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Attractive Invariant Submanifold-based Coupling Controller Design

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***Abstract*— In this paper, we provide an algorithm for the design of a coupling controller for a nonlinear input-affine system. The resulting controller renders the maximal locally controlled invariant output-nulling submanifold locally attractive for the controlled system. The connections to the constrained dynamics algorithm and the triangular decoupling problem are presented, and necessary and sufficient conditions for the success of the new algorithm are derived.**

I. Introduction

In the design of multivariable control systems, much effort has been put in the investigation of decoupling control in the past. Crucial for the development was the work of Falb and Wolovich [1] presenting a solution for some classes of linear systems. Especially the appearance of unstable, invariant zeros restricts the possibilities of the design. Solutions avoiding this restriction by achieving only partial decoupling can be found in [10] and [5]. The results achieved in the decoupling of linear systems have been extended to the nonlinear case employing differential geometric concepts [3]. The requirement that the system has relative degree and stable zero dynamics is very limiting for reaching a stable decoupled system.

In contrast, achieving decoupling is not always the objective in controller design. In many applications the specific coupling of some states or outputs is necessary, such as controlling the engines for steel rolling mills, paper machines, automotive test benches, and a very contemporary task, synchronizing the engines of a wheel individually actuated electric vehicle. Although the design of coupling controllers is very useful in many cases, few attention is paid to the coupling problem compared to the decoupling problem. For linear time-invariant systems, a solution based on the Complete Modal Synthesis is given in [4] which was extended to dynamic output feedback in [6].

Obviously, nonlinear coupling can also be achieved via decoupling the system and setting the corresponding inputs to zero. But the restrictions are too demanding for solving the coupling problem only. With differential geometric concepts for linear systems [11] also nonlinear problems can be solved. In [7] a geometric approach is given for the triangular decoupling problem for linear systems which was extended to the nonlinear case in [8]. Again in this context, coupling can be achieved by setting some of the inputs to zero. Indeed, the requirements for triangular decoupling are less restrictive than for complete decoupling. Although necessary and sufficient conditions for the solvability of the triangular decoupling problem are stated in [8], no constructive method for designing the control law is given.

If the coupling conditions are described by the outputs of a nonlinear system, the remaining dynamics of the coupled system live on a locally controlled invariant output-nulling submanifold of the state manifold only. The maximal locally controlled invariant output-nulling submanifold can be computed via the constrained dynamics algorithm described in [9]. Although a suitable control law is derived during the algorithm, the generated submanifold will not be an attractive submanifold in general.

In this contribution, we want to draw the connection of the constrained dynamics algorithm to the triangular decoupling problem to create a coupling controller which produces the same submanifold as the constrained dynamics algorithm and additionally renders this submanifold locally attractive. We will show that the success of our approach coincides with the solvability of the triangular decoupling problem, and we will give a constructive algorithm deriving the control law.

This paper is organized as follows. In section II we will introduce the problem to be solved and define some basic nomenclature. The design of the proposed control law is carried out in section III, and the main results are stated in section IV. We exemplify our approach with a model of three heated rooms in section V and finally terminate this paper with the conclusion in section VI.

II. Problem Statement and Preliminaries

We will consider nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + \sum_{l=1}^{p} g_l(x)u^l = f(x) + g(x)u \qquad (1)\\ y &= h(x), \end{aligned}$$

where $x \in \mathcal{M}$ describes the state variables and $u \in \mathbb{R}^p$ the inputs. The manifold $\mathcal{M}$ is assumed to be smooth, $T_x\mathcal{M}$ denotes the tangent space at x, and $T\mathcal{M}$ =

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$\bigcup_{x\in\mathcal{M}} T_x\mathcal{M}$ denotes the tangent bundle of $\mathcal{M}$. $f(x)$ and $g_l(x)$ are $n\times 1$-dimensional smooth vector fields on $\mathcal{M}$. The map $h:\mathcal{M}\to\mathbb{R}^q$ with $q<p$ defines the coupling conditions. Therefore a system (1) is said to be coupled if the conditions

$$0 = h(x) \tag{2}$$

are fulfilled. We will also depict y as the output of the system. The control law

$$u = \alpha(x) + \beta(x)v \tag{3}$$

is described by a $p\times 1$ dimensional vector function $\alpha(x)$ and a $p\times m$ dimensional matrix function $\beta(x)$. We want to design the control law, such that the controlled system locally around x_0 with $h(x_0)=0$ fulfills the coupling conditions asymptotically and independently of the reference input $v\in\mathbb{R}^m$. As v should be used to control further control variables the remaining dynamics of the coupled system (1) together with (2) should be as large as possible. Therefore, we are seeking for the maximal locally controlled invariant output-nulling submanifold.

