2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Pattern generation in diffusive networks: how do those brainless centipedes walk?

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Abstract— **In this paper we study the existence and stability of linear invariant manifolds in a network of diffusively coupled identical dynamical systems. Symmetry under permutation of different units of the network is helpful to construct explicit formulae for linear invariant manifolds of the network, in order to classify them, and to examine their stability through Lyapunovs direct method. A particular attention is drawn to the situation when all the subsystems without interconnections are globally asymptotically stable and the oscillatory behavior is forced via diffusive coupling.**

I. INTRODUCTION

The high number of scientific contributions in the field of synchronization of coupled dynamical systems reflects the importance of this subject. The reason for this importance appears to be threefold: synchronization is common in nature, coupled dynamical systems display a very rich phenomenology and, finally, it can find applications.

First of all, many situations can be modelled as ensembles of coupled oscillators: a large number of examples of synchronization in nature can be found in [1], [2], and references therein.

The rich phenomenology constitutes another reason for the importance of these studies. Coupled dynamical systems have been shown to give rise to rather complex phenomena. Milton Erikson, used to tell a story, that a centipede was asked how it was able to move all the hundred legs in such a synchronous way. After this question had been put to the poor creature, it had been unable to make a step ever since [3]. Apparently a very primitive and distributed nervous system can generate complex wave-like patterns and this problem will be addressed in the paper. The centipede tale is not so meaningless as it seems from a first glance. There is an evidence [1] that wave-like motions of centipede's legs are generated by a spatially distributed neural network rather than by a local generator. Questions like this motivated studies of the so called *central pattern generators*, see, e.g. [4] and references therein.

Synchronization is therefore important, so it is especially important to develop criteria that guarantee its stability, if applications are sought. In this paper we consider networks of identical systems coupled through diffusion, and we give conditions that guarantee asymptotic stability of a particular invariant manifold (a synchronous state) of a given network. The diffusion which is very important for cooperative behavior of living cells, was usually considered as a smoothening or trivializing process. However it turns out that it can result in nontrivial oscillatory behavior in different systems. In this paper we are concerned with oscillatory phenomena occurring in systems consisting of diffusively coupled subsystems described by ordinary differential equations. A motivation of our study is the paper by Smale [5] who proposed an example of two 4th order diffusively coupled systems. Each system describes a mathematical cell and by itself is inert or dead in the sense that it is globally asymptotically stable. In interaction, however "the cellular system pulses (or expressed perhaps overdramatically, becomes alive!) in the sense that the concentrations of the enzymes in each cell will oscillate infinitely". In his paper Smale posed the problem to find conditions under which globally asymptotically stable systems being diffusively coupled will oscillate. For related results see [6], [7], [8]. Our approach is based on exploiting concepts related to stability such as passivity and minimum-phaseness to study synchronization [9], [10], [11], [12] (for an input-output approach, see [13]).

Synchronous motion is most often understood as the equality of corresponding variables of two identical systems. In other words, the trajectories of two (or more) identical systems will follow, after some transient, the same path in time. This situation is not, of course, the only commonly understood situation of synchronization. Other different relationships between coupled systems can be considered synchronous. In this paper we consider a situation when two different kinds of symmetries in the network, i.e. global and internal, can result in two types of synchronization: in-phase and anti-phase.

Symmetry considerations are helpful to classify several invariant sets, and a possible hierarchy to accommodate them. The symmetry generated by the coupling only has been termed *global*, to distinguish it from the additional symmetries brought upon by the dynamical systems modelling each unit, that has been termed *internal*. This terminology has been introduced in [14] where it is studied how these two groups of symmetries interact.

The research has received partial funding from the CONACYT project (Mexico) Analisis de localizacion de conjuntos compactos invariantes de systemas no lineales con dinamica compleja y sun aplicaciones, N. 000000000078890 and from Federal program "Research and Pedagogical Cadre for Innovative Russia" (N 14.740.11.0946), A.P acknowledges financial support from the European Community's Seventh Framework Program (FP7/2007-2013) under grant agreement no. INFSO-ICT-223844.

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[1]The authors thank Prof. D. Tentler from the Institute of Cytology (INC RAS) for a convincing experiment: two parts of a leach cut apart are still able to reproduce wave-like oscillations.

In this paper we exploit symmetry under permutation of a given network of dynamical systems coupled through diffusion in order to classify some linear invariant manifolds and investigate their stability. More specifically, we see that to any specific symmetry it is associated a linear invariant manifold, and we show how to construct a Lyapunov function to determine its stability, from the same specific symmetry. Therefore, under the conditions formulated in this work, stability in the network descends from its topology.

