2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

# On Casimir Functionals for Field Theories in Port-Hamiltonian Description for Control Purposes

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***Abstract*— We consider infinite dimensional Port-Hamiltonian systems in an evolutionary formulation. Based on this system representation conditions for Casimir densities (functionals) will be derived where in this context the variational derivative plays an extraordinary role. Furthermore the coupling of finite and infinite dimensional systems will be analyzed in the spirit of the control by interconnection problem. Our Hamiltonian representation differs significantly from the well-established one using Stokes-Dirac structures that are based on skew-adjoint differential operators and the use of energy variables. We mainly base our considerations on a bundle structure with regard to dependent and independent coordinates as well as on differential-geometric objects induced by that structure.**

## I. Introduction

Port controlled Hamiltonian systems with Dissipation (PCHD-Systems) enjoy great popularity in the control and modeling community. In many applications the physics behind the equations becomes apparent in a remarkable way by the use of this system structure, see for example [1]. A key benefit of this system class lies in the possibility to couple several systems via (energy) ports [1] which is advantageous for modeling of networks. Concerning control issues the control by interconnection method is based on such couplings. The study of these systems is not limited to the finite dimensional case and regarding infinite dimensional systems an approach based on Stokes-Dirac structures (also known from the lumped parameter scenario) was proposed by many authors, see for instance [2], [3], [4], [5] and references therein.

Our approach for field theories in a Hamiltonian setting is based on a bundle structure with respect to independent and dependent coordinates (not focusing on the properties of the underlying Stokes-Dirac structures), such that the variational derivative is interpreted differently compared to [2], [3], [4], [5], i.e. the Hamiltonian density explicitly depends on derivative coordinates. See also [6] as well as [7], [8] where the ideas presented in [6] were adapted to control purposes. Therefore our system class is not necessarily based on the use of skew-adjoint differential operators and the use of energy variables.

In this contribution we focus on the derivation of Casimir functionals for distributed parameter systems in an evolutionary Hamiltonian formulation and it will be demonstrated that the variational derivative plays an important role for the system representation and the extraction of the Casimir functionals. Furthermore the coupling of finite and infinite dimensional systems will be analyzed and Casimir functionals/functions will be derived for the coupled systems, where we restrict our considerations to one dimensional spatial domains for the partial differential equations.

Inspired by the work of [4], [5], [9], [10] where similar investigations have been performed using the method of Stokes-Dirac structures, we analyze the concepts of interconnection and Casimir functionals based on a system representation as in [6] which we generalized for our purposes, see [8] such that control inputs (in the domain and/or the boundary) and nontrivial boundary conditions important in concrete physical/engineering applications appear. We believe that the formulation not based on energy variables is advantageously in some aspects. The choice of the dependent variables is essentially different in the two approaches (Stokes-Dirac structure [4] versus evolutionary Hamiltonian as in [8]). In mechanical applications for example we use the displacement as a coordinate and not the strain, see [9], and this is of course beneficial when position control is the objective, i.e. it is not necessary to consider the partial differential equation as a kind of transmission system between two finite dimensional ones. This has the consequence that for example in mechanics the interconnection tensor is not necessarily a differential operator in our setting, and the Hamiltonian density depends on derivative coordinates which has the implication that the variational derivative does not degenerate to a partial one.

This paper is organized as follows. In Section II some notation is presented and in the third section we review the finite dimensional case to introduce some basic concepts as well as a coordinate free system representation. The fourth section then deals with the infinite dimensional case where the evolutionary Hamiltonian picture is introduced and the conditions for the Casimir functionals are derived. The interconnection of finite and infinite dimensional systems as well as the control by interconnection method are discussed in the fifth section. A specific application, a heavy chain system is analyzed in the sixth section where also some simulation studies are presented.

## II. Notation

We will use differential geometric methods for our considerations and the notation is similar to the one in [11], where the interested reader can find much more details about

M. Schöberl is an APART fellowship holder of the Austrian Academy of Sciences. His work was also partly supported by the Austrian Center of Competence in Mechatronics (ACCM)

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this geometric machinery. To keep the formulas short and readable we will use tensor notation and especially Einsteins convention on sums. We use the standard symbol $\otimes$ for the tensor product, $\wedge$ denotes the exterior product (wedge product), $\mathrm{d}$ is the exterior derivative, $\rfloor$ the natural contraction between tensor fields. By $\partial_\alpha^B$ are meant the partial derivatives with respect to coordinates with the indices ${}^\alpha_B$ and $[m^{\alpha\beta}]$ corresponds to the matrix representation of the tensor $m$ with components $m^{\alpha\beta}$. Furthermore $C^\infty(\cdot)$ denotes the set of the smooth functions on the corresponding manifold. Moreover we will not indicate the range of the used indices when they are clear from the context. Additionally pull back bundles are not indicated to avoid exaggerated notation.

## III. The Finite dimensional Case

In this section the well known class of time-invariant Port Controlled Hamiltonian systems (with dissipation, PCH(D)-systems) is considered.