Before we can proceed, we need some definitions and lemmas.

Definition 1: A submanifold $\mathcal{N}=\{x\in\mathcal{M}\mid\gamma(x)=0\}$ of $\mathcal{M}$ is called locally attractive for the system (1) in the neighborhood U of x_0 with $\gamma(x_0)=0$ if for every $x(t_0)\in U$, $\gamma(x(t))=0$ holds for $t\to\infty$.

Definition 2: (see [8]) An involutive distribution D on $\mathcal{M}$ is locally controlled invariant for the system (1) if locally there exists a feedback of the form (3) with $m=p$, such that the modified dynamics $\dot{x}=\bar{f}(x)+\sum_{i=1}^{p}\bar{g}_i(x)v^i$ with $\bar{f}(x)=f(x)+g(x)\alpha(x)$ and $\bar{g}(x)=g(x)\beta(x)$ leave D invariant, i.e.

$$\begin{aligned}&\left[\bar{f},D\right]\subset D\\ &[\bar{g}_i,D]\subset D,\quad \forall\, 1\le i\le p.\end{aligned}$$

D is additionally a regular local controllability distribution if it is also an involutive closure of $\left\{ad_{\bar{f}}^{k}\bar{g}_i \mid k\in\mathbb{N}, i\in I\right\}$ for a certain subset $I\subset\{1,\ldots,p\}$.

Lemma 2.1: Let $\mathcal{N}=\{x\in\mathcal{M}\mid\gamma(x)=0\}$ be a submanifold of $\mathcal{M}$. Assume that the rank of the s-dimensional mapping $\gamma(x)$ is equal to s in a neighborhood of x_0. Suppose there exists a diffeomorphism $\left(\xi_1^T,\xi_2^T\right)=\Phi(x)^T=\left(\gamma(x)^T,\Psi(x)^T\right)$ around x_0 with $\xi_1=\gamma(x)$ and $\xi_2=\Psi(x)$, and the system (1) transformed with Φ leads to the system

$$\begin{aligned}\begin{pmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{pmatrix} &= \begin{pmatrix} A\xi_1 \\ \hat{f}_2(\xi_1,\xi_2)+\sum_{l=1}^{p}\hat{g}_{2l}(\xi_1,\xi_2)u^l\end{pmatrix} \\ &= \hat{f}(\xi)+\sum_{l=1}^{p}\hat{g}_l(\xi)u^l\end{aligned} \tag{4}$$

around $\xi_0=\Phi(x_0)$. If all eigenvalues of the $s\times s$ matrix A are in $\mathbb{C}^-$, $\mathcal{N}$ is a locally attractive invariant submanifold around x_0 for system (1).

Proof: Because of the special structure of (4), $\dot{\xi}_1$ is independent of ξ_2. As $t\to\infty$, $\xi_1=0$ holds, and therefore $\gamma(x)=0$. As $\left[\hat{f}(\xi),T_\xi\mathcal{N}\right]\subset T_\xi\mathcal{N}$ and $[\hat{g}_l(\xi),T_\xi\mathcal{N}]\subset T_\xi\mathcal{N}$ for all $\xi\in\mathcal{N}$ and $1\le l\le p$, $\mathcal{N}$ is a locally attractive and invariant submanifold for system (1) around x_0. ■

The aim of this contribution is finding a maximal locally controlled invariant output-nulling submanifold and a control law that renders this submanifold locally attractive for the controlled system. The approach is related to the constrained dynamics algorithm in [9]. The constrained dynamics algorithm computes the maximal locally controlled invariant output-nulling submanifold that fulfills the coupling conditions (2) and an appropriate control law. But this manifold will generally not be attractive. As the control law has some degrees of freedom, we will use them, if possible, to make the manifold locally attractive for the controlled system.

III. Constrained Dynamics Algorithm with Stability

Step 0

Set $\mathcal{M}_0=\mathcal{M}$, $i=0$, $p_0=p$, $p_1=p-1,\ldots,p_q=p-q$. Denote the system (1) by Σ_0, and set $z_0=h(x)$. The elements of z_0 are denoted by $z_0^1,\ldots,z_0^q$.

