(\varepsilon) \tag{15}$$

[6] See Appendix I for the coefficients' definitions. Note that all the coefficients tend to 1 when $\varepsilon \to 0$.

3) Closed loop system: From (5), (9) and (15), the closed loop system is represented by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{array}{c} \varepsilon^2 x_{1,1} \\ x_{1,2} \\ \varepsilon^3 x_{1,3} \\ \hline x_{2,1} \\ x_{2,2} \\ \varepsilon\, x_{2,3} \\ \hline x_{a,1} \\ x_{a,2} \\ x_{a,3} \end{array}\right] = \left[\begin{array}{c|c|c} X_{11}(\varepsilon) & X_{12}(\varepsilon) & A_{13} \\ \hline A_{21} & X_{22} & A_{23} \\ \hline 0 & A_{32} & X_{33} \end{array}\right] x + \left[\begin{array}{c} \varepsilon^3 v^T_{1,1}x \\ 0 \\ \varepsilon^4 v^T_{1,3}x \\ \hline 0 \\ 0 \\ \varepsilon^2 v^T_{2,3}x \\ \hline 0 \\ 0 \\ 0 \end{array}\right]$$

$$X_{11}(\varepsilon) = \left[\begin{array}{ccc} -2\varepsilon & -3\varepsilon & 0 \\ 0 & 0 & 1 \\ \varepsilon^3 & -(3\varepsilon - 7\varepsilon^3) & -3\varepsilon^2 \end{array}\right],\ X_{12}(\varepsilon) =$$
$$\left[\begin{array}{ccc} -(1 - 3\varepsilon^2) & -(3\varepsilon - 6\varepsilon^2) & -\varepsilon^2 \\ 1 & 1 & 1 \\ 5\varepsilon^3 & -(1 - 3\varepsilon^2 - 6\varepsilon^3) & -(\varepsilon + \varepsilon^2 - 6\varepsilon^3) \end{array}\right],$$
$$X_{22} = A_{22} - B_2B_2^T,\ X_{33} = A_{33} + B_3F_3,\ F_3 = \left[\begin{array}{ccc} 0 & -1 & -2 \end{array}\right] \tag{16}$$

where: $v_{i,j} \in \mathbb{R}^9[\varepsilon]$.

IV. Singularly Perturbed Model

Let us express the closed loop representation (16) by means of a singularly perturbed model [9]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{array}{c} x_{1,2} \\ \hline x_{2,1} \\ x_{2,2} \\ \hline x_{a,1} \\ x_{a,2} \\ x_{a,3} \end{array}\right] = \left[\begin{array}{c|cc|ccc} 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{array}\right]\left[\begin{array}{c} x_{1,2} \\ \hline x_{2,1} \\ x_{2,2} \\ \hline x_{a,1} \\ x_{a,2} \\ x_{a,3} \end{array}\right] + \left[\begin{array}{cc|c} 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} x_{1,1} \\ x_{1,3} \\ \hline x_{2,3} \end{array}\right] \tag{17}$$

$$\left[\begin{array}{cc|c} \varepsilon^2 & 0 & 0 \\ 0 & \varepsilon^3 & 0 \\ \hline 0 & 0 & \varepsilon \end{array}\right]\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{array}{c} x_{1,1} \\ x_{1,3} \\ \hline x_{2,3} \end{array}\right] = \left[\begin{array}{c|cc|ccc} -3\varepsilon^2 & -(1 - 3\varepsilon^2) & -(3\varepsilon - 6\varepsilon^2) & 0 & 0 & 0 \\ -(3\varepsilon - 7\varepsilon^3) & 5\varepsilon^3 & -(1 - 3\varepsilon^2 - 6\varepsilon^3) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} x_{1,2} \\ \hline x_{2,1} \\ x_{2,2} \\ \hline x_{a,1} \\ x_{a,2} \\ x_{a,3} \end{array}\right] + \left[\begin{array}{cc|c} -2\varepsilon & 0 & -\varepsilon^2 \\ \varepsilon^3 & -3\varepsilon^2 & -(\varepsilon + \varepsilon^2 - 6\varepsilon^3) \\ \hline 0 & 0 & -1 \end{array}\right]\left[\begin{array}{c} x_{1,1} \\ x_{1,3} \\ \hline x_{2,3} \end{array}\right] + \left[\begin{array}{c} \varepsilon^3 v^T_{1,1} \\ \varepsilon^4 v^T_{1,3} \\ \hline \varepsilon^2 v^T_{2,3} \end{array}\right] x \tag{18}$$

