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An Anti-windup Scheme with Embedded Internal Model Control Anti-windup for Improved Performance

Wei Wu

***Abstract*— This paper considers the synthesis of an anti-windup(AW) compensator which comprises an internal model control(IMC) antiwindup compensator and a linear anti-windup compensator, for stable linear plants subject to input saturation. Built on the conventional internal model control antiwindup scheme which preserves the stability and stability robustness of the unconstrained linear closed-loop system, the proposed antiwindup scheme aims to improve the performance of the constrained closed-loop system by including another anti-windup compensator. The $\mathcal{L}_2$ gain of the constrained system from the exogenous input to the performance output is used as the performance objective for the the synthesis of the extra antiwindup compensator, resulting in nonconvex matrix conditions. For certain orders of the extra antiwindup compensator, the nonconvex matrix conditions are reduced into convex linear matrix inequalities(LMIs). Especially when the order the extra anti-windup compensator is zero, it is proven that there always exists such an anti-windup compensator guaranteeing a finite $\mathcal{L}_2$ gain for the antiwindup compensated constrained closed-loop system. This important result leads to the design of an overall plant order antiwindup compensator which improves system performance compared to the IMC anti-windup, as demonstrated by the simulation results.**

I. INTRODUCTION

Actuator saturation leads to performance degradation and, sometimes, instability in the feedback systems. A large amount of research, aiming at mitigating input saturation's adverse effects on control systems' stability and performance, has been continuously conducted since 1950's(see [15],[22] and references therein). The linear antiwindup augmentation is popular in the literature. In this approach, a linear controller is first designed based on the linear control theory ignoring the fact that the actuator can saturate, then add the antiwindup compensation to counteract saturation when it occurs. The philosophy of designing the AW compensator is twofold [5],[4]: (1) to preserve the stability and performance properties of the unconstrained linear system in the absence of saturation during normal operations and; (2)to swiftly recover the linear response when saturation occurs while maintaining the stability of the closed-loop system. The approach in this paper is within this two step linear antiwindup design paradigm.

Antiwindup research has developed over the years from treating specific controllers at the early stages, such as a PI controller [24], to offering more general solutions with formally proven stability and performance guarantees since the 1990's. Recently, linear matrix inequality formulations of antiwindup problems have become popular in the literature, leading to convex optimization problems for the antiwindup compensator synthesis with $\mathcal{L}_2$ performance objectives [1],[2],[3],[6],[5],[10],[12],[19]. [10] gave a LMI formulation of the static antiwindup design problem, where the LMIs may not always be feasible. [2] gave a system theoretic limit for the $\mathcal{L}_2$ performance of the full authority antiwindup compensation: a lower bound on the $\mathcal{L}_2$ gain achievable by the augmented system is the maximum of the $\mathcal{L}_2$ gain of the open-loop plant(with zero control input) and that of the unconstrained closed-loop and reduced nonconvex matrix constraints into convex LMI constraints when the antiwindup compensator order is zero or the order of the plant. It was shown that the plant order antiwindup synthesis is always feasible (for a large enough $\mathcal{L}_2$ gain).

The scheme proposed in this paper is motivated by the IMC anti-windup scheme. IMC [23] was not intended for anti-windup, however, the IMC guarantees the closed-loop stability under input saturation provided that both the plant and the controller are stable. The IMC anti-windup scheme was improved in [9], also refer to [18] for the relation between the model-based conditioning and the IMC-like scheme. But many examples have shown it to be a poorly performing anti-windup scheme [14], [6] when the plant has slow dynamics or non-minimum phase zeros. [6],[13] showed that the IMC anti-windup scheme is optimal in the sense of the retention of the linear system's stability with respect to the additive plant uncertainty. [11] that the IMC AW achieves the lowest global performance bound of the AW compensated constrained closed-loop system on the gain from the output signal of the deadzone nonlinearity with the plant input in the linear closed-loop as it's input, to the deviation of the nonlinear plant response from the corresponding linear one.