Step 1

Increase i by 1 and set $\gamma_1=\left(z_0^1,\ldots,z_{i-1}^1\right)^T$. Assume that the rank of γ_1 is constant around x_0 and

$$\mathrm{rank}(\gamma_1)=i. \tag{5}$$

If $L_{g_l}z_{i-1}^1=0\;\forall\,1\le l\le p_0$ holds, set $z_i^1=L_f z_{i-1}^1$.

Step 2

Repeat step 1 until $\exists\, l,\,1\le l\le p_0$, such that $L_{g_l}z_{i-1}^1\neq 0$. Set $\delta_1=i$ and $\mathcal{M}_1=\{x\in\mathcal{M}_0\mid\gamma_1=0\}$. Choose parameters ν_j^1, such that the roots of the polynomial

$$P^1(s)=1+\sum_{j=1}^{\delta_1}\nu_j^1\cdot(s)^j \tag{6}$$

are all in $\mathbb{C}^-$. It is possible to find a control law $u=\alpha_1(x)+\beta_1(x)u_1$, with the p_0-dimensional vector $\alpha_1(x)$, the $p_0\times p_1$ matrix $\beta_1(x)$, and $u_1\in\mathbb{R}^{p_1}$, such that

$$\begin{aligned}&z_0^1+\nu_1^1 z_1^1+\cdots+\nu_{\delta_1-1}^1 z_{\delta_1-1}^1\\ &\qquad+\nu_{\delta_1}^1\left(L_f z_{\delta_1-1}^1+\sum_{l=1}^{\nu_0}L_{g_l}z_{\delta_1-1}^1\alpha_1^l\right)=0\end{aligned}$$

and

$$L_{g_l}z_{i-1}^1\beta_1^l(x)=0\quad\forall\,1\le l\le p_0$$

hold on $\mathcal{M}_1$ in a neighborhood of x_0. Denote the system (1) controlled by $u=\alpha_1(x)+\beta_1(x)u_1$ by Σ_1. In the following step we proceed with the second output, and therefore set $k=2$.

Step 3

Let $\dot{x} = \tilde{f}(x) + \sum_{l=1}^{p_{k-1}} \tilde{g}_l(x) u_{k-1}^l$ with $u_{k-1} \in \mathbb{R}^{p_{k-1}}$ be the description of the system Σ_{k-1} and set $i = 0$.

Step 4

Increase i by 1 and set $\gamma_k = \left(z_0^k, \ldots, z_{i-1}^k\right)^T$. Assume that the rank of γ_k is constant around x_0 and

$$\operatorname{rank}\left(\gamma_1^T, \ldots, \gamma_k^T\right)^T = i + \sum_{j=1}^{k-1} \delta_j. \tag{7}$$

If $L_{\tilde{g}_l} z_{i-1}^k = 0,\ \forall\, 1 \leq l \leq p_{k-1}$ holds, set $z_i^k = L_{\tilde{f}} z_{i-1}^k$.

Step 5

Repeat step 4 until $\exists\, l,\ 1 \leq l \leq p_{k-1}$, such that $L_{\tilde{g}_l} z_{i-1}^k \neq 0$. Set $\delta_k = i$ and $\mathcal{M}_k = \{x \in \mathcal{M}_{k-1} \mid \gamma_k = 0\}$. Choose parameters ν_j^k, such that the roots of the polynomial

$$P^k(s) = 1 + \sum_{j=1}^{\delta_k} \nu_j^k \cdot (s)^j \tag{8}$$

are all in $\mathbb{C}^-$. It is possible to find a control law $u_{k-1} = \alpha_k(x) + \beta_k(x) u_k$, with the p_{k-1}-dimensional vector $\alpha_k(x)$, the $p_{k-1} \times p_k$ matrix $\beta_k(x)$, and $u_k \in \mathbb{R}^{p_k}$, such that

$$z_0^k + \nu_1^k z_1^k + \cdots + \nu_{\delta_k - 1}^k z_{\delta_k - 1}^k + \nu_{\delta_k}^k \left(L_{\tilde{f}} z_{\delta_k - 1}^k + \sum_{l=1}^{p_{k-1}} L_{\tilde{g}_l} z_{\delta_k - 1}^k \alpha_k^l \right) = 0$$

and

$$L_{\tilde{g}_l} z_{\delta_k - 1}^k \beta_k^l(x) = 0 \quad \forall\, 1 \leq l \leq p_{k-1}$$

hold on $\mathcal{M}_k$ in a neighborhood of x_0. Denote the system Σ_{k-1} controlled with $u_{k-1} = \alpha_k(x) + \beta_k(x) u_k$ by Σ_k.