Let us note that:

$$\det\left(\lambda\left[\begin{array}{cc|c} \varepsilon^2 & 0 & 0 \\ 0 & \varepsilon^3 & 0 \\ \hline 0 & 0 & \varepsilon \end{array}\right] - \left[\begin{array}{cc|c} -2\varepsilon & 0 & -\varepsilon^2 \\ \varepsilon^3 & -3\varepsilon^2 & -(\varepsilon + \varepsilon^2 - 6\varepsilon^3) \\ \hline 0 & 0 & -1 \end{array}\right]\right) = \varepsilon^3(\varepsilon\lambda + 1)(\varepsilon\lambda + 2)(\varepsilon\lambda + 3) \tag{19}$$

A. Slow Model

The *slow model* is obtained doing $\varepsilon = 0$ in (17) and (18), namely:

$$\begin{aligned}\bar{x}_2 &= 0\\ \tfrac{\mathrm{d}}{\mathrm{d}t}\bar{x}_a &= \begin{bmatrix}-1&0&0\\1&0&1\\0&-1&-2\end{bmatrix}\bar{x}_a\\ \bar{x}_1 &= \begin{bmatrix}-1&-1&-1\\-1&-1&-1\\-(\frac{\mathrm{d}}{\mathrm{d}t}+1)&-(\frac{\mathrm{d}}{\mathrm{d}t}+1)&-(\frac{\mathrm{d}}{\mathrm{d}t}+1)\end{bmatrix}\bar{x}_a\end{aligned}\tag{20}$$

The trajectories solution of (20) are:

$$\begin{aligned}\bar{x}_2(t) &= 0\\ \bar{x}_a(t) &= \mathrm{e}^{-t}\begin{bmatrix}1&0&0\\(t+\frac{1}{2}t^2)&(1+t)&t\\-\frac{1}{2}t^2&-t&(1-t)\end{bmatrix}\bar{x}_a(0)\\ \bar{x}_1(t) &= -\mathrm{e}^{-t}\begin{bmatrix}(1+t)&1&1\\(1+t)&1&1\\1&0&0\end{bmatrix}\bar{x}_a(0)\end{aligned}\tag{21}$$

B. Fast Model

For obtaining the *fast model* we define the *boundary layer* correction variables: $\hat{x}_{1,1}=x_{1,1}-\bar{x}_{1,1}$, $\hat{x}_{1,3}=x_{1,3}-\bar{x}_{1,3}$ and $\hat{x}_{2,3}=x_{2,3}-\bar{x}_{2,3}$, and the fast time scale: $\tau=t/\varepsilon$, namely (see (18) and (19)):

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}\hat{x}_{1,1}\\\hat{x}_{1,3}\\\hline\hat{x}_{2,3}\end{bmatrix}=\left[\begin{array}{cc|c}-2&0&-\varepsilon\\\varepsilon&-3&-(1/\varepsilon+1-6\varepsilon)\\\hline 0&0&-1\end{array}\right]\begin{bmatrix}\hat{x}_{1,1}\\\hat{x}_{1,3}\\\hline\hat{x}_{2,3}\end{bmatrix}\tag{22}$$

The trajectories solution of (22) are:

$$\begin{bmatrix}\hat{x}_{1,1}(\tau)\\\hat{x}_{1,3}(\tau)\\\hline\hat{x}_{2,3}(\tau)\end{bmatrix}=\mathrm{e}^{-\tau}\begin{bmatrix}\mathrm{e}^{-\tau}&0&-\varepsilon\left(1-\mathrm{e}^{-\tau}\right)\\\varepsilon\left(\mathrm{e}^{-\tau}-\mathrm{e}^{-2\tau}\right)&\mathrm{e}^{-2\tau}&-\left(\frac{1}{\varepsilon}+1-6\varepsilon+\varepsilon^2\right)-\varepsilon^2\mathrm{e}^{-\tau}+\left(\frac{1}{\varepsilon}+1-6\varepsilon-\varepsilon^2\right)\mathrm{e}^{-2\tau}\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}\hat{x}_{1,1}(0)\\\hat{x}_{1,3}(0)\\\hline\hat{x}_{2,3}(0)\end{bmatrix}\tag{23}$$