Since it is desirable to improve the performance of the IMC anti-windup scheme. [7],[13] introduced an extra linear AW compensator into the IMC AW structure to improve the performance, where the resulting AW configuration is similar to that in [4] and [8] for the $\mathcal{L}_2$ AW problem and synthesis. Based on the earlier results, the AW configuration combining the IMC AW and the full authority linear AW is proposed in these paper to preserve the optimal stability and performance properties of the IMC AW and to better

Senior Control System Engineer, Lexmark International, Lexington, KY, 40550, USA wu_esi@yahoo.com

AW performance with the addition of a linear, full authority, AW compensator. The particular AW arrangement is a subset of the linear, full authority AW compensator in [2] having a beneficial internal structure that will be shown. In the proposed AW configuration, the AW compensator is designed by considering the quadratic stability and $\mathcal{L}_2$ performance of the AW closed-loop system using the Lyapupov analysis tools in [2]. Many results, parallel to those in [2], are derived in terms of matrix conditions. For some special cases where the AW compensator has a particular order, nonconvex matrix conditions are recast into LMIs. Especially when the proposed AW compensator is static(zero order), the AW synthesis problem in terms of LMIs is always feasible. The solution may lead to a better AW performance compared to the optimal antiwindup compensator chosen over the complete set of plant order, full authority, linear antiwindup compensators as demonstrated by the numerical example.

This paper is organized as follows. In Section 2, we describe the proposed AW problem. In section 3, we address the AW synthesis problem, deriving matrix conditions. Especially the plant order AW synthesis is discussed. A numerical example is given in Section 4, along with the comparisons to two AW approaches from the literature. Finally, conclusions are stated in Section 5.

II. Antiwindup Problem

A. Constrained System Description

Consider a linear exponentially stable plant

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_{p,u} u + B_{p,w} w \\ y &= C_{p,y} x_p + D_{p,yu} u + D_{p,yw} w \\ z &= C_{p,z} x_p + D_{p,zu} u + D_{p,zw} w \end{aligned} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the plant input, $w \in \mathbb{R}^{n_w}$ is the external input(including all reference, disturbance, and sensor noise considered), $y \in \mathbb{R}^{n_y}$ is the measured plant output, $z \in \mathbb{R}^{n_z}$ is the performance output and all matrices have suitable dimensions.

Assume that a linear controller has already been designed

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_{c,y} y_a + B_{c,w} w + v_1 \\ y_c &= C_c x_c + D_{c,y} y_a + D_{c,w} w + v_2 \end{aligned} \tag{2}$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state, $y_c \in \mathbb{R}^{n_u}$ is the controller output, $y_a \in \mathbb{R}^{n_y}$ is the modified measurement by the IMC AW, and v_1 and v_2 are inputs generated by the extra linear AW compensator and all matrices are of suitable dimensions, such that the linear closed-loop with the interconnection

$$u = y_c \qquad v_1 = v_2 = 0 \qquad y_a = y \tag{3}$$

is well-posed and internally stable.

The plant input is limited by a decentralized saturation function

$$u = sat(y_c) \tag{4}$$

where

$$sat(y_c) = [sat_1(y_{c1}), sat_2(y_{c2}), \ldots, sat_{n_u}(y_{cn_u})]^T \tag{5}$$

where $sat(y_{ci}) = sign(y_{ci}) \times min\{|y_{ci}|, M_i\}, M_i > 0 \; \forall i = 1, \ldots, n_u$. M_i is the saturation limit for the ith input. The decentralized saturation function belongs to the sector[0 I] and has the following property

$$sat_i(u)^2 \le sat_i(u)u \le u^2 \quad \forall u \in \mathbb{R} \quad \forall i \in \{1, \ldots, n_u\}. \tag{6}$$

B. Anti-windup Configuration

First the constrained closed-loop is compensated by the IMC AW given by

$$\begin{aligned} \dot{x}_{im} &= A_p x_{im} + B_{p,u}(y_c - u) \\ y_{im} &= C_{p,y} x_{im} + D_{p,yu}(y_c - u) \end{aligned} \tag{7}$$

where $x_{im} \in \mathbb{R}^{n_p}$ is the state of the IMC AW compensator, $y_{im} \in \mathbb{R}^{n_y}$ is the IMC AW compensator output, and $A_p, B_{p,u}, C_{p,y}, D_{p,yu}$ are plant matrices, through the interconnection

$$y_a = y + y_{im}. \tag{8}$$

Next, we introduce a linear, full authority AW compensation to improve AW performance in this way