Step 6

Increase k by 1 and repeat steps 3 – 6 until $k = q$.

IV. Main Results

If the constant rank assumptions at step 1 and 4 are satisfied, x_0 is a regular point for this algorithm.

If (5) or (7) are not fulfilled in the k-th step for any i, the k-th coupling condition is not controllable by any of the remaining inputs. As in general, $z_0^k = 0$ is not an asymptotical stable equilibrium, the coupling conditions can not be fulfilled asymptotically stable with a static feedback by this approach. In this case the algorithm is not successful.

If the algorithm is successful, it leads to the manifold $\mathcal{M}_q$ and the control law

$$u = \alpha_1 + \beta_1 \left(\alpha_2 + \beta_2 \left(\ldots \left(\alpha_q + \beta_q v\right) \ldots\right)\right), \tag{9}$$

with the reference input $v = u_q \in \mathbb{R}^{p-q}$.

The presented algorithm is very much related to the constrained dynamics algorithm in [9]. The following theorem shows the similarity of the results.

Theorem 4.1: If the constrained dynamics algorithm with stability is successful, $\mathcal{M}_q$ is the maximal locally controlled invariant output-nulling submanifold around x_0.

Before proofing this theorem, we need some additional results.

Lemma 4.1: The manifold $\mathcal{M}_q$ is invariant against any permutation of the coupling conditions.

Proof: Assume that two sequent coupling conditions $h^k(x)$ and $h^{\tilde{k}}(x)$, $\tilde{k} = k + 1$, are permuted. Therefore $\mathcal{M}_{k-1}$ does not depend on the permutation. Let Σ_{k-1} be $\dot{x} = \bar{f}(x) + \bar{g}(x) u_{k-1}$ with $u_{k-1} \in \mathbb{R}^{p_{k-1}}$.

As long as $L_{\bar{g}} z_i^k = 0$, $z_{i+1}^k = L_{\bar{f}} z_i^k$ holds, and as long as $L_{\bar{g}} z_{\tilde{i}}^{\tilde{k}} = 0$, $z_{\tilde{i}+1}^{\tilde{k}} = L_{\bar{f}} z_{\tilde{i}}^{\tilde{k}}$ holds regardless of a permutation. Now assume that $L_{\bar{g}} z_i^k \neq 0$ and $L_{\bar{g}} z_{\tilde{i}}^{\tilde{k}} \neq 0$ and denote by B the $2 \times p_{k-1}$ matrix

$$B = \begin{pmatrix} L_{\bar{g}} z_i^k \\ L_{\bar{g}} z_{\tilde{i}}^{\tilde{k}} \end{pmatrix}.$$

There are two possibilities:

1) $\operatorname{rank} B = 2$,
2) $\operatorname{rank} B = 1$.

If the first possibility holds, the same control law can be used to fulfill $z_{i+1}^k = 0$ and $z_{\tilde{i}+1}^{\tilde{k}} = 0$ on $\mathcal{M}_{k-1}$ independent of the permutation. Therefore Σ_{k+1} is also independent of it. This proves the invariance of $\mathcal{M}_{k+1}$.

If the second possibility holds, there exists a function $K(x) \neq 0$, such that $L_{\bar{g}} z_i^k = K(x) L_{\bar{g}} z_{\tilde{i}}^{\tilde{k}}$. Note that the permutation of the coupling conditions is important for the choice of the control law.