### A. System Structure

To study the time-invariant case of Hamiltonian control systems in a geometric fashion we introduce the state manifold $\mathcal{X}$ equipped with coordinates $(x^\alpha)$. The tangent bundle $\mathcal{T}(\mathcal{X})$ and the cotangent bundle $\mathcal{T}^*(\mathcal{X})$, which possess the induced coordinates $(x^\alpha, \dot{x}^\alpha)$ and $(x^\alpha, \dot{x}_\alpha)$ with respect to the holonomic bases $\partial_\alpha$ and $\mathrm{d}x^\alpha$ can be introduced in a standard manner. Typical elements of $\mathcal{T}(\mathcal{X})$ (*vector fields*) and $\mathcal{T}^*(\mathcal{X})$ (*1-forms*) read in local coordinates as $w = \dot{x}^\alpha(x)\partial_\alpha$ and $\omega = \dot{x}_\alpha(x)\mathrm{d}x^\alpha$, respectively. To introduce inputs and outputs we consider the vector bundle $\mathcal{U} \to \mathcal{X}$ with the coordinates $(x^\alpha, u^i)$ for $\mathcal{U}$ and the basis $e_i$ for the fibres as well as the dual output vector bundle $\mathcal{Y} \to \mathcal{X}$ possessing the coordinates $(x^\alpha, y_i)$ and the fibre basis $e^i$. The Greek indices $\alpha, \beta, \gamma$ will correspond to the components of the coordinates of the state manifold and induced structures. The Latin indices $i, j$ correspond to the components of the input and the output variables (fibres of the dual bundles $\mathcal{U} \to \mathcal{X}$ and $\mathcal{Y} \to \mathcal{X}$). Let us consider the maps $J, R : \mathcal{T}^*(\mathcal{X}) \to \mathcal{T}(\mathcal{X})$ which are contravariant tensors where $J$ (interconnection tensor) is skew-symmetric, and $R$ (dissipation tensor) is symmetric and positive-semidefinite. Furthermore we introduce the bundle map $G : \mathcal{U} \to \mathcal{T}(\mathcal{X})$ and its dual (adjoint) $G^* : \mathcal{T}^*(\mathcal{X}) \to \mathcal{Y} = \mathcal{U}^*$. Having the maps $J, R$ and $G$ at our disposal a time-invariant port controlled Hamiltonian system with dissipation, see [1], [12] can be constructed as

$$\begin{aligned} \dot{x} &= (J - R)\rfloor \mathrm{d}H + u\rfloor G \\ y &= G^*\rfloor \mathrm{d}H \end{aligned} \tag{1}$$

where the function $H \in C^\infty(\mathcal{X})$ denotes the Hamiltonian. The local coordinate expression of the system (1) can be deduced as

$$\begin{aligned} \dot{x}^\alpha &= \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta H + G_i^\alpha u^i \\ y_i &= G_i^\alpha \partial_\alpha H \end{aligned} \tag{2}$$

where the maps $J, R, G$ have the coordinate representation

$$J = J^{\alpha\beta}\partial_\alpha \otimes \partial_\beta\,, \quad R = R^{\alpha\beta}\partial_\alpha \otimes \partial_\beta\,, \quad G = G_i^\alpha e^i \otimes \partial_\alpha,$$

and the tensor representation of the map $G^*$ equals the one of $G$.

Let us consider a vector field $v$ on $\mathcal{X}$ (possibly depending on $u$), i.e $v : \mathcal{X} \to \mathcal{T}(\mathcal{X})$, $v = v^\alpha(x, u)\partial_\alpha$, then the change of $H$ in the direction of $v$ (Lie-derivative) reads as $v(H) = v\rfloor \mathrm{d}H$. If $v$ is chosen such that $v = \dot{x}$ from (1) then we obtain

$$v(H) = -R\rfloor \mathrm{d}H\rfloor \mathrm{d}H + u\rfloor y \tag{3}$$

where we write $\dot{H} = v(H)$ in this special case. Obviously the relation (3) shows how the Hamiltonian is affected along solutions of the system, namely by dissipation and the collocation of the inputs and outputs.

### B. Casimir Functions

A Casimir function for the System (1) is a function $C \in C^\infty(\mathcal{X})$ such that

$$\dot{C} = \dot{x}\rfloor \mathrm{d}C = (u\rfloor G)\rfloor \mathrm{d}C \tag{4}$$

holds independently of the Hamiltonian $H$. This leads to the partial differential equation $\partial_\alpha C(J^{\alpha\beta} - R^{\alpha\beta}) = 0$ in the unknown functions $C$. If in particular $(u\rfloor G)\rfloor \mathrm{d}C = 0$ then $C$ is a constant of the motion since then $\dot{C} = 0$.

This section was a reminder on some well known properties concerning PCHD systems in the finite dimensional case and is mainly used to prepare for the generalization to first order field theories in a geometric fashion.

## IV. The Infinite dimensional Case

The generalization to the case of partial differential equations will be performed by adapting the maps and spaces introduced in section III to cope with field theories[1], see also [7], [8]. Hamiltonian methods for evolutionary equations have been discussed also in [6] but in contrast to our exposure the focus is not on control topics (i.e. there are no inputs present, neither on the domain nor on the boundary) and the boundary conditions are chosen to be trivial which has an impact also on the study of Casimir densities, which are called distinguished functionals in [6].