C. Closed Loop Trajectories

From (21), (23) and Theorem 5.1-[9], there exists $\varepsilon^*>0$ (recall (19)) such that, for all $\varepsilon\in(0,\varepsilon^*]$, the closed loop trajectories solution of the singularly perturbed model (17) and (18), are approximated for all $t>0$ by:

$$x_a(t)=\mathrm{e}^{-t}\begin{bmatrix}1&0&0\\(t+\frac{1}{2}t^2)&(1+t)&t\\-\frac{1}{2}t^2&-t&(1-t)\end{bmatrix}x_a(0)+\mathcal{O}(\varepsilon)$$

$$x_2(t)=\mathrm{e}^{-t/\varepsilon}\begin{bmatrix}0\\0\\1\end{bmatrix}\hat{x}_{s,3}(0)+\mathcal{O}(\varepsilon)$$

$$x_1(t)=-\mathrm{e}^{-t}\begin{bmatrix}(1+t)&1&1\\(1+t)&1&1\\1&0&0\end{bmatrix}x_a(0)+\mathcal{O}(\varepsilon)+$$

$$\mathrm{e}^{-\frac{t}{\varepsilon}}\begin{bmatrix}\mathrm{e}^{-\frac{t}{\varepsilon}}&0&-\varepsilon\left(1-\mathrm{e}^{-\frac{t}{\varepsilon}}\right)\\0&0&0\\\varepsilon\left(\mathrm{e}^{-\frac{t}{\varepsilon}}-\mathrm{e}^{-2\frac{t}{\varepsilon}}\right)&\mathrm{e}^{-2\frac{t}{\varepsilon}}&-\left(\frac{1}{\varepsilon}+1-6\varepsilon+\varepsilon^2\right)-\varepsilon^2\mathrm{e}^{-\frac{t}{\varepsilon}}+\left(\frac{1}{\varepsilon}+1-6\varepsilon-\varepsilon^2\right)\mathrm{e}^{-2\frac{t}{\varepsilon}}\end{bmatrix}\cdot\begin{bmatrix}\hat{x}_{1,1}(0)\\\hat{x}_{1,3}(0)\\\hline\hat{x}_{2,3}(0)\end{bmatrix}$$

V. Trentelman's P.D. Feedback

Let us note that the same average behavior, $(\bar{x}_1,\,\bar{x}_2,\,\bar{x}_a)$, of the *slow model* (20), with trajectories solution (21), is also obtained by means of the following P.D. feedback:

$$u_\infty=\underbrace{\left[\begin{array}{c|c|c}0&B_1^TX_{12}(0)&0\\\hline 0&-B_2^T&0\\\hline 0&0&F_3\end{array}\right]}_{F_p}x+\underbrace{\left[\begin{array}{c|c|c}B_1^T&0&0\\\hline 0&B_2^T&0\\\hline 0&0&0\end{array}\right]}_{F_d}\frac{\mathrm{d}}{\mathrm{d}t}x\tag{24}$$

Indeed, applying (24) to (5) and (9), we get the closed loop system described by the *implicit representation*:

$$\underbrace{\left[\begin{array}{c|c|c}Y_1&0&0\\\hline 0&Y_2&0\\\hline 0&0&\mathrm{I}_3\end{array}\right]}_{E^*}\frac{\mathrm{d}}{\mathrm{d}t}x=\underbrace{\left[\begin{array}{c|c|c}X_{11}(0)&X_{12}(0)&A_{13}\\\hline A_{21}&X_{22}&A_{23}\\\hline 0&A_{32}&X_{33}\end{array}\right]}_{A^*}x\tag{25}$$

$Y_1=\mathrm{I}_3-B_1B_1^T$, $Y_2=\mathrm{I}_3-B_2B_2^T$

Comparing (25) with (16), we realize that (*c.f.* (20)): *If we do* $\varepsilon=0$ *in (16), we precisely get the* slow model *(25).* Let us note that the P.D. feedback (24) is also directely obtained from the Trentelman's high gain feedback (15). Indeed, rewritting (15) as follows:

$$u=\underbrace{\left[\begin{array}{c|c|c}0&0&0\\\hline 0&0&0\\\hline 0&0&F_3\end{array}\right]}_{F_{p1}}x+\underbrace{\left[\begin{array}{c|c}\mathrm{BDM}\left\{1/\varepsilon^2,\,1/\varepsilon^3,\,1/\varepsilon\right\}&0\\\hline 0&0\end{array}\right]}_{G(\varepsilon)}\cdot\left(\underbrace{\left[\begin{array}{c|c|c}0&B_1^TX_{12}(0)&0\\\hline 0&-B_2^T&0\\\hline 0&0&0\end{array}\right]}_{F_{p2}}+\mathcal{O}(\varepsilon)\right)x\tag{26}$$

we get:

$$F_p=F_{p1}+F_{p2}\quad\text{and}\quad F_d=G(1)B^T\tag{27}$$

From (27.b), we have that (see (25) and (9)):

$$\mathscr{X}=\mathrm{Im}\ E^*\oplus\mathrm{Ker}\ E^*\quad\text{and}\quad\mathrm{Ker}\ E^*=\mathscr{K}\cap\mathscr{B}\tag{28}$$