Throughout the paper we use the following notations. I_k denotes the $k \times k$ identity matrix. The Euclidean norm in $\mathbb{R}^n$ is denoted as $||\cdot||$, $||x||^2 = x^\top x$, where $^\top$ defines transposition. The notation $\mathrm{col}(x_1, \ldots, x_n)$ stands for the column vector composed of the elements $x_1, \ldots, x_n$. This notation will also be used in case where the components x_i are vectors again. For matrices A and B the notation $A \otimes B$ (the Kronecker product) stands for the matrix composed of submatrices $A_{ij}B$, where A_{ij}, $i, j = 1 \ldots n$, stands for the ij-th entry of the $n \times n$ matrix A.

II. Diffusive Cellular Networks

The subject of our research is the existence and stability of partial synchronization regimes in diffusive networks. To make the problem statement clearer, we start our discussion by introducing the concept of diffusive network. In 1976 Smale [5] proposed a model of two interacting cells based on two identical coupled oscillators, and noticed that diffusion, rather counterintuitively, does not necessarily smooth out differences between the two systems' outputs, giving the example of two stable systems that can display oscillations when connected via diffusive coupling. Taking inspiration from Smale's previous research, a *diffusive cellular network* describes a network composed of identical dynamical systems coupled through diffusive coupling that cannot be decomposed into two or more disconnected smaller networks.

To put these statements into a more mathematical description, let us consider k identical systems of the form

$$\dot{x}_j = f(x_j) + Bu_j, \quad y_j = Cx_j, \tag{1}$$

where f is a smooth vector field, $j = 1, \ldots, k$, $x_j(t) \in \mathbb{R}^n$ is the state of the j-th system, $u_j(t) \in \mathbb{R}^m$ and $y_j(t) \in \mathbb{R}^m$ are, respectively, the input and the output of the j-th system, and B, C are constant matrices of appropriate dimension. We assume that matrix CB is similar to a positive definite matrix, and the k systems are interconnected through mutual linear output coupling,

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \ldots - \gamma_{jk}(y_j - y_k) \tag{2}$$

where γ_{ij} are nonnegative constants. With no loss of generality we assume in the sequel that CB *is* a positive definite matrix.

Define the $k \times k$ matrix Γ as

$$\Gamma = \begin{pmatrix} \sum_{i=2}^{k} \gamma_{1i} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \sum_{i=1, i\neq 2}^{k} \gamma_{2i} & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \sum_{i=1}^{k-1} \gamma_{ki} \end{pmatrix} \tag{3}$$

where all row sums are zero. With definition (3), the collection of k systems (1) with feedback (2) can be rewritten in the more compact form

$$\begin{cases} \dot{x} = F(x) + (I_k \otimes B)u \\ y = (I_k \otimes C)x \end{cases} \tag{4}$$

with the feedback given by

$$u = -(\Gamma \otimes I_m)y, \tag{5}$$

where we denoted $x = \mathrm{col}(x_1, \ldots, x_k)$, $F(x) = \mathrm{col}(f(x_1), \ldots, f(x_k)) \in \mathbb{R}^{kn}$, $y = \mathrm{col}(y_1, \ldots, y_k)$, and $u = \mathrm{col}(u_1, \ldots, u_k) \in \mathbb{R}^{km}$.

If matrix Γ is symmetric, has only one zero eigenvalue and $\gamma_{ij} \geq 0$, system (1,2) is referred to as *diffusive cellular network*.

All main points have now been introduced in order to formulate a clear problem statement. Can we exploit symmetry in the network to identify its linear invariant manifolds, and benefit from a representation of the system as (1,2), and/or (4,5), typical for control purposes, in order to give conditions that *guarantee* stability of some linear invariant manifolds?

III. Symmetries and invariant manifolds

If a given network possesses a certain symmetry, this symmetry must be present in matrix Γ. In particular, the network may contain some repeating patterns, when considering the arrangements of constants γ_{ij}, hence the permutation of some elements will leave the network unchanged. The matrix representation of a permutation σ of the set $\{1, 2, ..., k\}$ is a permutation matrix $\Pi \in \mathbb{R}^{k \times k}$. Permutation matrices are orthogonal, *i.e.* $\Pi^\top \Pi = I_k$, and they form a group with respect to the multiplication.