### A. System Structure

Instead of the Hamiltonian $H$, a function, we now consider a Hamiltonian density $\mathfrak{H}$ (as well as its integrated quantity, also denoted by $H$) defined on a bundle $\mathcal{X} \to \mathcal{D}$, $(X^A, x^\alpha) \to (X^A)$. The first jet manifold $\mathcal{J}^1(\mathcal{X})$ can be introduced possessing the coordinates $(X^A, x^\alpha, x_A^\alpha)$, where the capital Latin indices $A, B$ are used for the base manifold $\mathcal{D}$ (independent coordinates) and $x_A^\alpha$ denote derivative coordinates of first order (derivative of the dependent coordinates with respect to the independent ones). The jet structure also induces the so-called total derivative

$$d_A = \partial_A + x_A^\alpha \partial_\alpha + x_{AB}^\alpha \partial_\alpha^B$$

[1]Let us again point out that pull backs are omitted throughout this paper in order to enhance the readability. This means that when evaluating integrals the jet-prolongation of the corresponding section has to be plugged in. Furthermore tensors are indicated without the pull backs to corresponding manifolds.

acting on elements including first order derivatives and $x^\alpha_{AB}$ correspond to derivative coordinates of second order living in $\mathcal{J}^2(\mathcal{X})$, the second jet manifold. Then a Hamiltonian density of first order (we restrict ourselves to this case) reads as $\mathfrak{H} = \mathcal{H}\mathrm{d}X$ with $\mathcal{H} \in C^\infty(\mathcal{J}^1(\mathcal{X}))$ where $\mathrm{d}X$ denotes the volume element on the manifold $\mathcal{D}$, i.e. $\mathrm{d}X = \mathrm{d}X^1 \wedge \ldots \wedge \mathrm{d}X^{A_d}$ with $\dim(\mathcal{D}) = A_d$. Additionally we denote by $H = \int_\mathcal{D} \mathfrak{H}$ the integrated quantity. The maps $J, R, G$ as in (1) now take the form of

$$\mathcal{J}, \mathcal{R} : \mathcal{T}^*(\mathcal{X}) \wedge (\overset{A_d}{\wedge} \mathcal{T}^*(\mathcal{D})) \to \mathcal{V}(\mathcal{X}), \;\; \mathcal{G} : \mathcal{U} \to \mathcal{V}(\mathcal{X})$$

with the vertical tangent bundle $\mathcal{V}(\mathcal{X}) \to \mathcal{X}$ (tangent vectors meeting $\dot{X}^A = 0$), where in general these maps are differential operators, but within this contribution we exclude this case. A port controlled Hamiltonian system then takes the form of

$$\begin{array}{rcl} \dot{x} & = & (\mathcal{J} - \mathcal{R})(\delta\mathfrak{H}) + u \rfloor \mathcal{G} \\ y & = & \mathcal{G}^* \rfloor \delta\mathfrak{H} \end{array} \tag{5}$$

with the variational derivative

$$\delta\mathfrak{H} : \mathcal{J}^2(\mathcal{X}) \to \mathcal{T}^*(\mathcal{X}) \wedge (\overset{A_d}{\wedge} \mathcal{T}^*(\mathcal{D}))$$

and additional boundary conditions (possibly including boundary inputs, optionally leading to so-called boundary ports). In coordinates we obtain

$$\delta\mathfrak{H} : (\delta_\alpha \mathcal{H}) \to (\delta_\alpha \mathcal{H}) \mathrm{d}x^\alpha \wedge \mathrm{d}X \,, \;\; \delta_\alpha = \partial_\alpha - d_A \partial^A_\alpha .$$

The input and the output bundles[2] are given as $\mathcal{U} \to \mathcal{X}$ and $\mathcal{Y} \to \mathcal{X}$. The local coordinate representation of (5) reads as

$$\begin{array}{rcl} \dot{x}^\alpha & = & \left(\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}\right) \delta_\beta \mathcal{H} + \mathcal{G}^\alpha_i u^i \\ y_i & = & \mathcal{G}^\alpha_i \delta_\alpha \mathcal{H}. \end{array}$$

Let us consider in analogy to the finite dimensional case a vector field which is used to measure the change of the Hamiltonian, i.e. the Hamiltonian density since we are dealing with field theories. We use a (generalized) vertical vector field $v : \mathcal{X} \to \mathcal{V}(\mathcal{X})$ locally given as $v = v^\alpha \partial_\alpha$ where $v^\alpha$ may depend on derivative coordinates, together with its first jet-prolongation $j^1(v)$ which reads as $j^1(v) = v^\alpha \partial_\alpha + d_A(v^\alpha) \partial^A_\alpha$. Then we obtain the relation

$$j^1(v)(\mathcal{H}\mathrm{d}X) = \left(v^\alpha(\partial_\alpha \mathcal{H} - d_A \partial^A_\alpha \mathcal{H}) + d_A(v^\alpha \partial^A_\alpha \mathcal{H})\right) \mathrm{d}X$$

and applying the Theorem of Stokes we find that

$$\int_\mathcal{D} j^1(v)(\mathcal{H}\mathrm{d}X) = \int_\mathcal{D} v^\alpha \left(\delta_\alpha \mathcal{H}\right) \mathrm{d}X + \int_{\partial\mathcal{D}} v^\alpha \left(\partial^A_\alpha \mathcal{H}\right) \mathrm{d}X_A \tag{6}$$

with $\mathrm{d}X_A = \partial_A \rfloor \mathrm{d}X$ is met. For many applications the boundary term can be used for the introduction of boundary ports[3].