VI. Approximation of Almost Controlled Invariant Subspaces

In this Section we show that the *Supremal Almost Controllability Subspace* contained in $\mathscr{K}=\mathrm{Ker}\,C$, $\mathscr{S}_{\mathscr{K}}^{\infty}$, is indeed the *Supremal Almost* (E^*,A,B) *Controllability Subspace* contained in $\mathscr{K}$, $\widehat{\mathscr{R}}_a^*$.

a) The Supremal Almost (E, A) *Controllability Subspace Contained in* $\text{Ker}\, C$ *(*$\widehat{\mathscr{R}}^*_{a0}$*):* Consider an *implicit representation*, $\Sigma^{imp}(E, A, B, C)$:

$$E\mathrm{d}x/\mathrm{d}t = Ax + Bu \quad \text{and} \quad y = Cx, \tag{29}$$

where: $E : \mathscr{X} \to \underline{\mathscr{X}}$, $A : \mathscr{X} \to \underline{\mathscr{X}}$, $B : \mathscr{U} \to \underline{\mathscr{X}}$ and $C : \mathscr{X} \to \mathscr{Y}$, and $u \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{U})$. The subspace,

$$\widehat{\mathscr{R}}^*_{a0} = \inf(\widehat{\mathfrak{R}}_0(E, A)),\ \widehat{\mathfrak{R}}_0(E, A) := \{\widehat{\mathscr{R}} \subset \mathscr{K} \mid \widehat{\mathscr{R}} = E^{-1}A\widehat{\mathscr{R}}\}, \tag{30}$$

is the limit of the non-decreasing geometric algorithm:

$$\widehat{\mathscr{R}}^0_0 = \mathscr{K} \cap \text{Ker}\ E;\ \ \widehat{\mathscr{R}}^{\mu+1}_0 = \mathscr{K} \cap E^{-1}A\widehat{\mathscr{R}}^\mu_0,\ \mu \in \mathbb{Z}^+ \tag{31}$$

$\widehat{\mathscr{R}}^*_{a0}$ characterizes (together with $A\widehat{\mathscr{R}}^*_{a0}$) the set of all the trajectories, $x \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{X})$, of (29) due to pure differential actions, $\mathrm{d}^iu/\mathrm{d}t^i$, with no influence on the input-output trajectories, namely: $x = U_0u + \sum_{j=1}^{\nu} U_j\mathrm{d}^ju/\mathrm{d}t^j$ and $x(t) \in \widehat{\mathscr{R}}^*_{a0} \subset \text{Ker}\ C$ for all $t \geq 0$. Bonilla *et al* [4] called $\widehat{\mathscr{R}}^*_{a0}$ the differential redundant subspace (see also [3]).

b) The Supremal Almost (E, A, B) *Controllability Subspace Contained in* $\text{Ker}\, C$ *(*$\widehat{\mathscr{R}}^*_a$*):* The subspace,

$$\begin{aligned}&\widehat{\mathscr{R}}^*_a = \inf(\widehat{\mathfrak{R}}(E, A, B)),\\ &\widehat{\mathfrak{R}}(E, A, B) := \{\widehat{\mathscr{R}} \subset \mathscr{K} \mid \widehat{\mathscr{R}} = E^{-1}(A\widehat{\mathscr{R}} + \mathscr{B})\},\end{aligned} \tag{32}$$

is the limit of the non-decreasing geometric algorithm:

$$\widehat{\mathscr{R}}^0 = \mathscr{K} \cap \text{Ker}\ E;\ \widehat{\mathscr{R}}^{\mu+1} = \mathscr{K} \cap E^{-1}(A\widehat{\mathscr{R}}^\mu + \mathscr{B}),\ \ \mu \in \mathbb{Z}^+ \tag{33}$$