$$\begin{aligned} \dot{x}_{aw} &= \Lambda_1 x_{aw} + \Lambda_2 (y_c - u) \\ v &= [v_1 \; v_2]^T = \Lambda_3 x_{aw} + \Lambda_4 (y_c - u) \end{aligned} \tag{9}$$

where $x_{aw} \in \mathbb{R}^{n_{aw}}$ is the anti-windup compensator state, $v \in \mathbb{R}^{n_v}$ with $v = n_c + n_u$ is the antiwindup compensator output, and matrices Λ_1,Λ_2,Λ_3,Λ_4 have suitable dimensions. The interconnection (1), (2), (7), (9), (4), and (8) is the complete antiwindup compensated constrained closed-loop system. The AW synthesis problem is to find a linear antiwindup compensator (9)(note that the IMC AW compensator is a copy of the plant) that guarantees the nonlinear stability and an acceptable $\mathcal{L}_2$ gain from the exogenous input w to the performance output z for the antiwindup compensated constrained closed-loop system.

The stability and $\mathcal{L}_2$ performance of the antiwindup compensated closed-loop system can be assured in terms of Lyapunov analysis as proposed in [2].

Definition 1 [2]: The antiwindup compensated closed-loop system is well-posed and has a quadratic performance of level γ for the decentralized saturation nonlinearity (4) if

1) the interconnection (1), (2), (7), (9), (4), and (8) is well-posed;
2) there exists a scalar $\epsilon > 0$ and a quadratic Lyapunov function $V(x) = x^T P x$ with $x = [x_p^T \; x_c^T \; x_{im}^T \; x_{aw}^T]^T$ and $P = P^T > 0$, such that its time derivative along the dynamics of the interconnection satisfies

$$\dot{V}(x) < -\epsilon x^T x - \frac{1}{\gamma} z^T z + \gamma w^T w \quad \forall (x, q, w) \neq 0. \tag{10}$$

C. Connections to other antiwindup configurations

The proposed antiwindup configuration is a particular combination of the IMC antiwindup ((7) and (8)) and the "linear antiwindup design" ((9) and (2)) [2],[1],[10] which has full authority toward modifying both the controller state and output. The IMC antiwindup can be showed as a special case of the linear antiwindup design with the following dynamics

$$\begin{aligned} \dot{x}_{im} &= A_p x_{im} + B_{p,u}(y_c - u) \\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} B_{c,y} \\ D_{c,y} \end{bmatrix} C_{p,y} x_{im} + \begin{bmatrix} B_{c,y} \\ D_{c,y} \end{bmatrix} D_{p,yu}(y_c - u). \end{aligned} \quad (11)$$

Define the overall antiwindup state $x' = [x_{im}^T \; x_{aw}^T]^T$, the proposed antiwindup compensation (7) and (9) has an overall dynamics as

$$\begin{aligned} \dot{x}' &= \begin{bmatrix} A_p 0 \\ 0 \quad \Lambda_1 \end{bmatrix} x' + \begin{bmatrix} B_{p,u} \\ \Lambda_2 \end{bmatrix} (y_c - u) \\ v &= \left[\begin{bmatrix} B_{c,y} C_{p,y} \\ D_{c,y} C_{p,y} \end{bmatrix} \Lambda_3 \right] x' + \left[\begin{bmatrix} B_{c,y} D_{p,yu} \\ D_{c,y} D_{p,yu} \end{bmatrix} + \Lambda_4 \right] (y_c - u), \end{aligned} \quad (12)$$

which also has the form of the "linear antiwindup design". In this work, the task is to properly design the antiwindup compensator of the order n_{aw} in (9), after the IMC antiwindup (7) is implemented. Note that this synthesis problem can not be solved by the algorithm in [2]. Regarding the overall antiwindup compensator with the specific structure in (12), the order of the overall antiwindup compensator is $n_{aw} + n_p$. Especially, when the linear antiwindup compensator (9) is static, $n_{aw} = 0$, the order of the overall AW compensator (12) is n_p. [2] proves that dynamic linear antiwindup of order n_p is always feasible for a large enough $\mathcal{L}_2$ gain if the plant is Hurwitz. Similarly, within the proposed antiwundup configuration, it is shown later that the linear antiwindup compensation (9)of order $n_{aw} = 0$ is always feasible for a large enough $\mathcal{L}_2$ gain if the plant is Hurwitz, that is a static linear antiwindup compensator (9) always exists.