Consequently setting $v = \dot{x}$ using (5) we can analyze the change of the Hamiltonian density along solutions of the partial differential equations (assuming their existence).

[2]It should be noted that the input and the output bundle are dual with respect to the interior product given as $(u^j e_j) \rfloor (y_i e^i \otimes \mathrm{d}X) = y_i u^i \mathrm{d}X$.

[3]Provided that $\mathcal{H}\mathrm{d}X$ corresponds to the energy density, a boundary port according to our definition allows for a non-zero energy flow through the boundary.

If furthermore the Hamiltonian density reflects the system energy a power balance relation can be stated using (6). In the forthcoming we set $\dot{H} = \int_\mathcal{D} j^1(v)(\mathcal{H}\mathrm{d}X)$ where we assume that the field $v$ generates a semi group and roughly speaking $\dot{\diamondsuit}$ represents the Lie-derivative of the object $\diamondsuit$ with respect to the group parameter.

*Example 1:* Let us consider the system of a heavy chain exposed to gravity (acceleration due to gravity is denoted by $g$). We introduce the following bundle structure $\mathcal{X} \to \mathcal{D}$, $(X, w, p) \to X$, where $X$ is the coordinate of the one-dimensional spatial domain, $w$ denotes the deflection and $p$ the temporal momentum. The first jet manifold $\mathcal{J}^1(\mathcal{X})$ additionally includes the derivative coordinates $w_X$ and $p_X$ and the boundary $\partial\mathcal{D}$ consists of two points only, namely $X = 0$ and $X = L$ where $L$ is the length of the chain. Approximately, the system can be modeled by the partial differential equation $\rho\ddot{w} = d_X(P(X) w_X)$ which can also be stated as

$$\begin{array}{rcl} \dot{w} & = & \frac{p}{\rho} \\ \dot{p} & = & d_X(P(X) w_X) \end{array} \tag{7}$$

with boundary conditions $P(X) w_X|_{\partial\mathcal{D}} = 0$ where $\rho$ is the mass (line)density and the force in the chain reads as $P(X) = g\rho X$. To rewrite this system in a Hamiltonian fashion we consider the Hamiltonian density $\mathfrak{H} = \mathcal{H}\mathrm{d}X$ corresponding to the energy density with

$$\mathcal{H} = \frac{1}{2\rho} p^2 + \frac{1}{2} P(X) w_X^2 . \tag{8}$$

The total energy can be evaluated by $H = \int_0^L \mathcal{H}\mathrm{d}X$. To obtain partial differential equations in the form as in (5) we can set $\mathcal{R}$ and $\mathcal{G}$ to zero (no damping and no inputs acting on the domain). A canonical choice for the map

$$[\mathcal{J}^{\alpha\beta}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

with $x = (w, p)$ together with the expression

$$\delta_w \mathcal{H} = \partial_w \mathcal{H} - d_X \partial^X_w \mathcal{H} = -d_X(P(X) w_X) \tag{9}$$

and

$$\delta_p \mathcal{H} = \partial_p \mathcal{H} - d_X \partial^X_p \mathcal{H} = \frac{p}{\rho}$$

gives the desired result $\dot{x} = \mathcal{J}(\delta\mathfrak{H})$. The boundary terms stem from $\partial^X_w \mathcal{H}\big|_{\partial\mathcal{D}} = P(X) w_X|_{\partial\mathcal{D}} = 0$ if no external forces act on the system.

*Remark 1:* It is worth mentioning at this stage that as the Example 1 shows the Hamiltonian in our setting depends explicitly on the derivative coordinates, in this case $w_X$, and therefore the variational derivative does not degenerate to a partial one, see (9). Furthermore the map $\mathcal{J}$ is no differential operator in our setting. These two properties are significantly different in a description based on Stokes-Dirac structures, see [3], [4], [5], [9] for an alternative Hamiltonian setting where in mechanical applications $w_X$ is a state variable and $\mathcal{J}$ is clearly a differential operator. In [3] the chain system (vibrating string) is formulated based on Stokes-Dirac structures and the differences to our approach become apparent, of course the resulting partial differential equations are the same, but their Hamiltonian representation differs.

### B. Casimir Functionals

In the case of partial differential equations we consider Casimir densities (or functionals) and we restrict ourselves to the first order case only, i.e. $\mathfrak{C} = \mathcal{C}\mathrm{d}X$ with $\mathcal{C} \in C^\infty(\mathcal{J}^1(\mathcal{X}))$. Using (6) where we replace $\mathcal{H}$ by $\mathcal{C}$ and setting $v = \dot{x}$ the relation (in analogy to (4))

$$\int_{\mathcal{D}} j^1(\dot{x})(\mathcal{C}\mathrm{d}X) = \int_{\mathcal{D}} (u\rfloor\mathcal{G})\rfloor\delta\mathfrak{C}$$

is obtained provided $\int_{\mathcal{D}}(\mathcal{J}-\mathcal{R})(\delta\mathcal{H})\rfloor\delta\mathfrak{C} = 0$ is met and an additional boundary expression vanishes. This leads to the following two conditions for the Casimir density

$$\delta_\alpha\mathcal{C}(\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) = 0 \tag{10}$$

$$(\dot{x}^\alpha\partial_\alpha^A\mathcal{C})\big|_{\partial\mathcal{D}} = 0 \tag{11}$$

which have to be fulfilled. If in addition $\int_{\mathcal{D}}(u\rfloor\mathcal{G})\rfloor\delta\mathfrak{C} = 0$ is met, then the density is again a conserved quantity.