Rewrite the dynamics of (4,5) in the closed loop form

$$\dot{x} = F(x) + Gx \tag{6}$$

where $G = -(I_k \otimes B)(\Gamma \otimes I_m)(I_k \otimes C) \in \mathbb{R}^{kn \times kn}$, that can be simplified as $G = -\Gamma \otimes BC$. Let us recall here that given a dynamical system as (6), the linear manifold $\mathcal{A}_M = \{x \in \mathbb{R}^{kn} \ : \ Mx = 0\}$, with $M \in \mathbb{R}^{kn \times kn}$, is *invariant* if $M\dot{x} = 0$ whenever $Mx = 0$. That is, if at a certain time t_0 a trajectory is on the manifold, $x(t_0) \in \mathcal{A}_M$, then it will remain there for all time, $x(t) \in \mathcal{A}_M$ for all t. The problem can be summarized in the following terms: given G and $F(\cdot)$ find a solution M to

$$MF(x(t_0)) + MGx(t_0) = 0 \tag{7}$$

for all $x(t_0)$ for which $Mx(t_0) = 0$. A natural way to solve (7) is to exploit the symmetry of the network.

A. Global symmetries

In representation (6), we can establish conditions to identify those permutations that leave a given network invariant. To this end we will establish conditions that guarantee that the set $\ker(I_{kn} - \Pi \otimes I_n)$ is invariant.

Let $\Sigma = \Pi \otimes I_n$ for simplicity, and assume that at time t_0 $x(t_0)$ satisfies $(I_{kn} - \Sigma)x(t_0) = 0$. Consider (6), and suppose that there is a solution X of the following system of linear equations:

$$(I_k - \Pi)\Gamma = X(I_k - \Pi). \tag{8}$$

Since Π is a permutation matrix, it also follows that $\Sigma F(x) = F(\Sigma x)$. If we multiply both sides of (6) by $I_{kn} - \Sigma$, we obtain, at time t_0,

$$\begin{aligned}(I_{kn} - \Sigma)\dot{x}(t_0) &= F(x(t_0)) - F(\Sigma x(t_0)) \\ &\quad -(X \otimes BC)(I_{kn} - \Sigma)x(t_0) = 0\end{aligned}$$

because we assumed $(I_{kn} - \Sigma)x(t_0) = 0$. Therefore, $(I_{kn} - \Sigma)x(t) = 0$ for all t, and we can reformulate this result as:

Lemma 3.1: Given a permutation matrix Π such that (8) has a solution X, the set

$$\ker(I_{kn} - \Pi \otimes I_n) \tag{9}$$

is a linear invariant manifold for system (6).

An important particular case arises when $X = \Gamma$, that is, Π and Γ commute. At this point one can make a remark on how to find a permutation Π that commutes with Γ. It is possible to characterize the coupling matrix Γ as an affine combination of permutation matrices.

Theorem 3.2: Let Γ be a $k \times k$ symmetric matrix with nonnegative off-diagonal elements and zero row sums. Let $\varrho > 0$ be the largest absolute value of diagonal elements. Then there are numbers $\tau_i \geq 0$, $\sum_i \tau_i = 1$ and permutation matrices Π_i so that

$$\Gamma = \varrho \left(I_k - \sum_i \frac{\tau_i}{2}(\Pi_i + \Pi_i^\top) \right) \tag{10}$$

To prove the result one can notice that $I_k - \varrho^{-1}\Gamma$ is bistochastic and invoke the seminal Birkhoff – von Neumann theorem, see.e.g. [15]. In many situations, representation (10) involves permutations from a commutative group, hence in this case any permutation matrix from this group commutes with Γ.

B. Internal symmetries

Additional internal symmetries in the differential equations governing the dynamics of the elements of the network will lead to the existence of additional linear invariant manifolds. Consider one uncoupled element of the network, $\dot{x}_j = f(x_j)$, with initial condition $x_j(0)$, generating the particular solution $x_j(t)$. It is easy to see that if

$$Jf(x_j) = f(Jx_j), \tag{11}$$

with $J \in \mathbb{R}^{n \times n}$ constant matrix, then $Jx_j(t)$ is a solution as well, generated by the initial condition $Jx_j(0)$. This property of $f(\cdot)$ defines an additional symmetry to the network, that originates additional invariant manifolds. In this paper we focus on a particular case of internal symmetries: we will assume that $f(\cdot)$ is an odd function, so $J = -I_n$. In this case, instead of (8) one can consider the following equation with respect to X:

$$(I_k + \Pi)\Gamma = X(I_k + \Pi). \tag{12}$$

As in the last argument, we can formulate the following statement:

Lemma 3.3: Suppose f is an odd function: for all x it follows that $f(-x) = -f(x)$. Given a permutation matrix Π such that (12) has a solution X, the set

$$\ker(I_{kn} + \Pi \otimes I_n) \tag{13}$$

is a linear invariant manifold for system (6).