III. Anti-windup Synthesis

To facilitate the analysis of the antiwindup compensated closed-loop system, we represent the system in a compact way. Define the deadzone function $Dz(\cdot) : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ as

$$q = Dz(y_c) = y_c - sat(y_c). \quad (13)$$

and, define the state

$$x = [x_p^T \quad x_c^T \quad x_{im}^T \quad x_{aw}^T]^T.$$

We simply write the dynamics of the plant, the controller, the IMC AW controller, and the extra linear AW controller together as

$$\begin{aligned} \dot{x} &= Ax + B_q q + B_w w \\ y_c &= C_y x + D_{yq} q + D_{yw} w \\ z &= C_z x + D_{zq} q + D_{zw} w \end{aligned} \quad (14)$$

where all matrices are suitably dimensioned and are completely determined by the matrices in (1),(2),(7), and (9). The interconnection of (13) and (14) represents the AW compensated constrained closed-loop system defined by (1),(2),(4),(7), (9) and (8). According to Theorem 1 in [2], we immediately have the following theorem.

Theorem 1. *The anti-windup closed-loop control system (14) and (13) is well-posed and has a quadratic performance of level γ if and only if there exist matrices $Q = Q^T > 0$, $U = diag(u_1, u_2, \ldots, u_{n_u})$ with $U > 0$, and a scalar $\gamma > 0$ satisfying the following LMI*

$$\begin{bmatrix} QA^T + AQ & B_q U + QC_y^T & B_w & QC_z^T \\ * & D_{yq}U + UD_{yq}^T - 2U & D_{yw} & UD_{zq}^T \\ * & * & -\gamma I & D_{zw}^T \\ * & * & * & -\gamma I \end{bmatrix} < 0 \quad (15)$$

Proof: See [2] and notice that U, as an unknown to be solved, is diagonal because we are concerned with decentralized saturation functions.

Theorem 1 can be used for system performance analysis when the antiwindup compensator is given. To use Theorem (1) for antiwindup compensator synthesis, another system representation is utilized. Define $x_{CL} = [x_p^T \quad x_c^T \quad x_{im}^T]^T$ and combine the plant, the controller, the IMC AW controller into a single dynamic system with the dynamics

$$\begin{aligned} \dot{x}_{CL} &= A_{CL} x_{CL} + B_{CL,q} q + B_{CL,w} w + B_{CL,v} v \\ y_c &= C_{CL,y} x_{CL} + D_{CL,yq} q + D_{CL,yw} w + D_{CL,yv} v \\ z &= C_{CL,z} x_{CL} + D_{CL,zq} q + D_{CL,zw} w + D_{CL,zv} v \end{aligned} \quad (16)$$

where all matrices are uniquely determined by the matrices of the plant, the linear controller, and the IMC antiwindup compensator. Together, (16), (13), and (9) describes the antiwindup compensated closed-loop system.

Theorem 2. *The antiwindup compensated closed-loop system is well-posed and has a quadratic performance of level γ if and only if the following matrix conditions in the unknowns (R, U, S, γ) are met:*

$$\begin{bmatrix} R_{11}A_p^T + A_p R_{11} & R_{13}A_p^T + A_p R_{13} - B_{p,u} U B_{p,u}^T & B_{p,w} & R_{11} C_{p,z}^T \\ * & R_{33}A_p^T + A_p R_{33} & 0 & -B_{p,u} U D_{p,zu}^T + R_{13}^T C_{p,z}^T \\ * & * & -\gamma I & D_{p,zw}^T \\ * & * & * & -\gamma I \end{bmatrix} < 0 \quad (17)$$

$$\begin{bmatrix} SA_{CL}^T + A_{CL} S & B_{CL,w} & SC_{CL,z}^T \\ * & -\gamma I & D_{CL,zw}^T \\ * & * & -\gamma I \end{bmatrix} < 0 \quad (18)$$

$$R = R^T = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^T & R_{22} & R_{23} \\ R_{13}^T & R_{23}^T & R_{33} \end{bmatrix} < 0 \tag{19}$$

$$S = S^T > 0 \tag{20}$$

$$R - S \geq 0 \tag{21}$$

$$rank(R - S) \leq n_{aw} \tag{22}$$

$$U = diag(u_1, u_2, \ldots, u_{n_u}) > 0 \tag{23}$$

Proof: See Appendix.