$\widehat{\mathscr{R}}^*_a$ characterizes the infimal subspace which can be done differential redundant by means of a proportional and derivative descriptor variable feedback, $u = F_px + F_d\mathrm{d}x/\mathrm{d}t$. The set of pairs, (F_p, F_d), for which $\widehat{\mathscr{R}}^*_a = \min(\widehat{\mathfrak{R}}_0((E - BF_d), (A + BF_p)))$ is called the friends set of $\widehat{\mathscr{R}}^*_a$, this set is denoted by $\mathbf{F}(\widehat{\mathscr{R}}^*_a)$.

c) Equivalence between $\mathscr{S}^\infty_{\mathscr{K}}$ *and* $\widehat{\mathscr{R}}^*_a$*:* Hereafter, we prove that for $E = E^*$:

$$\mathscr{S}^\infty_{\mathscr{K}} = \widehat{\mathscr{R}}^*_a \tag{34}$$

1) Let us first note that algorithms, (2) and (33), are invariants under proportional feedback.

2) **Let us show that** $\widehat{\mathscr{R}}^*_a \subset \mathscr{S}^\infty_{\mathscr{K}}$: Indeed, applying algorithm (33) to the *implicit representation* (29), we get (recall (28) and (2)): $\widehat{\mathscr{R}}^0 = \mathscr{K} \cap \text{Ker}\ E^* = \mathscr{K} \cap \mathscr{B} = \mathscr{S}^1$, and assuming that $\widehat{\mathscr{R}}^i \subset \mathscr{S}^{i+1}$ for all $1 \leq i \leq \mu$, $\widehat{\mathscr{R}}^{\mu+1} = \mathscr{K} \cap E^{*-1}(A\widehat{\mathscr{R}}^\mu + \mathscr{B}) \subset \mathscr{K} \cap E^{*-1}(A\mathscr{S}^{\mu+1} + \mathscr{B}) = \mathscr{K} \cap (\mathrm{I} - BF_d)^{-1}(A\mathscr{S}^{\mu+1} + \mathscr{B}) = \mathscr{K} \cap (A\mathscr{S}^{\mu+1} + \mathscr{B}) = \mathscr{S}^{\mu+2}$. Then: $\widehat{\mathscr{R}}^\mu \subset \mathscr{S}^{\mu+1}$, for all $\mu \geq 0$.

3) **Let us show the reverse inclusion** $\mathscr{S}^\infty_{\mathscr{K}} \subset \widehat{\mathscr{R}}^*_a$: Indeed, applying algorithm (33) to the *implicit representation* (29), we get (recall (28) and (2)): $\mathscr{S}^1 = \mathscr{K} \cap \mathscr{B} = \mathscr{K} \cap \text{Ker}\ E^* = \widehat{\mathscr{R}}^0$, and assuming that $\mathscr{S}^i \subset \widehat{\mathscr{R}}^{i-1}$ for all $2 \leq i \leq \mu$, $\mathscr{S}^{\mu+1} = \mathscr{K} \cap (A\mathscr{S}^\mu + \mathscr{B}) \subset \mathscr{K} \cap (A\widehat{\mathscr{R}}^{\mu-1} + \mathscr{B}) = \mathscr{K} \cap (\mathrm{I} - BF_d)^{-1}(A\widehat{\mathscr{R}}^{\mu-1} + \mathscr{B}) = \mathscr{K} \cap E^{*-1}(A\widehat{\mathscr{R}}^{\mu-1} + \mathscr{B}) = \widehat{\mathscr{R}}^\mu$. Then: $\mathscr{S}^\mu \subset \widehat{\mathscr{R}}^{\mu-1}$, for all $\mu \geq 1$. ■

Let us note that F_{p2} is indeed a projection on $\mathscr{X}_2$ (see (27), Lemma 4 and (9)). This fact guarantees that: $\widehat{\mathscr{R}}^*_a = \min(\widehat{\mathfrak{R}}_0(E^*, A^*))$ (see also (35) and (36)), thus $(F_p, F_d) \in \mathbf{F}(\widehat{\mathscr{R}}^*_a)$ (*c.f.* (24)).

In Appendix III, we show some subspace computations and numerical simulations for our illustrative example.

VII. Conclusion

In this paper, using singularly perturbed techniques, we have shown on a particular example that the Trentelman's high gain feedback (8), issued from Theorem 5, also tends (when ε tends to zero) to a P.D. feedback, which is directly obtained from (8) (see (27), (26) and (14)). Thus, an *Almost Controllability Subspace* can also be interpreted as a subspace that can be made unobservable (when $\mathscr{K} = \text{Ker}\ C$) by means of a P.D. feedback. In fact, there are works relating $\mathscr{S}^\infty_{\mathscr{K}}$ with P.D. feedbacks (*c.f.* [17] and [1]).