*Remark 2:* It is worth mentioning that a solution of $\delta_\alpha\mathcal{C} = 0$ evidently fulfilling (10) (independently of the precise form of $(\mathcal{J}-\mathcal{R})$) might be nontrivial, i.e $\mathcal{C}$ is not constant, this should be compared with the finite dimensional case where $\mathrm{d}C = 0$ only generates a trivial solution.

Let us again consider the simple heavy chain system to analyze the Casimir densities for this particular (academic) example.

*Example 2:* The relations (10) and (11) read as

$$\delta_w\mathcal{C} = 0\,, \;\; \delta_p\mathcal{C} = 0, \;\; (\dot{w}\,\partial_w^X\mathcal{C})\big|_{\partial\mathcal{D}} = 0\,, \;\; (\dot{p}\,\partial_p^X\mathcal{C})\big|_{\partial\mathcal{D}} = 0.$$

If we restrict ourselves to the case where $\mathcal{C} = \mathcal{C}(p, p_X)$ then we have to fulfill the condition

$$\partial_p\mathcal{C} = d_X\partial_p^X\mathcal{C} \tag{12}$$

subjected to the boundary expressions

$$\dot{p}\,\partial_p^X\mathcal{C}\big|_{X=0} = 0\,, \quad \dot{p}\,\partial_p^X\mathcal{C}\big|_{X=L} = 0.$$

Considering the boundary condition $\dot{p}|_{X=L} = 0$, i.e. the velocity at $X = L$ is constant and $w_X|_{X=L} = 0$ which means that we have to guarantee $\partial_p^X\mathcal{C}\big|_{X=0} = 0$ only. Consequently we choose

$$\mathcal{C} = \frac{1}{L}(p + Xp_X) = d_X(\frac{1}{L}pX)$$

which meets (12) as well as the boundary condition $\partial_p^X\mathcal{C}\big|_{X=0} = 0$ and integrating we obtain $C = \int_0^L \mathcal{C}\mathrm{d}X = p\big|_{X=L}$. The conserved quantity is the momentum at $X = L$ which is in accordance with the choice $\dot{p}|_{X=L} = 0$ at the boundary.

This simple example was just of academic nature to show the meaning of the conditions (10) and (11). The concept of Casimir densities will be fully exploited in the forthcoming section when the interconnection of systems with the purpose to design controllers is analyzed.

## V. INTERCONNECTION

In this section we discuss the interconnection (in a power conserving manner) of finite dimensional Hamiltonian systems with an infinite dimensional one, where we restrict ourselves to spatial domains which are one dimensional $X = [0, L]$ as well as to boundary control. The boundary of the infinite dimensional system is decomposed such that $\mathcal{D}_\partial = \mathcal{D}_a \cup \mathcal{D}_u$ is met, where $\mathcal{D}_a$ denotes the actuated boundary ($X = L$) and $\mathcal{D}_u$ is the unactuated boundary ($X = 0$). The coupling will be performed by interconnecting the systems via energy ports. We will analyze the coupling of a finite dimensional controller system with the infinite dimensional system at $\mathcal{D}_a$. However, it is worth mentioning that an additional finite dimensional system connected at $\mathcal{D}_u$ (i.e. a system corresponding to a load system, not interacting via a control input) can be considered in the same spirit, but due to lack of space this is not the focus of the present contribution.

The infinite dimensional system is a partial differential equation in Hamiltonian representation modeled on a bundle $\mathcal{X} \to \mathcal{D}$

$$\dot{x} = (\mathcal{J} - \mathcal{R})(\delta\mathfrak{H}) \tag{13}$$

where the control enters through the actuated boundary $\mathcal{D}_a$. From (6) it becomes obvious that the energy port at the boundary (if it exists) can be expressed by

$$\dot{x}^\alpha\partial_\alpha^X\mathcal{H}\big|_{\partial\mathcal{D}} = u_\partial\rfloor y^\partial. \tag{14}$$

Here $u_\partial$ and $y^\partial$ denote collocated inputs and outputs where the assignment of input or output is not unique, see [8] for more details. In the sequel $a$ and $u$ will always denote actuated and unactuated, i.e. $a$ and $u$ are no indices which are used for summation (in the sense of Einstein's sum convention).