In Theorem (2) only the rank condition (22) is nonlinear with respect to the unknowns. Because the IMC antiwindup of order n_p is embedded in the closed-loop for antiwindup purpose *a priori*, the full order of the antiwindup compensator (9) in this case is $n_{aw} = n_p + n_c + n_p$. When the antiwindup compensator is full order, the rank condition is automatically satisfied and a LMI eigenvalue problem can be formulated to solve for the antiwindup compensator which optimizes the performance level γ based on LMIs in Theorem (2). However, the reduced order antiwindup compensator of $n_{aw} < 2n_p + n_c$ is more practical. For the special reduced order cases $n_{aw} = 0$ and $n_{aw} \geq 2n_p$, the nonlinear rank condition can be reduced into equivalent linear conditions.

Theorem 3. *For an antiwindup compensator of order $n_{aw} = 0$, the antiwindup closed-loop system is well-posed and has a performance level of γ if and only if there exist a solution (R, U, γ) to the following LMI conditions:*

$$\begin{bmatrix} R_{11}A_p^T + A_pR_{11} & R_{13}A_p^T + A_pR_{13} - B_{p,u}UB_{p,u}^T & B_{p,w} & R_{11}C_{p,z}^T \\ * & R_{33}A_p^T + A_pR_{33} & 0 & -B_{p,u}UD_{p,zu}^T + R_{13}^TC_{p,z}^T \\ * & * & -\gamma I & D_{p,zw}^T \\ * & * & * & -\gamma I \end{bmatrix} < 0 \tag{24}$$

$$\begin{bmatrix} RA_{CL}^T + A_{CL}R & B_{CL,w} & RC_{CL,z}^T \\ * & -\gamma I & D_{CL,zw}^T \\ * & * & -\gamma I \end{bmatrix} < 0 \tag{25}$$

$$R = R^T = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^T & R_{22} & R_{23} \\ R_{13}^T & R_{23}^T & R_{33} \end{bmatrix} < 0 \tag{26}$$

$$U = diag(u_1, u_2, \ldots, u_{n_u}) > 0 \tag{27}$$

Proof: When $n_{aw} = 0$, the rank condition in Theorem (2) requires that $R = S$.

Theorem 4. *For an antiwindup compensator of order $n_{aw} \geq 2n_p$, the antiwindup closed-loop system is well-posed and has a performance level of γ if and only if there exist a solution $(R_{11}, R_{33}, R_{13}, U, S, \gamma)$ to the following LMI conditions:*

$$\begin{bmatrix} R_{11}A_p^T + A_pR_{11} & R_{13}A_p^T + A_pR_{13} - B_{p,u}UB_{p,u}^T & B_{p,w} & R_{11}C_{p,z}^T \\ * & R_{33}A_p^T + A_pR_{33} & 0 & -B_{p,u}UD_{p,zu}^T + R_{13}^TC_{p,z}^T \\ * & * & -\gamma I & D_{p,zw}^T \\ * & * & * & -\gamma I \end{bmatrix} < 0 \tag{28}$$

$$\begin{bmatrix} SA_{CL}^T + A_{CL}S & B_{CL,w} & SC_{CL,z}^T \\ * & -\gamma I & D_{CL,zw}^T \\ * & * & -\gamma I \end{bmatrix} < 0 \tag{29}$$

$$S = S^T = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12}^T & S_{22} & S_{23} \\ S_{13}^T & S_{23}^T & S_{33} \end{bmatrix} < 0 \tag{30}$$