We have also shown that the *Supremal Almost Controllability Subspace* contained in $\mathscr{K} = \text{Ker}\ C$, $\mathscr{S}^\infty_{\mathscr{K}}$, is indeed the *Supremal Almost* (E^*, A, B) *Controllability Subspace* contained in $\text{Ker}\ C$, $\widehat{\mathscr{R}}^*_a$. The importance of this fact is that the subspace $\widehat{\mathscr{R}}^*_a$, is also the limit of the sequences of (A, B)*–Invariant Subspaces*, $\{\mathfrak{L}_i(\varepsilon)\}$. So, the high gain state feedback (8) is also an effective approximation of a given P.D. state feedback ((26) for instance).

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Appendix I
Coefficients' Definitions

$\alpha_1 = \varepsilon^3 - 2\varepsilon^2 + \varepsilon + 1,\ \alpha_2 = -\frac{\varepsilon^3}{2} + \frac{\varepsilon}{2} + 1,\ \alpha_3 = -\varepsilon^2 + \varepsilon + 1,$ $\alpha_4 = -\varepsilon^3 + 3\varepsilon^2 + 3\varepsilon + 1,\ \alpha_5 = -\varepsilon + 1,\ \alpha_6 = -\frac{1}{6}\varepsilon^4 + \varepsilon^3 - \varepsilon^2 - \frac{2}{3}\varepsilon + 1,\ \alpha_7 = \varepsilon^4 - \frac{3\varepsilon^3}{2} + 1,\ \alpha_8 = -\frac{2\varepsilon^2}{3} + \frac{2\varepsilon}{3} + 1,\ \alpha_9 = -\frac{3\varepsilon^3}{2} + \varepsilon^2 + 2\varepsilon + 1,\ \beta_1 = -\frac{\varepsilon^5}{3} + 2\varepsilon^4 - \varepsilon^3 - \frac{5\varepsilon^2}{3} - \varepsilon + 1,\ \beta_2 = \frac{2\varepsilon^5}{3} - \varepsilon^4 - 2\varepsilon^3 + \varepsilon^2 + \varepsilon + 1, \beta_3 = 3\varepsilon^6 - 3\varepsilon^5 - 9\varepsilon^4 - 2\varepsilon^3 + 9\varepsilon^2 + 3\varepsilon + 1,\ \beta_4 = \varepsilon^2 - \varepsilon + 1, \beta_5 = \varepsilon + 1,\ \beta_6 = -2\varepsilon + 1,\ \beta_7 = 3\varepsilon^3 + 4\varepsilon^2 + \varepsilon + 1,\ \delta_1 = 3\varepsilon + 1,\ \gamma_9 = 3\varepsilon^2 + 3\varepsilon + 1,\ \eta_5 = 1 - \frac{\varepsilon}{2}$

Appendix II
Convergence of Subspaces

In Trentelman's thesis [12] is provided the following useful criterion for the convergence of subspace:[7]

Lemma 6 (Lemma 2.29–[12]): Let $\{\mathscr{V}_\varepsilon\}_{\varepsilon \in \mathbb{R}^{*+}}$ and $\mathscr{V}$ be subspaces of $\mathscr{X}$ of a given dimension. Then $\lim_{\varepsilon\to 0} \mathscr{V}_\varepsilon = \mathscr{V}$ if and only if there is a basis $\{v_1, \ldots, v_q\}$ for $\mathscr{V}$ and bases $\{v_1(\varepsilon), \ldots, v_q(\varepsilon)\}$ for $\mathscr{V}_\varepsilon$ such that $\lim_{\varepsilon\to 0} v_i(\varepsilon) = v_i,\ i = 1, \ldots, q$.

Appendix III
Subspaces computation and simulations

In order to light computations, let us define the isomorphism, $T = \left[\begin{array}{c|c|c} T_{11} & X_{11}(0) & 0 \\ \hline T_{21} & T_{22} & 0 \\ \hline T_{31} & X_{11}(0) & I_3 \end{array}\right]$, where: $T_{11} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, $T_{21} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $T_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, and $T_{31} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, then (see (25)):[8]

$$TA^* = \left[\begin{array}{c|c|c} X_{11}(0) & \overline{X}_{12} & A_{13} \\ \hline A_{21} & \overline{X}_{22} & A_{23} \\ \hline 0 & 0 & X_{33} \end{array}\right] =, \ TE^* = E^*, \qquad (35)$$