As the default sorting is used, the control law guarantees that $z_{i+1}^k = 0$ on $\mathcal{M}_{k-1}$. Therefore $L_{\bar{g}} z_i^k u_{k-1} = -L_{\bar{f}} z_i^k$ holds on $\mathcal{M}_{k-1}$, which leads to $K(x) L_{\bar{g}} z_{\tilde{i}}^{\tilde{k}} u_{k-1} = -L_{\bar{f}} z_i^k$. Thus, the new constraint is

$$z_{\tilde{i}+1}^{\tilde{k}} = L_{\bar{f}} z_{\tilde{i}}^{\tilde{k}} - \frac{1}{K(x)} L_{\bar{f}} z_i^k.$$

If the permuted sorting is used, the control law guarantees that $z_{\tilde{i}+1}^{\tilde{k}} = 0$. Therefore $L_{\bar{g}} z_{\tilde{i}}^{\tilde{k}} u_{k-1} = -L_{\bar{f}} z_{\tilde{i}}^{\tilde{k}}$ holds, which leads to $L_{\bar{g}} z_i^k u_{k-1} = -K(x) L_{\bar{f}} z_{\tilde{i}}^{\tilde{k}}$ on $\mathcal{M}_{k-1}$. Hence, the new constraint is

$$z_{i+1}^k = L_{\bar{f}} z_i^k - K(x) L_{\bar{f}} z_{\tilde{i}}^{\tilde{k}}.$$

Since $K(x) z_{\tilde{i}+1}^{\tilde{k}} = -z_{i+1}^k$ on $\mathcal{M}_{k-1}$, the sorting has no influence on the manifold $\mathcal{M}_{k+1}$, but the control laws are different on $\mathcal{M}_{k-1}$. Though $L_{\bar{f}} z_i^k - K(x) L_{\bar{f}} z_{\tilde{i}}^{\tilde{k}} = 0$ on $\mathcal{M}_{k+1}$, the control laws are the same on $\mathcal{M}_{k+1}$, and therefore Σ_{k+1} is also invariant.

As in both possibilities Σ_{k+1} and $\mathcal{M}_{k+1}$ are independent of the permutation, $\mathcal{M}_q$ is also invariant against a permutation of sequent coupling conditions. A permutation of any two coupling conditions can be reached by sequent permutations, hence $\mathcal{M}_q$ is invariant against any permutation of coupling conditions. ■

Now we are ready to proof theorem 4.1.

Proof: (of theorem 4.1) Nijmeijer and van der Schaft [9] have shown that the maximal locally controlled invariant output-nulling submanifold $\mathcal{N}^*$ around x_0 is independent of the particular choice of the control law in the constrained dynamics algorithm. Permuting the coupling conditions such that $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_q$ holds the introduced algorithm coincides with the constrained dynamics algorithm choosing a particular control law. With lemma 4.1 it follows that $\mathcal{M}_q = \mathcal{N}^*$. ■

Now we want to draw the connection to the triangular decoupling problem presented in [8]. Find, if possible, regular local controllability distributions $R_1, \ldots, R_q$, such that

$$
\begin{aligned}
&R_1 \subset T\mathcal{M} \\
&R_i \subset \bigcap_{j=1}^{i-1} \ker h_*^j, \quad \forall\, 2 < i \leq q \\
&R_i + \ker h_*^i = T\mathcal{M},
\end{aligned}
$$

where h_*^i denotes the pushforward of h^i. R_i^* is the supremal regular local controllability distribution in $\bigcap_{j=1}^{i-1} \ker h_*^j$.

Proposition 4.1: The presented algorithm terminates successfully if and only if the triangular decoupling problem (see [8]) is locally solvable for the system (1), which is equivalent to the requirement

$$
R_i^* + \ker h_*^i = T\mathcal{M}, \quad \forall\, 1 \leq i \leq q \tag{10}
$$

around x_0.

Proof: Sufficiency of (10)*:*

As shown in [8], if (10) holds, there are local coordinates $w^1, \ldots, w^{q+1}$, each w^i possibly being a vector, such that

$$
\begin{aligned}
R_q^* &= \operatorname{span}\left\{\frac{\partial}{\partial w^1}\right\} \\
R_{q-1}^* &= \operatorname{span}\left\{\frac{\partial}{\partial w^1}, \frac{\partial}{\partial w^2}\right\} \\
&\vdots \\
R_1^* &= \operatorname{span}\left\{\frac{\partial}{\partial w^1}, \ldots, \frac{\partial}{\partial w^q}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
z_0^1 &= h^1(w^q, w^{q+1}) \\
z_0^2 &= h^2(w^{q-1}, w^q, w^{q+1}) \\
&\vdots \\
z_0^q &= h^q(w^1, \ldots, w^{q+1})
\end{aligned}
$$

holds, and there is a control law of the form (3) with $\tilde{p} = p$ and a partitioning of the new inputs v into q disjoint subsets I_j, such that the controlled system has the form