### A. Finite-Infinite Interconnection

The relation (14) for this configuration reads as

$$\dot{x}^\alpha\partial_\alpha^X\mathcal{H}\big|_{\mathcal{D}_a} = u_{\partial,a}\rfloor y^{\partial,a}\,, \quad \dot{x}^\alpha\partial_\alpha^X\mathcal{H}\big|_{\mathcal{D}_u} = 0. \tag{15}$$

A power conserving interconnection has to fulfill the relation

$$u_c\rfloor y^c + u_{\partial,a}\rfloor y^{\partial,a} = 0 \tag{16}$$

where $y^c$ and $u_c$ denote the collocated port variables of the finite dimensional controller system that reads as

$$\begin{aligned} \dot{x}_c &= (J_c - R_c)\rfloor\mathrm{d}H_c + u_c\rfloor G_c \\ y^c &= G_c^*\rfloor\mathrm{d}H_c \end{aligned} \tag{17}$$

which is modeled on a manifold $\mathcal{X}_c$ with local coordinates $(x^{\alpha_c})$ and is equipped with dual in and output bundles $\mathcal{U}_c \to \mathcal{X}_c$ and $\mathcal{Y}_c \to \mathcal{X}_c$. The interconnection is chosen according to (16) as a feedback interconnection in the form

$$u_{\partial,a} = -D^*\rfloor y^c\,, \;\; u_c = D\rfloor y^{\partial,a} \tag{18}$$

with an appropriate map $D$ and its dual $D^*$. Analyzing (14) we have to fix the role of the collocated inputs and outputs. We make the identification

$$u_{\partial,a}\rfloor B^a = u_{\partial,a}^i B_{\alpha i}^a = \partial_\alpha^X\mathcal{H}\big|_{\mathcal{D}_a} \tag{19}$$

as well as

$$\bar{B}_a \rfloor y^{\partial,a} = y_i^{\partial,a} \bar{B}_a^{\alpha i} = \dot{x}^\alpha|_{\mathcal{D}_a} \qquad (20)$$

with a suitable map meeting $B^a \rfloor \bar{B}_a = I$ ($I$ denotes the identity tensor) at the actuated part of the boundary $\mathcal{D}_a$ such that

$$\dot{x}^\alpha \partial_\alpha^X \mathcal{H}\big|_{\mathcal{D}_a} = u_{\partial,a} \rfloor B^a \rfloor \bar{B}_a \rfloor y^{\partial,a} = u_{\partial,a} \rfloor y^{\partial,a}$$

is met.

Now we are able to exploit the benefits of the control by interconnection technique originally developed for coupling of finite dimensional systems, see [1] and references therein. To this end we analyze Casimir functionals for the coupled systems (13,17) via (15,18) in order to relate the states of the controller with the plant. We now have to find quantities of the form $C_I = C_c + \int_\mathcal{D} \mathcal{C} \mathrm{d}X$ with $C_c \in C^\infty(\mathcal{X}_c)$ and $\mathcal{C} \in \mathcal{J}^1(\mathcal{X})$. The change of $C_I$ along solutions of the interconnected system (we assume again the existence of a solution) should vanish, i.e $\dot{C}_I = 0$ since we are in a closed loop scenario and no further inputs are present which gives in local coordinates

$$\dot{C}_I = (\partial_{\alpha_c} C_c)\, \dot{x}_c^{\alpha_c} + \int_\mathcal{D} \dot{x}^\alpha \left(\delta_\alpha \mathcal{C}\right) \mathrm{d}X + \dot{x}^\alpha \left(\partial_\alpha^X \mathcal{C}\right)\big|_{\partial\mathcal{D}} .$$

Using (13,17,18,20) leads to the conditions

$$\delta_\alpha \mathcal{C}(\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) = 0 \qquad (21)$$

$$\partial_{\alpha_c} C_c (J_c^{\alpha_c\beta_c} - R_c^{\alpha_c\beta_c}) = 0 \qquad (22)$$

$$(\partial_{\alpha_c} C_c)\, G_{c,i}^{\alpha_c} D^{ij} + \bar{B}_a^{\beta j} \partial_\beta^X \mathcal{C}\big|_{\mathcal{D}_a} = 0 \qquad (23)$$

$$\dot{x}^\alpha\, \partial_\alpha^X \mathcal{C}\big|_{\mathcal{D}_u} = 0 \qquad (24)$$

which have to be met, where $D^{ij}$ denote the components of $D$. To find solutions of this set of equations (21-24) one can perform the following strategy for instance.

1) Motivated by the finite dimensional case we make the ansatz $C_I^\alpha = x_c^\alpha + \int_\mathcal{D} \mathcal{C}^\alpha \mathrm{d}X$.
2) Choose $J_c$ and $R_c$ of the finite dimensional controller system in order to satisfy (22).
3) Solve $\delta_\alpha \mathcal{C}(\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) = 0$ subjected to the boundary conditions (23,24) with the design parameters $G_c$ and $D$.

The remarkable fact of this procedure and in special the choice for $C_I^\alpha$ is the relation of (some of) the controller states with quantities of the plant to be controlled since $C_I$ is a constant along the solutions of the interconnected system, i.e. $x_c = -\int_\mathcal{D} \mathcal{C} \mathrm{d}X + \bar{c}$ where $\bar{c}$ depends on the initial conditions only. The Hamiltonian of the interconnected system $H_I = H_c + \int_\mathcal{D} \mathcal{H} \mathrm{d}X$ can be used for stability investigations provided that it serves as a Lyapunov function candidate, where in this context the connecting term $x_c = -\int_\mathcal{D} \mathcal{C} \mathrm{d}X + \bar{c}$ plays an extraordinary important role.

