2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Distributed Containment Control of Linear Multi-Agent Systems with Multiple Leaders and Reduced-Order Controllers

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Abstract— **This paper considers the containment control problems for both continuous-time and discrete-time multi-agent systems with general linear dynamics under directed communication topologies. Distributed reduced-order observer-based containment controllers relying on the relative outputs of neighboring agents are constructed for both continuous-time and discrete-time cases, under which the states of the followers will asymptotically converge to the convex hull formed by those of the leaders if for each follower there exists at least one leader that has a directed path to that follower. Sufficient conditions on the existence of these containment controllers are given.**

I. INTRODUCTION

Consensus control of a group of agents has received compelling attentions from various scientific communities, for its potential applications in spacecraft formation flying, sensor networks, cooperative surveillance, and so forth [1], [2]. The main idea of consensus is to develop distributed control policies that enable the agents to reach an agreement on a state of interest. Consensus algorithms are studied in [3], [4], [5], [6] for a group of single-, double-, and high-order integrators with fixed and switching communication topologies. Different static and dynamic consensus protocols are designed in [7], [8], [9], [10] to reach consensus for multi-agent systems with general linear dynamics. Distributed H_∞ consensus and control problems are investigated in [11], [12] for networks of agents subject to external disturbances.

The above-mentioned references mainly focus on consensus for a group of agents without any leader. However, in some practical applications, there might exist one or even multiple leaders in the agent network. Tracking control problem for multi-agent consensus with an active leader is considered in [13] by using a distributed neighbor-based estimator. Distributed tracking algorithms are proposed, respectively, in [14] and [15] for a network of continuous-time and discrete-time agents to track a time-varying leader. In the presence of multiple leaders, [16] considers the containment control problem, by proposing a hybrid control law to drive the followers into the convex hull spanned by the leaders. Distributed containment control problems are studied in [17], [18] for a group of autonomous first- and second-order agents. The authors in [19], [20] study the containment control problem for a collection of rigid bodies with multiple stationary leaders while [20] discusses the case of dynamic leaders with finite-time convergence.

This paper considers the containment control problems for both continuous-time and discrete-time multi-agent systems with general linear dynamics under directed communication topologies, by extending our previous work [21] where static and full-order dynamic containment controllers are designed. Unlike the full-order controllers in [21], distributed reduced-order observer-based containment controllers relying on the relative outputs of neighboring agents are proposed for both the continuous-time and discrete-time cases, which is motivated by the consensus protocols in [9]. In the continuous-time case, a multi-step algorithm is presented to construct a reduced-order containment controller, under which the states of the followers will asymptotically converge to the convex hull formed by those of the leaders, if for each follower there exists at least one leader that has a directed path to that follower. It is shown that a sufficient condition on the existence of such a controller is that each agent is stabilizable and detectable. In the discrete-time case, in light of the modified algebraic Riccati equation, an algorithm is given to design a containment controller that solves the containment control problem. Sufficient conditions on the existence of such a discrete-time controller are also given. Different from the continuous-time case, in the discrete-time case, the eigenvalues of the stochastic matrix of the communication graph have to satisfy a constraint related to the unstable eigenvalues of the state matrix A, when A has a least one eigenvalue outside the unit circle. In contrast to [16], [17], [18] where the agent dynamics are restricted to be single or double integrators and to [19], [20] which considers second-order Euler-Lagrange systems, the results obtained in the current paper are applicable to multi-agent systems with general linear dynamics. Compared to the dynamic controllers in [21], the proposed containment controllers here are reduced-order and hence have lower dimensions.

The rest of this paper is organized as follows. Some useful results of the graph theory are reviewed in Section II. The containment control problems for continuous-time and discrete-time multi-agent systems are considered, respectively, in Sections III and IV. Simulation examples are presented for illustration in Section V. Conclusions are drawn

This work was supported in part by the National Natural Science Foundation of China under grant 61104153, China Postdoctoral Science Foundation under grant 20100480211, Singapore-MIT Alliance for Research and Technology (SMART) Centre under grant FM-06, and National Science Foundation under grant ECCS-1002393.

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in Section VI.