IV. On global asymptotic stability of the linear invariant manifolds

A permutation matrix Π satisfying (8) for some X defines a linear invariant manifold of system (6), given by (9). This expression stands for a set of linear equations of the form

$$x_i - x_j = 0 \tag{14}$$

for some i and j that can be read off from the nonzero elements of the Π matrix under consideration. Therefore, we can identify a particular manifold associated with a particular matrix Π by the correspondent set $\mathcal{I}_\Pi$ of pairs i, j for which (14) holds. In this case it is natural to refer (9) to as the *in-phase synchronization manifold*.

In a similar way, if the internal symmetries are taken into account, the set $\ker(I_{kn} + \Pi \otimes I_n)$ can be represented in a form of linear equations

$$x_i + x_j = 0$$

and the corresponding invariant manifold (13) is referred to as *anti-phase synchronization manifold*.

We begin with the in-phase synchronization manifolds. From now on we consider a particular class of symmetries, for which $X = \Gamma$, i.e. we assume that there is a permutation Π that commutes with Γ. For such a Π consider the following problem: find the smallest λ so that

$$(I_k - \Pi)^\top \Gamma (I_k - \Pi) \leq \lambda (I_k - \Pi)^\top (I_k - \Pi). \tag{15}$$

Due to Gershgorin theorem, Γ is positive semi-definite. Since Γ commutes with Π it follows that Γ commutes with $(I_k - \Pi)^\top (I_k - \Pi)$. Due to commutativity, any eigenvector of Γ is either in $\mathrm{range}(I_k - \Pi)$ or in $\ker(I_k - \Pi)$. So, λ is the largest eigenvalue of Γ taken under restriction that the corresponding eigenvector lies in range of $I_k - \Pi$.

Theorem 4.1: Suppose that

i) The system (4, 5) is ultimately bounded.
ii) There exists a positive definite matrix P and positive ε such that for all $x \in \mathbb{R}^n$ the following inequality

$$P\frac{\partial f(x)}{\partial x} + \left(\frac{\partial f(x)}{\partial x}\right)^\top P \leq -\varepsilon I_n$$

holds.
iii) There is a $k \times k$ permutation Π which commutes with Γ: $\Pi\Gamma = \Gamma\Pi$.

Let λ' be the largest eigenvalue of Γ under restriction that the corresponding eigenvector lies in the range of $I_k - \Pi$. Then there exists a positive $\bar{\lambda}$ such that if $\lambda' < \bar{\lambda}$ the set $\ker(I_{kn} - \Pi \otimes I_n)$ contains a globally asymptotically stable compact subset.

Before giving a proof of this statement, it is worth giving some clarifying remarks on the assumptions imposed. First of all, it is not difficult to notice that Assumption ii implies global asymptotic stability of the system without coupling (when all the factors γ_{ij} are taken zero). The Lyapunov function that proves global asymptotic stability is quadratic and since the coupling appears in the system equations in a linear way, it is not surprising that for "small enough" γ_{ij}'s, the stability will be preserved. If the coupling factors are small, yet non-zero to ensure the existence of a permutation matrix from the theorem statement, then Lemma 3.1 guarantees that there is a linear invariant manifold. Clearly, this manifold contains a globally asymptotically stable compact subset - the origin. Now let us consider the coupled system for larger γ_{ij}'s. It can be a situation when for such a coupling, the origin is unstable, but the linear invariant manifold (9) is stable in the sense that it contains a globally asymptotically stable compact subset that differs from the origin. This idea explains that a possible proof of the result is based on treating the coupling as a linear "destabilizing" feedback, and one simply has to find conditions to ensure that this feedback can not destabilize a particular linear invariant manifold.