### B. Concluding Remark

The key observation of the presented strategy is the fact that a solution to the problem takes the form of a partial differential equation involving the variational derivative subjected to boundary conditions. Here it is worth noting that total divergences are annihilated by the variational derivative, i.e. a solution to $\delta f = 0$ can be constructed by $f = d_X \bar{f}$ (one independent coordinate $X$) due to the special properties of the variational derivative $\delta$. Furthermore it is worth mentioning that in our setting (where $\mathcal{J}$ and $\mathcal{R}$ are no differential operators) the extraction of the conditions concerning the Casimir functionals/functions needs no further integration by part in contrast to the system description explicitly using Stokes-Dirac structures as in [5].

## VI. Application

In this section we demonstrate the proposed method using the heavy chain system from Example 1 which will be slightly modified in order to include the possibility of a boundary control. The focus of the example is to show how the interconnection procedure as in section V-A works on a concrete application, namely we couple the heavy chain system with a controller system acting on the actuated boundary. The focus is laid on the system representation and the interconnection procedure, thus stability arguments are only touched and are not worked out in detail.

### A. The heavy chain system

The partial differential equations (see Example 1) read as in (7) where we consider the boundary conditions of the form

$$P(X) w_X|_{X=0} = 0\,, \quad P(X) w_X|_{X=L} = F \qquad (25)$$

where $F$ serves as the control input at the actuated boundary at $X = L$. The Hamiltonian density reads as in (8) and

$$\dot{H} = \dot{w} P(X) w_X|_0^L = \dot{w} F|_{X=L}$$

where we used the boundary conditions (25). The boundary map $B^a$ can now be constructed from $u_{\partial,a}^i B_{\alpha i}^a = \partial_\alpha^X \mathcal{H}\big|_{\mathcal{D}_a}$ in the following manner

$$[B_{\alpha i}^a] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; [\bar{B}_a^{\alpha i}] = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Let us now investigate the control by interconnection problem where we use a finite dimensional controller system of the form (17) where we choose $\dim(\mathcal{X}_c) = 2$ with $x_c = (q_c, p_c)$. For $C_I$ we make the ansatz $C_I = q_c + \int_0^L \mathcal{C} \mathrm{d}X$. Then the condition (22) can be fulfilled by the choice

$$[J_c^{\alpha_c\beta_c}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; [R_c^{\alpha_c\beta_c}] = \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}, \; r > 0.$$

The design Parameters $G_c$ and $D$ will be chosen in a trivial manner together with the choice $\dim(\mathcal{U}_c) = 1$, i.e $G_{c,1}^1 = G_{c,1}^2 = D = 1$. This implies

$$F = u_{\partial,a} = -y_c\,, \;\; u_c = y^{\partial,a} = \dot{w}|_{X=L}$$

and from (23) and (24) we obtain

$$1 + \partial_w^X \mathcal{C}\big|_{X=L} = 0\,, \;\; \dot{w}\, \partial_w^X \mathcal{C}\big|_{X=0} + \dot{p}\, \partial_p^X \mathcal{C}\big|_{X=0} = 0. \quad (26)$$

The equation (21) reads $\delta_w \mathcal{C} = 0$ and $\delta_p \mathcal{C} = 0$ subjected to the boundary expressions (26) and a solution can be found as

$$\mathcal{C} = -\frac{1}{L} d_X(Xw) = -\frac{1}{L}(w + X w_X)$$

such that

$$C_I = q_c - \frac{1}{L} X w \Big|_{X=0}^{X=L} = q_c - w|_{X=L} = \bar{c} \quad (27)$$

holds, where $\bar{c}$ depends on the initial conditions only since $\dot{C} = 0$ is met. The Hamiltonian of the finite dimensional controller system can be chosen as

$$H_c = \frac{1}{2} k_1 p_c^2 + \frac{1}{2} k_2 (q_c - q_{c_d})^2 \,, \quad k_1, k_2 > 0$$

which has a minimum at $(q_c, p_c) = (q_{c_d}, 0)$. Furthermore $q_{c_d} = \bar{c} + w_d|_{X=L}$ (with a constant desired displacement $w_d$) can be deduced from (27). The total energy of the interconnected system can be written as

$$H_I = \int_{\mathcal{D}} \left( \frac{1}{2\rho} p^2 + \frac{1}{2} P(X) w_X^2 \right) \mathrm{d}X + H_c$$

and from

$$\dot{H}_I = u_{\partial,a} \, \dot{w}|_{X=L} - k_1^2 p_c^2 r + k_1 p_c u_c + k_2 (q_c - q_{c_d}) u_c$$

together with $y_c = k_2(q_c - q_{c_d}) + k_1 p_c$ and $\dot{w}|_{X=L} = \dot{q}_c = u_c$ we obtain $\dot{H}_I = -r k_1^2 p_c^2 \leq 0$.

*Remark 3:* Detailed stability investigations are not in the scope of the presented contribution. It can be shown that $H_I$ is positive definite where a coordinate change of the type $\bar{w} = w - w_d$ has to be applied. Therefore $H_I$ serves as a Lyapunov function candidate. Provided the solution of the closed-loop system is well-posed in the sense of Hadamard, from $\dot{H}_I \leq 0$ the stability of the equilibrium can be deduced where these considerations have to be carried out on suitable function spaces (with an inner product corresponding to $H_I$ which induces a suitable equivalent norm). For further detailed information we refer to [13] and references therein.