$$R_{11} = R_{11}^T > 0 \tag{31}$$

$$R_{11} - S_{11} > 0 \tag{32}$$

$$R_{33} = R_{33}^T > 0 \tag{33}$$

$$R_{33} - S_{33} > 0 \tag{34}$$

$$U = diag(u_1, u_2, \ldots, u_{n_u}) > 0 \tag{35}$$

Proof: *Sufficiency:* Given a solution $(R_{11}, R_{33}, R_{13}, U, S, \gamma)$ to the LMIs in Theorem (4), let $R_{12} = S_{12}$, $R_{22} = S_{22}$ and $R_{23} = S_{23}$. The rank condition of R and S is satisfied. $R_{11} - S_{11} > 0$ and $R_{33} - S_{33} > 0$ implies $R - S \geq 0$. Thus (R, U, S, γ) is a solution to the matrix conditions in Theorem (2). *Necessity:* Given a solution (R, U, S, γ) to the matrix conditions in Theorem (2), $R - S \geq 0$ implies that $R_{11} - S_{11} \geq 0$ and $R_{33} - S_{33} \geq 0$. Then there exist $\bar{R}_{11} = \bar{R}_{11}^T > 0$ and $\bar{R}_{33} = \bar{R}_{33}^T > 0$ such that $\bar{R}_{11} - S_{11} > 0$, $\bar{R}_{33} - S_{33} > 0$ and the first LMI of Theorem (4) is satisfied. Thus $(\bar{R}_{11}, \bar{R}_{33}, R_{13}, U, S, \gamma)$ is a solution to LMIs in Theorem (4).

A. Plant order antiwindup compensator

In our antiwindup configuration, a n_{aw} order antiwindup compensator (9) results in a $n_{aw} + n_p$ order antiwindup compensator according to the configuration for the "linear antiwindup design". We are more interested in low order linear AW compensation for obvious practical reasons, of which We are particularly interested in the zero order antiwindup compensators which correspond to plant order AW compensators in [2]. We have the following theory regarding LMIs for zero order antiwindup compensators.

Theorem 5. *In the proposed AW configuration, a zero order antiwindup comepensator (9) guarantees the* *well-posedness and a quadratic performance level of γ if and only if there exist a matrix R such that*

$$R_{11}A_p^T + A_pR_{11} < 0 \tag{36}$$

$$RA_{CL}^T + A_{CL}R < 0 \tag{37}$$

$$R = R^T = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^T & R_{22} & R_{23} \\ R_{13}^T & R_{23}^T & R_{33} \end{bmatrix} < 0 \tag{38}$$

An important feature of the n_p order linear, full authority antiwindup compensation in [2] is that the n_p antiwindup synthesis problem is always feasible if the plant is Hurwitz. Our zero order antiwindup compensator corresponds to a special n_p order antiwindup compensator of [2]. It is therefore interesting to investigate if the zero order antiwindup synthesis for the proposed antiwindup scheme is feasible or not. It is found that because of the special structure of the proposed antiwindup compensation, an important feature of this scheme is that a zero order antiwindup compensator always exists for a large enough $\mathcal{L}_2$ gain γ provided that the plant is stable. A trivial solution for the zero order antiwindup compensator synthesis problem is no linear antiwindup, that is $\Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda_4 = 0$. No linear antiwindup compensation leads to the IMC antiwindup compensator, which guarantees the stability of the constrained closed-loop and a finite $\mathcal{L}_2$ gain if the linear system without saturation is internally stable. However, the IMC antiwindup might induce poor system performance, so a non zero zero order linear antiwindup compensator which improves the system performance, is desirable and should be found. The next theorem shows that a non zero zero order linear antiwindup compensator always exists.

Theorem 6. *For the proposed antiwindup compensation, a non zero static antiwindup compensator (9) that guarantees the well-posedness and a finite quadratic performance level of the antiwindup compensated closed-loop system always exists.*

IV. SIMULATION STUDY

This example is from [1]. We use this example to demonstrate the feasibility of the zero order/static antiwindup under the proposed scheme and, to compare it to the $\mathcal{L}_2$ AW synthesis approach proposed in [2]. Consider a damped mass-spring system with a state space equation

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k/m & -f/m \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,$$

$$z = y - r,$$

$$m = 0.1kg,\ k = 1kg/s^2,\ f = 0.005kg/s,$$

where $x = [q\ \dot{q}]$ represents position and speed of the body connected to the spring, m is the mass of the body, k is the spring constant, f is the damping coefficient, u represents the control force exerted on the mass, r is the desired mass position, y is the measurement for feedback, and z is the performance output. The linear controller and the prefilter designed by ignoring the input saturation are

$$K(s) = 200\frac{(s+5)^2}{s(s+80)},\ F(s) = \frac{5}{s+5}.$$

The linear design guarantees a fast response and zero steady-state error to step references in the linear mode. The plant input u is limited to ± -1Newton.
Responses of the linear system and the constrained system to a double pulse switching between ± 0.9meter every 5 seconds and going back to zero permanently after 10 seconds, are shown in Fig. 1. The response of the linear system is very desirable, but the response of the constrained system with input limitation converges to a stable limit cycle with a large peak value.