where: $\overline{X}_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $\overline{X}_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let us compute $(TE^*)^{-1}TA^*\widehat{\mathscr{R}}^*_a$ (recall that $\mathscr{S}^\infty_{\mathscr{K}} = \widehat{\mathscr{R}}^*_a$):

$$\begin{aligned} (TE^*)^{-1}TA^*\widehat{\mathscr{R}}^*_a &= (TE^*)^{-1}TA^*\mathrm{span}\{e_1, e_2, e_3, e_4, e_5, e_6\} \\ &= (TE^*)^{-1}\mathrm{span}\{f_4, f_5, f_2, f_1, f_3, f_6\} = \widehat{\mathscr{R}}^*_a \end{aligned} \qquad (36)$$

Let us compute $\widehat{\mathscr{R}}^*_{a0}$, $\mathscr{S}^\infty_{\mathscr{K}}$ and $\widehat{\mathscr{R}}^*_a$ (see (31), (33), (2), (35), (25) and (10), and also (9) and (11)):

$$\begin{aligned} \widehat{\mathscr{R}}^0 &= \mathscr{K} \cap \mathrm{Ker}\ E^* = \mathrm{span}\{e_1, e_3, e_6\} \\ \widehat{\mathscr{R}}^1 &= \mathscr{K} \cap (E^*)^{-1}(A_{F^*}\widehat{\mathscr{R}}^0 + \mathscr{B}) = \mathrm{span}\{e_4, e_2; e_1, e_3, e_6\} \\ \widehat{\mathscr{R}}^2 &= \mathscr{K} \cap (E^*)^{-1}(A_{F^*}\widehat{\mathscr{R}}^1 + \mathscr{B}) = \mathrm{span}\{e_5, e_4, e_2; e_1, e_3, e_6\} \\ \widehat{\mathscr{R}}^3 &= \mathscr{K} \cap (E^*)^{-1}(A_{F^*}\widehat{\mathscr{R}}^2 + \mathscr{B}) = \widehat{\mathscr{R}}^2 = \widehat{\mathscr{R}}^*_a \\ \widehat{\mathscr{R}}^0_0 &= \mathscr{K} \cap \mathrm{Ker}\ E^* = \mathrm{span}\{e_1, e_3, e_6\} = \mathrm{span}\{b_1, b_2, b_3\} = \mathscr{S}^1 \\ \widehat{\mathscr{R}}^1_0 &= \mathscr{K} \cap (TE^*)^{-1}TA^*\widehat{\mathscr{R}}^0_0 = \mathrm{span}\{e_4, e_2; e_1, e_3, e_6\} \\ &= \mathrm{span}\{b_1, A_{F^*}b_1, b_2, A_{F^*}b_2, b_3\} = \mathscr{S}^2 \\ \widehat{\mathscr{R}}^2_0 &= \mathscr{K} \cap (TE^*)^{-1}TA^*\widehat{\mathscr{R}}^1_0 = \mathrm{span}\{e_5, e_4, e_2; e_1, e_3, e_6\} \\ &= \mathrm{span}\{b_1, A_{F^*}b_1, b_2, A_{F^*}b_2, A^2_{F^*}b_2, b_3\} = \mathscr{S}^3 = \mathscr{S}^\infty_{\mathscr{K}} \\ \widehat{\mathscr{R}}^3_0 &= \mathscr{K} \cap (TE^*)^{-1}TA^*\widehat{\mathscr{R}}^2_0 = \widehat{\mathscr{R}}^2_0 = \widehat{\mathscr{R}}^*_{a0} = \mathscr{S}^\infty_{\mathscr{K}} = \widehat{\mathscr{R}}^*_a \end{aligned} \qquad (37)$$

In Figs. 1 and 2, we show the behaviors of the trajectory, x, in $\mathscr{S}^\infty_{\mathscr{K}} = \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \mathfrak{L}_3$ and $\mathfrak{L}(\varepsilon) = \mathfrak{L}_1(\varepsilon) \oplus \mathfrak{L}_2(\varepsilon) \oplus \mathfrak{L}_3(\varepsilon)$, of the system represented by (9) and fed back by (14)-(13), $\varepsilon = 1/100$, with the initial condition: $x(0)^T = \overline{X}(\varepsilon)\begin{bmatrix} 1 & \frac{25}{10^3} & \frac{1}{\sqrt{2}} & \frac{25}{\sqrt{2}\times 10^3} & \frac{5}{\sqrt{2}\times 10^4} & 1 & 0 & 0 & 0 \end{bmatrix} \in \mathfrak{L}(\varepsilon)$.