$$
\begin{aligned}
\begin{pmatrix} \dot{w}^1 \\ \vdots \\ \dot{w}^q \\ \dot{w}^{q+1} \end{pmatrix} &= \begin{pmatrix} \bar{f}^1(w^1, \ldots, w^{q+1}) \\ \vdots \\ \bar{f}^q(w^q, w^{q+1}) \\ \bar{f}^{q+1}(w^{q+1}) \end{pmatrix} + \\
&+ \sum_{j \in I_1} \begin{pmatrix} \bar{g}_j^1(w^1, \ldots, w^{q+1}) \\ \vdots \\ \bar{g}_j^q(w^q, w^{q+1}) \\ 0 \end{pmatrix} v^j + \cdots + \\
&+ \sum_{j \in I_q \setminus I_{q-1}} \begin{pmatrix} \bar{g}_j^1(w^1, \ldots, w^{q+1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix} v^j.
\end{aligned}
$$

Together with the output controllability [8], (5) and (7) are fulfilled.

Necessity of (10)*:*

Consider the control law $u = \alpha_1(x) + \beta_1(x)v$, $v \in \mathbb{R}^{p-1}$ in the first step of the algorithm. Choose a p-dimensional function $\tilde{\beta}_1(x)$, such that $\left(\tilde{\beta}_1, \beta_1\right)$ is a nonsingular $p \times p$ matrix around x_0. Modify the control law to $u = \alpha_1(x) + \tilde{\beta}_1(x)\tilde{v}_1 + \beta_1(x)v$. Then the input $\tilde{v}_1$ has influence on the first output and possibly on some of the following.

Consider the control law $u = \alpha_2(x) + \beta_2(x)v$, $v \in \mathbb{R}^{p-2}$ in the second step of the algorithm. Choose a $(p-1)$-dimensional function $\tilde{\beta}_2(x)$ such that $\left(\tilde{\beta}_2, \beta_2\right)$ is a nonsingular $(p-1) \times (p-1)$ matrix around x_0. Modify the control law to $u = \alpha_2(x) + \tilde{\beta}_2(x)\tilde{v}_2 + \beta_2(x)v$. Then the input $\tilde{v}_2$ has influence on the second output and possibly on some of the following, but not on the first output. Proceeding this way leads to the input $\tilde{v}_i$ that has influence on the i-th output and possibly the following, but not on the previous. This exactly describes a triangular decoupled system defined in [8]. Therefore, if the algorithm succeeds, system (1) can be triangular decoupled and equivalently (10) holds. ■

Theorem 4.2: The submanifold $\mathcal{M}_q$ is locally attractive around x_0 for the system (1) controlled by (9).

Proof: Let $\xi_1 = \gamma_1, \ldots, \xi_q = \gamma_q$ together with $\xi_{q+1} \in \mathbb{R}^r$, $r = n - \sum_{i=1}^q \delta_i$ be local coordinates for the controlled system around x_0. The system has the form

$$
\begin{pmatrix} \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_q \\ \dot{\xi}_{q+1} \end{pmatrix} = \begin{pmatrix} A_1\xi_1 \\ \vdots \\ A_q\xi_q \\ \tilde{f}(\xi_1, \ldots, \xi_{q+1}) + \tilde{g}(\xi_1, \ldots, \xi_{q+1})v \end{pmatrix}.
$$

As the roots of the polynomial $P^k(s)$ (see (8)) coincide with the eigenvalues of the matrix A_k, with lemma 2.1 $\mathcal{M}_q$ is a locally attractive submanifold for the controlled system around x_0. ■

Remark 4.2: If a systems fails to have relative degree, hence no static decoupling is possible, a static coupling controller can still be derived if (10) holds. This will be exemplified in the following example.