Proof: Consider the following Lyapunov function candidate with $\xi := \Pi \otimes I_n x$

$$\begin{aligned} V(x) &= x^\top (I_{kn} - \Pi \otimes I_n)^\top (I_k \otimes P)(I_{kn} - \Pi \otimes I_n)x \\ &= (x-\xi)^\top (I_k \otimes P)(x - \xi) \end{aligned}$$

Its time derivative along the solutions of the coupled system satisfies

$$\begin{aligned} \dot{V} &= (F(x) + Gx)^\top (I_{kn} - \Pi \otimes I_n)^\top (I_k \otimes P) \\ &\times \ (I_{kn} - \Pi \otimes I_n)x \\ &\quad + x^\top (I_{kn} - \Pi \otimes I_n)^\top (I_k \otimes P) \\ &\times \ (I_{kn} - \Pi \otimes I_n)(F(x) + Gx) \end{aligned}$$

Since Π is a permutation it follows that

$$(I_{kn} - \Pi \otimes I_n)F(x) = F(x) - F((\Pi \otimes I_n)x) = F(x) - F(\xi).$$

Assumption ii implies that

$$\begin{aligned} &(x-\xi)^\top (I_k \otimes P)(F(x) - F(\xi)) \\ &+ (F(x) - F(\xi))^\top (I_k \otimes P)(x - \xi) \le -\varepsilon ||x - \xi||^2. \end{aligned}$$

Using definition of G, one has

$$\begin{aligned} (I_{kn} - \Pi \otimes I_n)G &= -(I_{kn} - \Pi \otimes I_n)(\Gamma \otimes BC) \\ &= -(I_k \otimes I_n - \Pi \otimes I_n)(\Gamma \otimes BC) \\ &= -(I_k - \Pi)\Gamma \otimes BC \\ &= -\Gamma(I_k - \Pi) \otimes BC \end{aligned}$$

Let β be the smallest solution of the following generalized eigenvalue problem $\det(PBC + (BC)^\top P - \beta P) = 0$.

It can be derived that

$$\dot{V} \le (-\varepsilon + 2\lambda' |\beta|) V$$

and the result follows. ■

Similarly, stability of the anti-phase synchronization manifold can be established by the following result.

Theorem 4.2: Suppose that f is an odd function: for all $x \in \mathbb{R}^n$ it follows that $f(-x) = -f(x)$ and all assumptions i-iii of the previous theorem hold. Let λ' be the largest eigenvalue of Γ under restriction that the corresponding eigenvector lies in the range of $I_k + \Pi$.

Then there exists a positive $\bar{\lambda}$ such that if $\lambda' < \bar{\lambda}$ the set $\ker(I_{kn} + \Pi \otimes I_n)$ contains a globally asymptotically stable compact subset.

V. DIFFUSION DRIVEN OSCILLATIONS

From the assumptions imposed on f one can conclude that without interconnections between the subsystems all the solutions converge to a globally asymptotically stable equilibrium point (it follows from the assumption ii). So, an oscillatory behavior can be brought via the way the systems are interconnected.

In this section we present conditions that guarantee that the network of diffusively coupled systems exhibits oscillatory behavior. We begin with a definition of oscillatory system that will be in use in the sequel.

Consider the following nonlinear system

$$\dot{x} = F(x), \ y = h(x), \quad x(t) \in \mathbb{R}^n, \quad y(t) \in \mathbb{R}^1. \tag{16}$$

where F satisfies assumptions guaranteeing existence of a unique solution on the infinite time interval, and y represents the output of the dynamical system (16). The system (16) is called *oscillatory with respect to a scalar output* y in the sense of Yakubovich if it is ultimately bounded and for almost all initial conditions there is no limit $\lim_{t\to\infty} y(t)$. We call the system oscillatory if it is oscillatory with respect to at least one of the components of the vector x.

The following result can be proved similarly to the proof of Theorem 1.1, $3°, 5°$ [6].

Theorem 5.1: Assume that

i) The equation $F(x) = 0$ has only isolated solutions $\bar{x}_j, j = 1, 2....$
ii) The system (16) is ultimately bounded.
iii) $\bar{x}_j$ are hyperbolic fixed points and each matrix $\frac{\partial F}{\partial x}(\bar{x}_j)$ has at least one eigenvalue with positive real part.

Then the system (16) is oscillatory in the sense of Yakubovich.

The proof of the statement is based on the Hartman-Grobman theorem.

The goal of this section is to build an example of a network of diffusively coupled systems that has the following property: i) each uncoupled system is globally asymptotically stable at the origin ii) the origin is the unique equilibrium of the network iii) each subsystem of the network has odd symmetry iv) for strong enough coupling the network is oscillatory in the sense of Yakubovich.