II. Notations and Graph Theory

Let $\mathbf{R}^{n\times n}$ and $\mathbf{C}^{n\times n}$ be the set of $n\times n$ real matrices and complex matrices, respectively. The superscript T means transpose for real matrices and H means conjugate transpose for complex matrices. I_N represents the identity matrix of dimension N. Matrices, if not explicitly stated, are assumed to have compatible dimensions. $\mathrm{Re}(\zeta)$ denotes the real part of $\zeta\in\mathbf{C}$. $A\otimes B$ denotes the Kronecker product of the matrices A and B. A matrix is Hurwitz (in the continuous-time case) if all of its eigenvalues have negative real parts, while is Schur stable (in the discrete-time case) if all of its eigenvalues have magnitude less than 1. For a set $X=\{x_1,\cdots,x_n\}$ in $V\subseteq\mathbf{R}^p$, its convex hull $\mathrm{co}(X)$ is defined as $\mathrm{co}(X)=\{\sum_{i=1}^n\alpha_i x_i|x_i\in V,\alpha_i\geq 0,\sum_{i=1}^n\alpha_i=1\}$.

A directed graph $\mathcal{G}$ is a pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{v_1,\cdots,v_N\}$ is a nonempty finite set of nodes and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is a set of edges, in which an edge is represented by an ordered pair of distinct nodes. For an edge (v_i,v_j), node v_i is called the parent node, node v_j the child node, and v_i is a neighbor of v_j. A graph with the property that $(v_i,v_j)\in\mathcal{E}$ implies $(v_j,v_i)\in\mathcal{E}$ is said to be undirected. A path from node v_{i_1} to node v_{i_l} is a sequence of ordered edges of the form $(v_{i_k},v_{i_{k+1}})$, $k=1,\cdots,l-1$. A directed graph contains a directed spanning tree if there exists a node called the root, which has no parent node, such that the node has a directed path to every other node in the graph.

The adjacency matrix $\mathcal{A}=[a_{ij}]\in\mathbf{R}^{N\times N}$ associated with the directed graph $\mathcal{G}$ is defined by $a_{ii}=0$, $a_{ij}=1$ if $(j,i)\in\mathcal{E}$ and $a_{ij}=0$ otherwise. The Laplacian matrix $\mathcal{L}=[\mathcal{L}_{ij}]\in\mathbf{R}^{N\times N}$ is defined as $\mathcal{L}_{ii}=\sum_{j\neq i}a_{ij}$ and $\mathcal{L}_{ij}=-a_{ij}$, $i\neq j$. Let $\mathcal{D}=[d_{ij}]\in\mathbf{R}^{N\times N}$ be a row-stochastic matrix associated with $\mathcal{G}$ with the additional assumption that $d_{ii}>0$, $d_{ij}>0$ if $(j,i)\in\mathcal{E}$ and $d_{ij}=0$ otherwise.

III. Containment Control of Continuous-Time Linear Multi-Agent Systems

Consider a group of N identical agents with general continuous-time linear dynamics, described by

$$\begin{aligned}\dot{x}_i&=Ax_i+Bu_i,\\ y_i&=Cx_i,\quad i=1,\cdots,N,\end{aligned}\tag{1}$$

where $x_i\in\mathbf{R}^n$, $u_i\in\mathbf{R}^p$, and $y_i\in\mathbf{R}^q$ are, respectively, the state, the control input, and the output of the i-th agent, and A, B, C, are constant matrices with compatible dimensions, where without loss of generality C is assumed to have full row rank.

A variety of static and dynamic consensus protocols were proposed to reach consensus for the agents with dynamics given by (1), e.g., in [7], [8], [9], [10]. However, these references only considered the case with at most one leader in the group. In the current paper, we consider the case with multiple leaders. Suppose that there are M ($M<N$) followers and $N-M$ leaders. An agent is called a leader if the agent has no neighbor. An agent is called a follower if the agent has at least one neighbor. Without loss of generality, we assume that the agents indexed by $1,\cdots,M$, are followers, while the agents indexed by $M+1,\cdots,N$, are leaders whose control inputs are set to be zero. We use $\mathcal{R}\triangleq\{M+1,\cdots,N\}$ and $\mathcal{F}\triangleq\{1,\cdots,M\}$ to denote, respectively, the leader set and the follower set. The communication topology among the N agents is represented by a directed graph $\mathcal{G}$. Note that here the leaders do not receive any information.

Assumption 1: Suppose that for each follower, there exists at least one leader that has a directed path to that follower.