### B. Simulation result

Finally in Figures 1 and 2 we present a simulation result of the controlled heavy chain system. The control objective is to stabilize the chain around a desired equilibrium (setpoint) where the initial condition does not correspond to the desired setpoint equilibrium.

We consider the simple but demonstrative case where the physical parameters $L, \rho, g$ are all set to 1 and the parameters of the controller are chosen as $k_1 = 200\,, k_2 = 0.8\,,\ r = 0.7$. Furthermore $w_d|_{X=L} = 1$. The initial conditions for the plant were chosen as $w|_{t=0} = 0\,,\ p|_{t=0} = 0$, and for the controller $q_c|_{t=0} = p_c|_{t=0} = 0$ such that $\bar{c} = 0$ follows immediately.

## VII. Conclusion

In this paper an alternative Hamiltonian representation for distributed parameter systems compared to the one based on Stokes-Dirac structures that are based on skew-adjoint differential operators and the use of energy variables is discussed. The control by interconnection method is reinterpreted in our setting and an example shows the applicability of the presented ideas. It turns out that the variational derivative acting on objects defined on jet-spaces is the key tool for our system representation as well as for the extraction of the Casimir functionals.

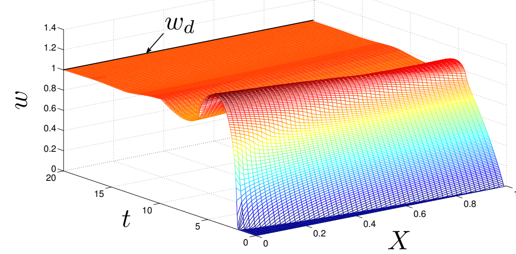


Fig. 1. Simulation results for the deflection $w$

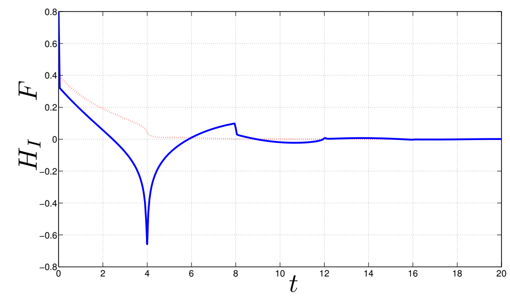


Fig. 2. Simulation results for the control force $F$ (solid-blue) and the closed loop Hamiltonian $H_I$ (dotted-red).

## References


[1] A. van der Schaft, *L2-Gain and Passivity Techniques in Nonlinear Control*. New York: Springer-Verlag, 2000.

[2] B. M. Maschke and A. van der Schaft, *Compositional Modelling of Distributed-Parameter Systems*, ser. Advanced Topics in Control Systems Theory. Springer Lect. Notes in Control and Information Sciences, London, 2005, pp. 115–154.

[3] A. van der Schaft and B. Maschke, "Hamiltonian formulation of distributed-parameter systems with boundary energy flow," *Journal of Geometry and Physics*, vol. 42, pp. 166–194, 2002.

[4] A. Macchelli, A. van der Schaft, and C. Melchiorri, "Port hamiltonian formulation of infinite dimensional systems: Part i modeling," in *Proc. 43rd IEEE Conf. Decision and Control (CDC)*, 2004, pp. 3762–3767.

[5] ——, "Port hamiltonian formulation of infinite dimensional systems: Part ii boundary control by interconnection," in *Proc. 43rd IEEE Conf. Decision and Control (CDC)*, 2004, pp. 3768–3773.

[6] P. Olver, *Applications of Lie Groups to Differential Equations*. New York: Springer-Verlag, 1986.

[7] K. Schlacher, "Mathematical modeling for nonlinear control: a hamiltonian approach," *Mathematics and Computers in simulation*, vol. 97, pp. 829–849, 2008.

[8] M. Schöberl, H. Ennsbrunner, and K. Schlacher, "Modelling of piezoelectric structures - a hamiltonian approach," *Mathematical and Computer Modelling of Dynamical Systems*, vol. 14, pp. 179–193, 2008.

[9] A. Macchelli and C. Melchiorri, "Modeling and control of the timoshenko beam. the distributed port hamiltonian approach," *SIAM J. Control Optim.*, vol. 43, pp. 743–767, 2004.

[10] R. Pasumarthy and A. van der Schaft, "Achievable casimirs and its implications on control of port-hamiltonian systems," *International Journal of Control*, vol. 80, pp. 1421–1438, 2007.

[11] G. Giachetta, L. Mangiarotti, and G. Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory*. Singapore: World Scientific, 1997.

[12] R. Ortega, A. van der Schaft, B. Maschke, and G. Escobar, "Interconnection and damping assignment passivity-based control of port hamiltonian systems," *Automatica*, vol. 36, pp. 585–596, 2002.

[13] Z. Luo, B. Guo, and O. Morgül, *Stability and Stabilization of Infinite Dimensional Systems with Applications*. London: Springer, 1999.