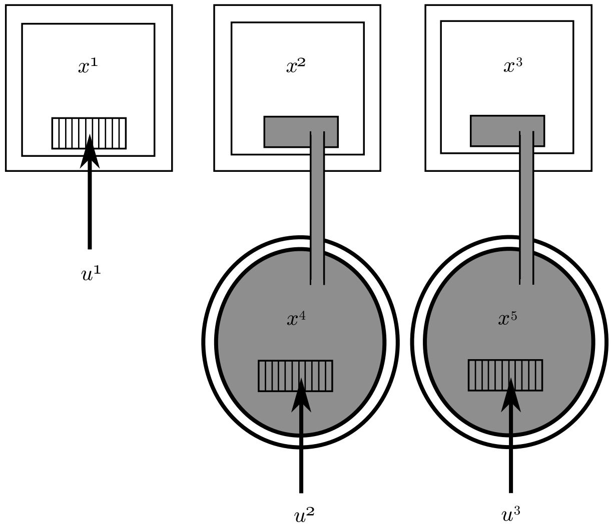

Fig. 1. Three heated rooms

V. Example

We consider three heated rooms depicted in Fig. 1. The first room can be heated directly by the input u^1, the heat transfer into the room. The other two rooms are heated via two boilers with the inputs u^2 and u^3 respectively. The temperature of each room is described by x^1, x^2, and x^3 respectively, the temperature of each boiler by x^4 and x^5. The heat emission of each room is considered as a nonlinear function of the room temperature. With $x = \left(x^1, \dots, x^5\right)^T$ the nonlinear system has the form

$$\dot{x} = \begin{pmatrix} -c_0(x^1 - T_0) - c_1(x^1 - T_0)^2 \\ -c_0(x^2 - T_0) - c_1(x^2 - T_0)^2 + c_2(x^4 - x^2) \\ -c_0(x^3 - T_0) - c_1(x^3 - T_0)^2 + c_2(x^5 - x^3) \\ -c_2(x^4 - x^2) \\ -c_2(x^5 - x^3) \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}, \tag{11}$$

where c_0, c_1, and c_2 are parameters for the heat transmission, and T_0 is the temperature outside of the rooms. We will choose our parameters so that the time is measured in hours and x^i is measured in kelvin. The state manifold is $\mathcal{M} = \mathbb{R}^5$, and the coupling conditions are

$$y = h(x) = \begin{pmatrix} x^1 - x^2 \\ x^1 - x^3 \end{pmatrix} = 0, \tag{12}$$

i.e. the temperature of the three rooms should be equal. As there are three inputs, an additional control variable $y_c = x^3$ is introduced. With the extended output $y_e = \left(y^T, y_c\right)^T$, it is easy to verify that system (11) has no well defined relative degree and therefore can not be decoupled with a static state feedback. But we will design a control law fulfilling the coupling conditions (12) and additionally manipulating the dynamics of y_c arbitrarily.

In the first step of the constrained dynamics algorithm with stability, z_0^1 is set to $h^1(x)$. As $L_{g_1} z_0^1 = 1$ holds, $\delta_1 = 1$, and $\mathcal{M}_1 = \left\{x \in \mathbb{R}^5 \mid x^1 - x^2 = 0\right\}$. The first control law fulfills $z_0^1 + \nu_1^1(L_f z_0^1 + \sum_{l=1}^3 L_{g_l} z_0^1 \alpha_1^l) = 0$, $L_{g_l} z_0^1 \beta_1^l = 0$,$\forall\, 1 \le l \le 3$. Set $\tilde{f}(x) = f(x) + g(x)\alpha_1(x)$ and $\tilde{g}(x) = g(x)\beta_1(x)$.

In the third step with $k = 2$, z_0^2 is set to $h^2(x)$. As $L_{\tilde{g}_l} z_0^2 = 0$ for $l = 1, 2$, $z_1^2 = L_{\tilde{f}} z_0^1$. Since $L_{\tilde{g}_l} z_1^2 \neq 0$ for $l = 1, 2$ and $\operatorname{rank}\left(z_0^1, z_0^2, z_1^2\right) = 3$, the second control law has to fulfill $z_0^2 + \nu_1^2 z_1^2 + \nu_2^2(L_{\tilde{f}} z_1^2 + \sum_{l=1}^2 L_{\tilde{g}_l} z_1^2 \alpha_2^l) = 0$, $L_{\tilde{g}_l} z_1^2 \beta_2^l = 0$ for $l = 1, 2$. As $k = q = 2$, the algorithm terminates. The restrictions lead to the manifold $\mathcal{M}_2 = \left\{x \in \mathcal{M}_1 \mid z_0^2 = 0,\ z_1^2 = 0\right\}$ identical to the manifold which can be found by the constrained dynamics algorithm. Hence $\mathcal{M}_2$ is the maximal locally controlled invariant output-nulling submanifold for the system (11) with the outputs $y = h(x)$.