It is assumed that each agent has access to only its own absolute output and the relative outputs with respect to its neighbors. We let $u_i=0$, $i\in\mathcal{R}$, and propose the following distributed reduced-order observer-based containment controller for each follower:

$$\begin{aligned}\dot{v}_i&=Fv_i+Gy_i+TBu_i,\\ u_i&=cKQ_1\sum_{j\in\mathcal{F}\cup\mathcal{R}}a_{ij}(y_i-y_j)\\ &\quad+cKQ_2\sum_{j\in\mathcal{F}\cup\mathcal{R}}a_{ij}(v_i-v_j),\quad i\in\mathcal{F},\end{aligned}\tag{2}$$

where $v_i\in\mathbf{R}^{n-q}$, $i\in\mathcal{F}$, are the states of the controllers corresponding to the followers, $v_j\in\mathbf{R}^{n-q}$, $j\in\mathcal{R}$, are the states of the following auxiliary systems:

$$\dot{v}_j=Fv_j+Gy_j,\quad j\in\mathcal{R},\tag{3}$$

$c>0$ is the coupling strength, a_{ij} is the (i,j)-th entry of the adjacency matrix $\mathcal{A}$ associated with a directed graph $\mathcal{G}$, $F\in\mathbf{R}^{(n-q)\times(n-q)}$ is Hurwitz and has no eigenvalues in common with those of A, $G\in\mathbf{R}^{(n-q)\times q}$, $T\in\mathbf{R}^{(n-q)\times n}$ is the unique solution to the following Sylvester equation:

$$TA-FT=GC,\tag{4}$$

which further satisfies that $\begin{bmatrix}C\\T\end{bmatrix}$ is nonsingular, $Q_1\in\mathbf{R}^{n\times q}$ and $Q_2\in\mathbf{R}^{n\times(n-q)}$ are given by $\begin{bmatrix}Q_1 & Q_2\end{bmatrix}=\begin{bmatrix}C\\T\end{bmatrix}^{-1}$, and $K\in\mathbf{R}^{p\times n}$ is the feedback gain matrix.

Denote by $\mathcal{L}$ the Laplacian matrix associated with $\mathcal{G}$. Because the leaders have no neighbors, $\mathcal{L}$ can be partitioned as

$$\mathcal{L}=\begin{bmatrix}\mathcal{L}_1 & \mathcal{L}_2\\ 0_{(N-M)\times M} & 0_{(N-M)\times(N-M)}\end{bmatrix},\tag{5}$$

where $\mathcal{L}_1\in\mathbf{R}^{M\times M}$ and $\mathcal{L}_2\in\mathbf{R}^{M\times(N-M)}$.

Lemma 1 ([20]): Under Assumption 1, all the eigenvalues of $\mathcal{L}_1$ has positive real parts, each entry of $-\mathcal{L}_1^{-1}\mathcal{L}_2$ is nonnegative, and each row of $-\mathcal{L}_1^{-1}\mathcal{L}_2$ has a sum equal to one.

Next, an algorithm is presented to select the control parameters in (2).

Algorithm 1: Under Assumption 1, a containment controller (2) can be constructed as follows:

1) Choose a Hurwitz matrix F having no eigenvalues in common with those of A. Select G such that (F,G) is stabilizable.

2) Solve (4) to get a solution T, which satisfies that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular. Then, compute matrices Q_1 and Q_2 by $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}$.

3) Solve the following linear matrix inequality (LMI):

$$AP + PA^T - 2BB^T < 0, \tag{6}$$

to get one solution $P > 0$. Then, choose the matrix $K = -B^T P^{-1}$.

4) Select the coupling strength $c \geq \frac{1}{\min\limits_{i=1,\cdots,M}\{\mathrm{Re}(\lambda_i)\}}$, where λ_i, $i = 1, \cdots, M$, are the eigenvalues of $\mathcal{L}_1$.

Remark 1: As pointed out in Remark 3.2 in [9], under the condition that (A, C) is detectable, the probability that steps 1) and 2) hold is 1. Furthermore, a necessary and sufficient condition on the existence of a $P > 0$ to the LMI (6) is that (A, B) is stabilizable [7]. Therefore, a sufficient condition for Algorithm 1 to successfully construct a controller (2) is that (A, B, C) is stabilizable and detectable.