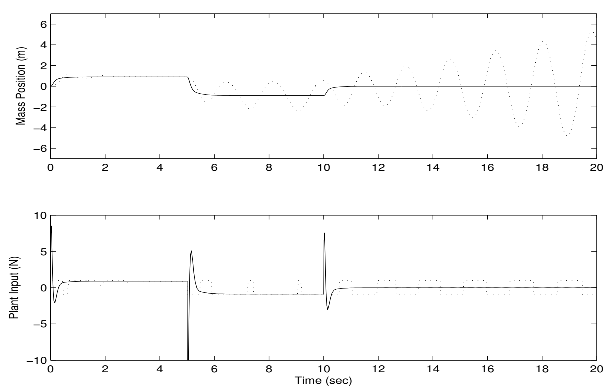

Fig. 1. Time responses of the linear system (solid), the constrained system (dotted) to a double pulse reference.

Next,we design the static AW gain for the proposed scheme using Theorem 6, giving

$$\Lambda_4 = [-3051.1,\ 64,\ 250.1,\ 0.4] \quad \text{and} \quad \gamma = 322.66.$$

We compared our approach with the approach proposed in [2]. In [2], the same $\mathcal{L}_2$ gain from w to z is minimized using the dynamic AW compensation as in (9) with no modification to the measurement input to the controller, that is $y_a = y$. A plant order dynamic AW compensator was given in [1] for this example. Note that our static AW also gives rise to an overall plant order antiwindup compensator. Fig. 2 shows the response of the AW compensated constrained system to a pulse reference signal which switches to 0.9meter for the first 5s, then switches back to 0, with the IMC AW, the proposed AW and the AW design proposed in [2], respectively. The IMC AW gives very poor performance with the large oscillations occurring in the steady state, the optimal dynamic AW compensation based on the method in [2] is not good neither, the response is very sluggish. Our result, however, is very good. It is also very interesting to compare the resultant $\mathcal{L}_2$ gains. For the IMC antiwindup it is 476.46 which is larger than 322.66 of the proposed scheme, and it is 21.0 for the optimal antiwindup compensator in [2]. The optimal antiwindup compensator [2] gives a much smaller global $\mathcal{L}_2$ bound compared to that given by our non-optimal antiwindup compensator. However, as demonstrated by this example, a smaller global $\mathcal{L}_2$ gain does not necessarily guarantee a better (local) performance for practical input signals. Furthermore, optimization leads to very large numbers in the state space matrices of the AW compensator, which causes implementation issues. Based on the simulation results, the proposed approach is superior to the other two AW approaches in this case.

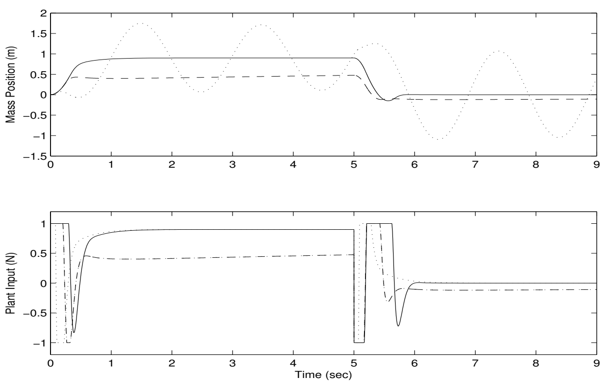

Fig. 2. Time responses of the constrained system with IMC anti-windup (dotted), the proposed AW (solid), and the AW from [2](dashed).

V. CONCLUSIONS

A new class of AW compensation has been proposed by introducing the linear AW compensation into the IMC AW scheme. The result is improved anti-windup performance whilst maintaining the stability and performance advantages of the IMC AW scheme. Antiwindup synthesis is given in nonconvex matrix conditions in general, and convex linear matrix inequalities for special lower order antiwindup compensators. Within this scheme, the zero order, or static, linear antiwindup synthesis problem is proven to be always feasible. The effectiveness and the superior performance of the proposed scheme are demonstrated through comparison to the IMC AW and another AW scheme in the literature using a numerical example.

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APPENDIX

Omitted.