To accomplish the goal consider the network (1,2) with

$$f(x_j) = Ax_j - B\phi(z_j), \quad x_j \in \mathbb{R}^n, \quad n \ge 3$$

with scalar ϕ, z_j and y_j and matrices A, B, C of the corresponding dimensions that satisfy the following assumption

Assumption 5.2: The following conditions hold

i) The matrix A is Hurwitz, so there is a positive definite matrix $P = P^\top$, so that $A^\top P + PA < 0$.

ii) $z_j = Zx_j$, where $Z^\top = PB$ with the matrix P as in i.

iii) ϕ is an odd smooth strictly increasing function with the following property:

$$\forall C > 0 \ \ \exists \sigma > 0 \ \ \forall z > \sigma \ \ \ \phi(z) > Cz.$$

iv) Let $W_y(s)$ be the transfer function of the linear part from u_j to $y_j = Cx$ taking $\phi(z_j) = 0$: $W_y(s) = C(sI - A)^{-1}B$. Then $W_y(s)$ is nondegenerate, it has relative degree one with even number of zeroes with positive real part and $W_y(0) > 0$.

v) Let $W_z(s)$ be the transfer function $W_z(s) = Z(sI - A)^{-1}B$. Then $W_z(0) > 0$.

Theorem 5.3: Consider a network of diffusively coupled systems (1,2) with a symmetric Γ as in (3) that satisfies Assumption 5.2.

There is a number $\bar{\lambda} > 0$ so that if the largest eigenvalue of Γ exceeds $\bar{\lambda}$ then the network is oscillatory in the sense of Yakubovich.

Proof: To prove the result one can verify conditions i-iii of Theorem 5.1. We sketch the prove.

i)We are going to prove that the origin is the unique equilibrium of the closed loop system. The Jacobi matrix of the right-hand side is given by

$$J(x) = I_k \otimes A - \Gamma \otimes BC - I_k \otimes BZ\phi'(z_j)$$

Since Γ is similar to a diagonal matrix with nonnegative entries there is a nonsingular $k \times k$ matrix S so that

$$(S \otimes I_n) J (S^{-1} \otimes I_n) = I_k \otimes A - \Lambda \otimes BC - I_k \otimes BZ\phi' \quad (17)$$

where Λ is a diagonal matrix with nonnegative entries λ_j. The right-hand side of (17) is a block-diagonal matrix of the form

$$J_j = A - \lambda_j BC - BZ\phi'(z_j)$$

To calculate its determinant recall the following identity $\det(Q + RT) = \det(Q)\det(I + RQ^{-1}T)$ that yields

$$\begin{aligned} \det J_j &= \det(A)(1 - BA^{-1}C\lambda_j - BA^{-1}Z\phi'(z_j)) \\ &= \det(A)(1 + W_y(0)\lambda_j + W_z(0)\phi'(z)). \quad (18) \end{aligned}$$

Since $W_y(0) > 0$, $W_z(0) > 0$, $\lambda_j \geq 0$ and $\phi'(z_j) > 0$ one concludes that the Jacobi matrix is nonsingular at any point.

Now consider an auxiliary system replacing Γ with $\varepsilon\Gamma$, where $\varepsilon \in [0, 1]$ is a parameter. As before, for any such an ε the Jacobian of the right-hand side is nonsingular. Suppose for some $\varepsilon = \varepsilon_* \in [0, 1]$ there is an equilibrium point x_{eq} of the auxiliary system which is different from the origin. Due to the implicit function theorem this point is determined as a solution of some equation $x_{eq} = \mathcal{F}(\varepsilon_*)$. Decreasing ε from ε_* to zero, one concludes that there is some equilibrium point different from the origin when $\varepsilon = 0$. However it contradicts with the global asymptotic stability of the free system ($u_j = 0$). This contradiction proves the uniqueness of the equilibrium point.

ii) To prove the ultimate boundedness, consider the following Lyapunov function $V(x) = x^\top (I_k \otimes P)x$ with P from item i of Assumption 5.2. Then using items ii and iii of Assumption 5.2 one can find such a number $\mathcal{C}$ so that $V(x) > \mathcal{C}$ implies $\dot{V} < 0$.