Theorem 1: Suppose that the directed communication graph $\mathcal{G}$ satisfies Assumption 1 and (A, B, C) is stabilizable and detectable. Then, the states of the followers under the controller (2) constructed by Algorithm 1 will asymptotically converge to the convex hull formed by those of the leaders, i.e., the containment control problem is solved. Specifically, $\lim_{t\to\infty}(x_f(t) - \varpi(t)) = 0$ and $\lim_{t\to\infty}(v_f(t) - GC\varpi(t)) = 0$, where $x_f = [x_1^T, \cdots, x_M^T]^T$, $v_f = [v_1^T, \cdots, v_M^T]^T$, and

$$\varpi(t) = (-\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes e^{At}) \begin{bmatrix} x_{M+1}(0) \\ \vdots \\ x_N(0) \end{bmatrix}. \tag{7}$$

Proof: Let $z_i = [x_i^T, v_i^T]^T$, $z_f = [z_1^T, \cdots, z_M^T]^T$, and $z_l = [z_{M+1}^T, \cdots, z_N^T]^T$. Then, the closed-loop network dynamics resulting from (1), (2), and (3) can be written as

$$\begin{aligned} \dot{z}_f &= (I_M \otimes \mathcal{S} + c\mathcal{L}_1 \otimes \mathcal{H})\, z_f + c(\mathcal{L}_2 \otimes \mathcal{H}) z_l, \\ \dot{z}_l &= (I_{N-M} \otimes \mathcal{S})\, z_l, \end{aligned} \tag{8}$$

where

$$\mathcal{S} = \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} BKQ_1C & BKQ_2 \\ TBKQ_1C & TBKQ_2 \end{bmatrix}.$$

Let $\xi_i = \sum_{j\in\mathcal{F}\cup\mathcal{R}} a_{ij}(z_i - z_j)$, $i \in \mathcal{F}$, and $\xi = [\xi_1^T, \cdots, \xi_M^T]^T$. Then, we have

$$\xi = (\mathcal{L}_1 \otimes I_{2n-q}) z_f + (\mathcal{L}_2 \otimes I_{2n-q}) z_l. \tag{9}$$

Considering the special structure of $\mathcal{L}$ as in (5), we can obtain from (8) and (9) that ξ satisfies [21]

$$\begin{aligned} \dot{\xi} &= (\mathcal{L}_1 \otimes I)\dot{z}_f + (\mathcal{L}_2 \otimes I)\dot{z}_l \\ &= (\mathcal{L}_1 \otimes I)((I_M \otimes \mathcal{S}) z_f + c(\mathcal{L}_1 \otimes \mathcal{H}) z_f \\ &\quad + c(\mathcal{L}_2 \otimes \mathcal{H}) z_l) + (\mathcal{L}_2 \otimes I)(I_{N-M} \otimes \mathcal{S}) z_l \\ &= (\mathcal{L}_1 \otimes \mathcal{S} + c\mathcal{L}_1^2 \otimes \mathcal{H})((\mathcal{L}_1^{-1} \otimes I)\xi \\ &\quad - (\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I) z_l) + (c\mathcal{L}_1\mathcal{L}_2 \otimes \mathcal{H} + \mathcal{L}_2 \otimes \mathcal{S}) z_l \\ &= (I_M \otimes \mathcal{S} + c\mathcal{L}_1 \otimes \mathcal{H})\xi. \end{aligned} \tag{10}$$

Under Assumption 1, it follows from Lemma 1 that all the eigenvalues of $\mathcal{L}_1$ have positive real parts. Let $U \in \mathbf{C}^{M\times M}$ be such a unitary matrix that $U^{-1}\mathcal{L}_1 U = \Lambda$, where Λ is an upper-triangular matrix with λ_i, $i = 1, \cdots, M$, as its diagonal entries. Let $\tilde{\xi} \triangleq [\tilde{\xi}_1^T, \cdots, \tilde{\xi}_N^T]^T = (U^{-1} \otimes I)\xi$. Then, it follows from (10) that

$$\dot{\tilde{\xi}} = (I_M \otimes \mathcal{S} + c\Lambda \otimes \mathcal{H})\tilde{\xi}. \tag{11}$$

By noting that Λ is upper-triangular, it is clear that (11) is asymptotically stable if and only if the following M systems

$$\dot{\tilde{\xi}}_i = (\mathcal{S} + c\lambda_i\mathcal{H})\tilde{\xi}_i, \quad i = 1, \cdots, M, \tag{12}$$

are simultaneously asymptotically stable. Multiplying the left and right sides of $\mathcal{S} + c\lambda_i\mathcal{H}$ by $T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$ and T^{-1}, respectively, we get

$$T(\mathcal{S} + c\lambda_i\mathcal{H})T^{-1} = \begin{bmatrix} A + c\lambda_i BK & c\lambda_i BKQ_2 \\ 0 & F \end{bmatrix}. \tag{13}$$