The system Σ_2, controlled by $u = \alpha_1 + \beta_1\left(\alpha_2 + \beta_2 w\right)$, can be transformed with $\xi = \Phi(x) = \left(z_0^1, z_0^2, z_1^2, x^3, x^5\right)^T$ and the parameters $\nu_1^1 = 1$, $\nu_1^2 = \frac{7}{3}$, and $\nu_2^2 = \frac{1}{3}$ to

$$\dot{\xi} = \begin{pmatrix} -\xi^1 \\ \xi^3 \\ -3\xi^2 - 3.5\xi^3 \\ c_0\left(T_0 - \xi^4\right) - c_1\left(T_0 - \xi^4\right)^2 + c_2\left(\xi^5 - \xi^4\right) \\ w + c_2(\xi^4 - \xi^5) \end{pmatrix}. \tag{13}$$

The manifold $\mathcal{M}_2$, described in the new coordinates ξ, leads to

$$\mathcal{M}_2 = \left\{\xi \in \mathbb{R}^5 \mid \xi^1 = \xi^2 = \xi^3 = 0\right\}.$$

With lemma 2.1 this manifold is a locally attractive invariant submanifold for the system (13).

As the system (11) is strongly accessible, $R_1^* = T\mathcal{M}$ [8]. The supremal controlled invariant distribution in $\ker h_*^1$ is

$$D = \operatorname{span}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & c_2 \\ 0 & 0 \\ 1 & -c_2 \end{pmatrix}\right\}.$$

As D is also a regular controllability distribution, $R_2^* = D$ holds, and therefore $R_2^* + \ker h_*^2 = T\mathcal{M}$. Hence the triangular decoupling problem is solvable, what is equal to the success of the presented algorithm.

Obviously, it is possible to manipulate the remaining dynamics on $\mathcal{M}_2$. We assumed the temperature x^3 to be the control variable y_c. For the system Σ_2 with the input w and the output y_c an input-to-output linearization [2] is possible as the relative degree is two. We choose the transfer function

$$G(s) = \frac{0.25}{s^2 + s + 0.25}$$

for the input-to-output behavior of the controlled system with reference input v_c. The parameters are chosen as

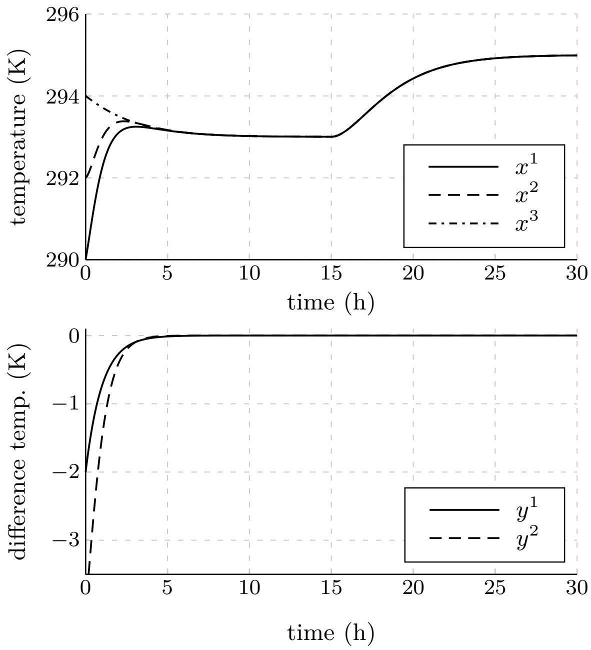

Fig. 2. Simulation of the controlled system

$c_0 = 0.1$, $c_1 = 0.01$, $c_2 = 1$, and $T_0 = 283$. The simulation of the controlled system is shown in Fig. 2. At the beginning, $x(0) = (290, 291, 294, 305, 317)^T$ and $v_c = 293$ are chosen. As depicted in Fig. 2, the coupling outputs tend to zero within less than 5 hours with the specified linear behavior. After 15 hours, the reference input is changed to 295. The room temperature of the three rooms follows this step with the same dynamics as $G(s)$. Note that the coupling outputs are not affected by v_c.

VI. Conclusion

We presented a constructive algorithm for deriving the maximal locally controlled invariant output-nulling submanifold and a static state feedback that renders this submanifold locally attractive for the controlled system. It was shown that the assumptions of our approach are fulfilled iff the triangular decoupling problem is solvable. Therefore necessary and sufficient conditions for the success of the new algorithm could be given.

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