iii) Similar to the proof of item i, to prove the instability of the origin it suffices to prove instability of the linear system

$$\dot{\xi} = (A - \lambda BC)\xi \quad (19)$$

for sufficiently large $\lambda > 0$. Due to Assumption 5.2.iv stability of this system is determined by the roots of the following polynomial $Q_{n-1}(s) + \varepsilon R_n(s) = 0$, where $\varepsilon = 1/\lambda$ and $Q_{n-1}(s)$ is the numerator of $W_y(s)$ of degree $n-1$ and $R_n(s)$ is a Hurwitz polynomial of degree n (denominator of $W_y(s)$). Since Q_{n-1} is unstable for sufficiently small ε system (19) is unstable. In particular, one can utilize the argument used in the proof of Lemma 2 in [16] to show that $n-1$ eigenvalues of the closed loop system will tend to the roots of $Q_{n-1}(s)$ as $\varepsilon \to 0$, while the rest eigenvalue will remain negative since $CB > 0$. The proof is completed. It is worth mentioning that we have not required (see Assumption 5.2.iv) that the zero dynamics of (A, B, C) has an even number of unstable zeroes. Note, however, that since A is Hurwitz $W_y(0)$ has the same sign as $\Pi_i^{n-1}(-\sigma_i)$, where σ_i are the zeroes of the numerator of $W_y(s)$. Therefore, $W_y(0) > 0$ necessarily implies that the number of unstable zeroes is even and, hence, $n \geq 3$. ■

The theorems presented in the paper allow one to design a network of diffusively coupled systems, so that each free system is globally asymptotically stable, the network is oscillatory in the sense of Yakubovich and there are different in- and anti-phase synchronous motions determined by the global and internal symmetries of the network. Such networks, if they possess circular topology can generate wave-like patterns similar to coordinated motion of centipedes' legs. In the next section we demonstrate how to make the theory operational via an example.

VI. An Example

Consider a ring of four diffusively coupled systems

$$\begin{aligned} \dot{x}_j &= Ax_j + B(u_j - z_j^3), \quad j = 1, \ldots, 4, \quad x(t) \in \mathbb{R}^3 \\ z_j &= Zx, \quad y_j = Cx, \quad u = -\Gamma y \end{aligned}$$

with

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -4 & 2 & -3 \end{pmatrix}$$

$$B = (0\ 0\ 1)^\top, \quad C = (0\ 0\ 1), \quad Z = B^\top P$$

where P is a solution of the following Lyapunov equation

$$A^\top P + PA = -I_3.$$

In this case we have

$$W_y(s) = \frac{s^2 - s + 1}{s^3 + 2s^2 + 2s + 1}, \ W_z(s) = \frac{s^2 + 1.5s + 1}{s^3 + 2s^2 + 2s + 1},$$

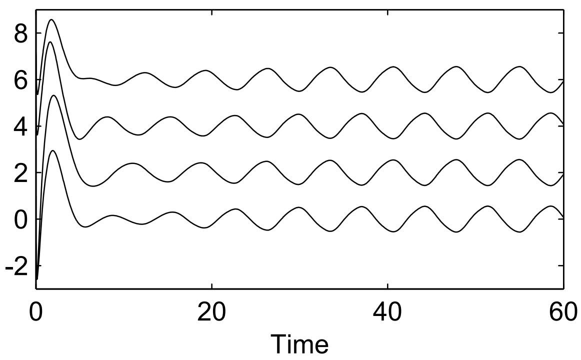

Fig. 1. Synchronous behavior in a ring of 4 systems, $y_j(t)+2(j-1),\ j = 1,\dots,4$ versus time. $\gamma = 1.5a$.

and therefore, all the conditions imposed in Assumption 5.2 are satisfied and $CB > 0$, as required from the definition of diffusive coupling. The coupling between the systems is defined by the following coupling matrix

$$\Gamma = \gamma \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

with a positive parameter γ. The largest eigenvalue of Γ is 4γ. Consider a symmetric permutation $1 \leftrightarrow 2,\ 3 \leftrightarrow 4$ and the corresponding permutation matrix Π_1, that commutes with Γ. The largest eigenvalue of Γ under restriction that the corresponding eigenvector is in the range of $I_4 + \Pi_1$ is 2γ. Consider a symmetric permutation $1 \leftrightarrow 3,\ 2 \leftrightarrow 4$ and the corresponding permutation matrix Π_2, that commutes with Γ. The largest eigenvalue of Γ under restriction that the corresponding eigenvector is in the range of $I_4 - \Pi_2$ is 2γ too. Finally, one concludes that there is a number $a > 0$ so that if $a < \gamma < 2a$ (a can be calculated in this case from the condition of Hopf bifurcation: $a = (\sqrt{13}-1)/8$) the network is oscillatory in the sense of Yakubovich and the intersection of sets $\ker(I_{12} + \Pi_1 \otimes I_3)$ and $\ker(I_{12} - \Pi_2 \otimes I_3)$ contains globally asymptotically stable compact subset. It means that the first and third subsystems oscillate in phase, the second and the fourth subsystems are in phase as well, while the first and the second subsystems are in anti-phase.