By steps 3) and 4) in Algorithm 1, it follows that there exists a $P > 0$ such that

$$\begin{aligned} (A &+ c\lambda_i BK)P + P(A + c\lambda_i BK)^H \\ &= AP + PA^T - 2c\mathrm{Re}(\lambda_i)BB^T \\ &\leq AP + PA^T - 2BB^T < 0, \quad i = 1, \cdots, M. \end{aligned}$$

That is, $A + c\lambda_i BK$, $i = 1, \cdots, M$, are Hurwitz. Therefore, using (11), (12), and (13), it follows that (10) is asymptotically stable. Then, it follows from (9) that $\|z_f(t) + (\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_{2n-q})z_l(t)\| \to 0$, as $t \to \infty$. By further noting $z_l(t) = (I_{N-M} \otimes e^{\mathcal{S}t})z_l(0)$, it is not difficult to get that $\lim_{t\to\infty}(x_f(t) - \varpi(t)) = 0$ and $\lim_{t\to\infty}(v_f(t) - GC\varpi(t)) = 0$. By Lemma 1, we know from (7) that the states of the followers asymptotically converge to the convex hull formed by those of the leaders. ■

Remark 2: Contrary to the previous results on containment control in [16], [17], [18], [19], [20], where the agent dynamics are restricted to be single or double integrators in [16], [17], [18] and to be second-order Euler-Lagrange systems in [19], [20], Theorem 1 is applicable to multi-agent systems with general linear dynamics. Compared to the dynamic controllers in [21], the proposed containment controller (2) is reduced-order and hence has a lower dimension. For the special case with only one leader, Theorem 1 implies that the states of the followers will asymptotically approach the state of the leader. In this case, the containment controller (2) will recover the reduced-order consensus protocol (2) in [9].

IV. Containment Control of Discrete-Time Linear Multi-Agent Systems

This section extends to discuss the discrete-time counterpart of the last section. Consider a group of N identical agents with general discrete-time linear dynamics, described by

$$\begin{aligned} x_i^+ &= Ax_i + Bu_i, \\ y_i &= Cx_i, \quad i = 1, \cdots, N, \end{aligned} \tag{14}$$

where $x_i = x_i(k) \in \mathbf{R}^n$, $x_i^+ = x_i(k+1)$, $u_i \in \mathbf{R}^p$, and $y_i \in \mathbf{R}^q$ are, respectively, the state at the current time instant, the state at the next time instant, the control input, the output of the i-th agent. It is assumed that B is of full column rank and C has full row rank.

Similar to the last section, we assume that the agents indexed by $1, \cdots, M$, are followers, while the agents indexed by $M+1, \cdots, N$, are leaders. The leader set and the follower set are denoted, respectively, by $\mathcal{R} = \{M+1, \cdots, N\}$ and $\mathcal{F} = \{1, \cdots, M\}$. The communication graph $\mathcal{G}$ among the N agents is directed and satisfies Assumption 1.

We again let $u_i = 0$, $i \in \mathcal{R}$. Similar to the continuous-time case, we propose the following distributed reduced-order containment controller for each follower as

$$\begin{aligned} \hat{v}_i^+ &= Fv_i + Gy_i + TBu_i, \\ u_i &= KQ_1 \sum_{j \in \mathcal{F} \cup \mathcal{R}} a_{ij}(y_i - y_j) \\ &\quad + KQ_2 \sum_{j \in \mathcal{F} \cup \mathcal{R}} a_{ij}(\hat{v}_i - \hat{v}_j), \quad i \in \mathcal{F}, \end{aligned} \tag{15}$$

where $\hat{v}_i \in \mathbf{R}^{n-q}$, $i \in \mathcal{F}$, are the states of the controllers corresponding to the followers, $\hat{v}_j \in \mathbf{R}^{n-q}$, $j \in \mathcal{R}$, are the states of the following systems:

$$\hat{v}_j^+ = F\hat{v}_j + Gy_j, \quad j \in \mathcal{R}, \tag{16}$$

$F \in \mathbf{R}^{(n-q)\times(n-q)}$ is Schur stable and has no eigenvalues in common with those of A, $G \in \mathbf{R}^{(n-q)\times q}$, $T \in \mathbf{R}^{(n-q)\times n}$ is the unique solution to (4), satisfying that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular, $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}$, $K \in \mathbf{R}^{p\times n}$ is the feedback gain matrix, and d_{ij} is the (i,j)-th entry of the row-stochastic matrix $\mathcal{D}$ associated with the graph $\mathcal{G}$.