[7]To be understood in the usual Grassmannian sense [6]; see Section 2.4–[12] for details.

[8]We change the co-domain basis, $\{e_1, \ldots, e_9\} \to \{f_1, \ldots, f_9\}$.

In Fig. 3, we compare the behaviors obtained with the high gain feedback (14)-(13) and with the P.D. feedback (24), (we use (21)), with the initial condition: $x(0)^T = \overline{X}(\varepsilon)\begin{bmatrix} 1 & \frac{25}{10^3} & \frac{1}{\sqrt{2}} & \frac{25}{\sqrt{2}\times 10^3} & \frac{5}{\sqrt{2}\times 10^4} & 1 & 1 & 1 & 1 \end{bmatrix}$.

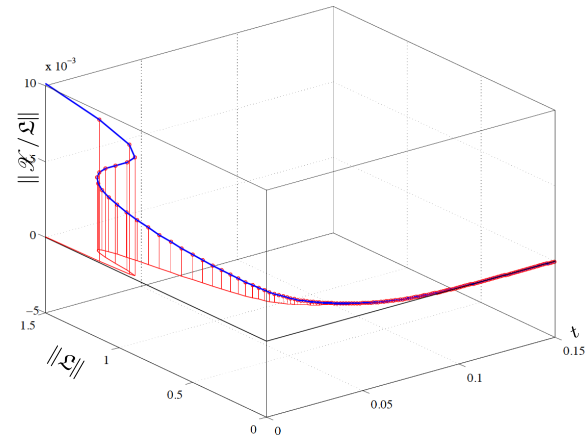

Fig. 1. Behavior of $\mathscr{S}^\infty_{\mathscr{K}} = \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \mathfrak{L}_3$

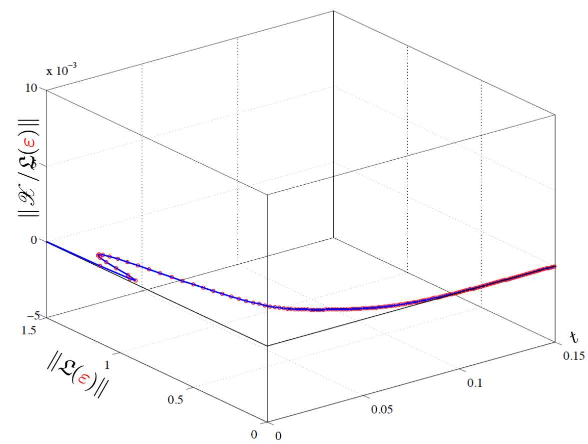

Fig. 2. Behavior of $\mathfrak{L}(\varepsilon) = \mathfrak{L}_1(\varepsilon) \oplus \mathfrak{L}_2(\varepsilon) \oplus \mathfrak{L}_3(\varepsilon)$.

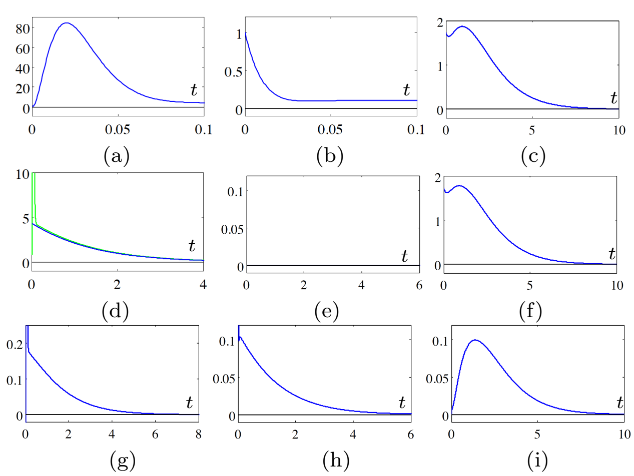

Fig. 3. (a) $\|x_1\|$, (b) $\|x_2\|$, (c) $\|x_a\|$, (d) $\|\bar{x}_1\|$ (blue trajectory) and $\|x_1\|$ (green trajectory), (e) $\|\bar{x}_2\|$ and $\|x_2\|$, (f) $\|\bar{x}_a\|$ and $\|x_a\|$, (g) $\|x_1\| - \|\bar{x}_1\|$, (h) $\|x_2\| - \|\bar{x}_2\|$, (i) $\|x_a\| - \|\bar{x}_a\|$.