The results of computer simulation are depicted on Fig. 1. It can be seen that after some transient period the synchronous regime is settled. Computer simulation reveals that for $\gamma > 2a$ this regime is stable as well. At the same time in this case the unstable manifold of the origin is transverse to the synchronous manifold, so no conclusion about the global stability can be derived in this case. A possible way to study synchronization in this case is to apply the method developed in [4].

A more relevant for the topic of the paper example with 24 neurons described by the previous model, that demonstrates wave-like oscillation will be analyzed in forthcoming publications, for the results of computer simulations, see

`http://www.youtube.com/watch?v=OlKLfY8MM6o`

VII. Conclusions

The paper addresses a problem of pattern generation in diffusive networks. A motivation for our study was an example studied by Smale [5]. He wrote: "There is a paradoxical aspect to the example. One has two dead (mathematically dead) cells interacting by a diffusion process which has a tendency in itself to equalize the concentrations. Yet in interaction, a state continues to pulse indefinitely". It is shown in this paper that globally asymptotically stable systems being interconnected via diffusive coupling can generate synchronous wave-like patterns if the network and individual subsystems possess some symmetries: if a coupling strength exceeds some bifurcation threshold, synchronous oscillations appear spontaneously like a Mexican wave in a football stadium.

References

[1] A. Pikovsky, M. Rosenblum and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Science*, Cambridge University Press, 2001.

[2] S. Strogatz, *SYNC: The Emerging Science of Spontaneous Order*, Hyperion, 2003.

[3] J. Haley, *Jay Haley On Milton H. Erikson*, Brunner/Mazel, Bristol, 1993.

[4] T. Iwasaki, "Multivariable harmonic balance for central pattern generators," *Automatica*, **44**(12), 3061–3069, 2008.

[5] S. Smale, "A mathematical model of two cells via Turing's equation", in Marsden, J.E. and M. McCracken (eds.) *The Hopf bifurcation and its applications*, pp. 354–367, Springer-Verlag, New York, 1976.

[6] E. Tomberg, V. Yakubovich, "Conditions for auto-oscillations in nonlinear systems", *Siberian Mathematical Journal*, **30**(4), 180–195, 1989.

[7] E. Tomberg, V. Yakubovich, "On one problem of Smale", *Siberian Mathematical Journal*, **41**(4), 771-774, 2000

[8] A. Pogromsky, T. Glad, and H. Nijmeijer, "On diffusion driven oscillations in coupled dynamical systems," *Int. J. Bifurcation and Chaos* **9**, 629-644, 1999.

[9] A. Pogromsky, "Passivity based design of synchronizing systems" *Int. J. Bifurcation and Chaos*, vol. **8**(2), pp. 295–319, 1998.

[10] A. Pogromsky, G. Santoboni, H. Nijmeijer, Partial synchronization: from symmetry towards stability, *Physica D,* **172**, pp. 65-87, 2002.

[11] A. Pogromsky, "A partial synchronization theorem", *Chaos*, **18**, 037107, see also erratum in *Chaos* **19**, 049901, 2009.

[12] G. Stan and R. Sepulchre, Analysis of interconnected oscillators by dissipativity theory, *IEEE Trans. Autom. Control*, **52**(2), 256270, 2007.

[13] L. Scardovi, M. Arcak, E. Sontag, "Synchronization of interconnected systems with applications to biochemical networks: an input-output approach", *IEEE Transactions Aut. Control,* **55**(6), 1367–1379, 2010.

[14] B. Dionne, M. Golubitsky and I. Stewart, "Coupled cells with internal symmetry," I, II, *Nonlinearity* **9**, 559-599, 1996.

[15] J.V. Romanovsky, 'A simple proof of Birkhoff-von Neumann theorem on bistochastic matrices', in *Discourses in Mathematics and its Applications. A tribute to Ilya Bakelman*, n3, Texas A & M University, 51–53, 1994.

[16] A. Fradkov, D. Hill, "Exponential feedback passivity and stabilizability of nonlinear systems," *Automatica*, **34**(6), 697–703, 1998.