Because the last $N-M$ agents are leaders that have no neighbors, $\mathcal{D}$ can be partitioned as

$$\mathcal{D} = \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 \\ 0_{(N-M)\times M} & I_{N-M} \end{bmatrix}, \tag{17}$$

where $\mathcal{D}_1 \in \mathbf{R}^{M\times M}$ and $\mathcal{D}_2 \in \mathbf{R}^{M\times(N-M)}$.

Lemma 2 ([21]): Under Assumption 1, all the eigenvalues of $\mathcal{D}_1$ lie in the open unit disk, each entry of $(I_M - \mathcal{D}_1)^{-1}\mathcal{D}_2$ is nonnegative, and each row of $(I_M - \mathcal{D}_1)^{-1}\mathcal{D}_2$ has a sum equal to one.

Before moving forward, we introduce the following modified algebraic Riccati equation (MARE) [22]:

$$P = A^TPA - \delta A^TPB(B^TPB)^{-1}B^TPA + Q, \tag{18}$$

where $P > 0$, $Q > 0$, and $\delta > 0 \in \mathbf{R}$. For $\delta = 1$, the MARE (18) is reduced to the commonly-used discrete-time Riccati equation.

The following lemma shows the existence of solutions for the MARE.

Lemma 3 ([22]): Let (A, B) be stablizable. Then, the following hold.

a) There exists a critical value $\delta_c \in [0, 1)$ such that the MARE (18) has a unique positive-definite solution P for any $\delta > \delta_c$. Moreover, $\delta_c = 0$ if A has no eigenvalues with magnitude larger than 1. For the case where A has a least one eigenvalue with magnitude larger than 1, $\delta_c = 1 - \frac{1}{\Pi_i |\lambda_i^u(A)|^2}$ if B is of rank one, and $\delta_c = 1 - \frac{1}{\max_i |\lambda_i^u(A)|^2}$ if B is invertible, where $\lambda_i^u(A)$ are the unstable eigenvalues of A.

b) If the MARE (18) has a unique positive-definite solution P, then $P = \lim_{k\to\infty} P_k$ for any initial condition $P_0 > 0$, where P_k satisfies

$$\begin{aligned} P(k+1) = A^TP(k)A &- \delta A^TP(k)B \\ &\times (B^TP(k)B)^{-1}B^TP(k)A + Q. \end{aligned}$$

Next, an algorithm for determining the control parameters in (15) is presented.

Algorithm 2: Under Assumption 1, a containment controller (15) can be constructed as follows:

1) Select a Schur stable matrix F having no eigenvalues in common with those of A, and G such that (F, G) is stablizable.
2) Solve (4) to get a solution T, satisfying that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular. Then, compute the matrices Q_1 and Q_2 by $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}$.
3) Choose $K = -(B^TPB)^{-1}B^TPA$, where $P > 0$ is the unique solution to the following MARE:

$$\begin{aligned} P = A^TPA - (1 - \max_{i=1,\cdots,M} |\hat{\lambda}_i|^2)A^TPB \\ \times (B^TPB)^{-1}B^TPA + Q, \end{aligned} \tag{19}$$

with $Q > 0$ and $\hat{\lambda}_i$, $i = 1, \cdots, M$, being the eigenvalues of $\mathcal{D}_1$.

Remark 3: According to Lemma 3 and Remark 1, for the case where A has eigenvalues outside the unit circle, a sufficient condition for the existence of the consensus protocol by using Algorithm 2 is that (A, B, C) is stabilizable and detectable, and $\max_{i=1,\cdots,M} |\hat{\lambda}_i|^2 < 1 - \delta_c$, where δ_c is defined in Lemma 3. For the case where A has no eigenvalues with magnitude larger than 1, the sufficient condition is reduced to that (A, B, C) is stabilizable and detectable.

Theorem 2: Assume that the directed communication graph $\mathcal{G}$ satisfies Assumption 1. Let (A, B, C) be stabilizable and detectable. Then, the controller given by Algorithm 2 solves the containment control problem for the agents in (15). Specifically, $\lim_{k\to\infty} (x_f(k) - \psi(k)) = 0$ and $\lim_{k\to\infty} (\hat{v}_f(k) - GC\psi(k)) = 0$, where $x_f = [x_1^T, \cdots, x_M^T]^T$, $\hat{v}_f = [\hat{v}_1^T, \cdots, \hat{v}_M^T]^T$, and

$$\psi(k) = \left((I_M - \mathcal{D}_1)^{-1}\mathcal{D}_2 \otimes A^k\right) \begin{bmatrix} x_{M+1}(0) \\ \vdots \\ x_N(0) \end{bmatrix}.$$

Proof: Let $\hat{z}_i = [x_i^T, \hat{v}_i^T]^T$, $\hat{z}_f = [\hat{z}_1^T, \cdots, \hat{z}_M^T]^T$, and $\hat{z}_l = [\hat{z}_{M+1}^T, \cdots, \hat{z}_N^T]^T$. Then, we can obtain from (14), (15), and (16) that the collective network dynamics can be written as

$$\begin{aligned} \hat{z}_f^+ &= (I_M \otimes \mathcal{S} + (I_M - \mathcal{D}_1) \otimes \mathcal{H})\, \hat{z}_f - (\mathcal{D}_2 \otimes \mathcal{H})\hat{z}_l, \\ \hat{z}_l^+ &= (I_{N-M} \otimes \mathcal{S})\hat{z}_l, \end{aligned} \tag{20}$$

where $\mathcal{S}$ and $\mathcal{H}$ are defined in (8). Let $\zeta_i = \sum_{j\in\mathcal{F}\cup\mathcal{R}} d_{ij}(\hat{z}_i - \hat{z}_j)$, $i \in \mathcal{F}$, and $\zeta = [\zeta_1^T, \cdots, \zeta_M^T]^T$. Then, we have

$$\zeta = ((I_M - \mathcal{D}_1) \otimes I_{2n-q})\,\hat{z}_f - (\mathcal{D}_2 \otimes I_{2n-q})\hat{z}_l. \tag{21}$$

By following similar steps to those in the proof of Theorem 1, we can obtain from (20) and (21) that ζ satisfies the following dynamics:

$$\zeta^+ = (I_M \otimes \mathcal{S} + (I_M - \mathcal{D}_1) \otimes \mathcal{H})\,\zeta. \tag{22}$$

Under Assumption 1, it follows from Lemma 2 that all the eigenvalues of $I_M - \mathcal{D}_1$ have positive real parts. Let $\hat{U} \in \mathbf{C}^{M\times M}$ be such a unitary matrix that $\hat{U}^{-1}(I_M - \mathcal{D}_1)\hat{U} = \hat{\Lambda}$, where $\hat{\Lambda}$ is an upper-triangular matrix with $1 - \hat{\lambda}_i$, $i = 1, \cdots, M$, on the diagonal. Let $\tilde{\zeta} \triangleq [\tilde{\zeta}_1^T, \cdots, \tilde{\zeta}_N^T]^T = (\hat{U}^{-1} \otimes I)\zeta$. Then, (22) can be rewritten as

$$\tilde{\zeta}^+ = (I_M \otimes \mathcal{S} + \hat{\Lambda} \otimes \mathcal{H})\tilde{\zeta}, \tag{23}$$

Clearly, (11) is asymptotically stable if and if the following M systems:

$$\tilde{\zeta}_i^+ = (\mathcal{S} + (1 - \hat{\lambda}_i)\mathcal{H})\tilde{\zeta}_i, \quad i = 1, \cdots, M, \tag{24}$$

are simultaneously asymptotically stable. In light of step 3) in Algorithm 2, we can obtain [21]

$$\begin{aligned}
&(A + (1 - \hat{\lambda}_i)BK)^H P(A + (1 - \hat{\lambda}_i)BK) - P \\
&= A^T PA - P + (-2\mathbf{Re}(1 - \hat{\lambda}_i) + |1 - \hat{\lambda}_i|^2)A^T PB \\
&\quad \times (B^T PB)^{-1} B^T PA \\
&= A^T PA - P + (|\hat{\lambda}_i|^2 - 1)A^T PB(B^T PB)^{-1} B^T PA \\
&\leq A^T PA - P - (1 - \max_{i=1,\cdots,M} |\hat{\lambda}_i|^2)A^T PB(B^T PB)^{-1} \\
&\quad \times B^T PA \\
&= -Q < 0.
\end{aligned} \tag{25}$$

Then, (25) implies that $A + (1 - \hat{\lambda}_i)BK$, $i = 1, \cdots, M$, are Schur stable. Therefore, considering (23), (24), (25) and (13), we obtain that (22) is asymptotically stable, which, by (21), implies that $\|z_f(k) - ((I_M - \mathcal{D}_1)^{-1}\mathcal{D}_2 \otimes I_{2n-q})z_l(k)\| \to 0$, as $k \to \infty$. Then, it is easy to get that $\lim_{k\to\infty} (x_f(k) - \psi(k)) = 0$ and $\lim_{k\to\infty} (\hat{v}_f(k) - GC\psi(k)) = 0$. In virtue of Lemma 2, the states of the followers asymptotically converge to the convex hull formed by those of the leaders. ■

V. Simulation Examples

In this section, a simulation example is provided to validate the effectiveness of the theoretical results.

Consider a network of harmonic oscillators described by (1), with

$$x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

A first-order containment controller in the form of (2) will be designed.

Take $F = -2$ and $G = -1$. Using the function `lyap` in Matlab to solve the Sylvester equation (4) gives $T = \begin{bmatrix} -0.4 & 0.2 \end{bmatrix}$, which obviously satisfies that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular. Then, the matrices Q_1 and Q_2 can be obtained as $Q_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $Q_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$. Solving the LMI (6) by using the LMI toolbox of Matlab gives the feedback gain matrix in (2) as $K = -\begin{bmatrix} 0.1251 & 0.5732 \end{bmatrix}$. For illustration, let the communication graph $\mathcal{G}$ be given by Fig. 1, where nodes $7, 8, 9$ are three leaders and the others are followers. It can be verified that $\mathcal{G}$ satisfies Assumption 1. Correspondingly, the matrix $\mathcal{L}_1$ in (5) is

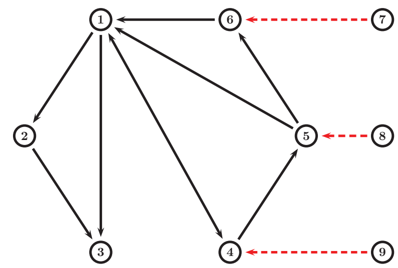

Fig. 1. The communication topology.

$$\mathcal{L}_1 = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

whose eigenvalues are 0.8213, 1, 2, $2.3329 \pm 0.6708i$, 3.5129. By Algorithm 1, we choose the coupling strength $c \geq 1.2176$. The state trajectories of the nine agents under the controller (2) designed as above and $c = 2$ are depicted in Fig. 2, from which it can be observed that the containment problem is indeed solved.

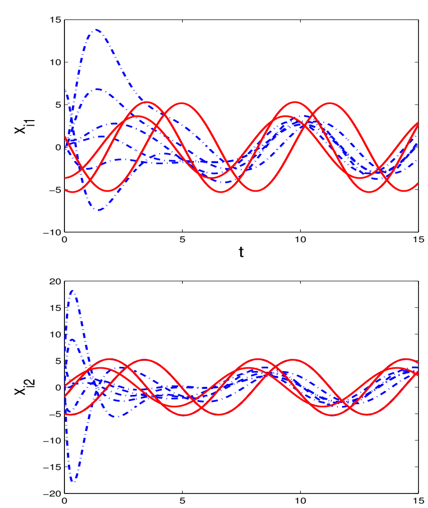

Fig. 2. The state trajectories of the agents (1) under (2). The solid and dashdotted lines denote, respectively, the trajectories of the leaders and the followers.

VI. Conclusions

In this paper, the containment control problems have been considered for both continuous-time and discrete-time multi-agent systems with general linear dynamics under directed communication topologies. Distributed reduced-order observer-based containment controllers relying on the relative outputs of neighboring agents have been constructed for both continuous-time and discrete-time cases, under which the states of the followers can asymptotically converge to the convex hull formed by those of the leaders if for each follower there exists at least one leader that has a directed path to that follower. Sufficient conditions for the existence of distributed reduced-order containment controllers have been given.

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