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Population Dynamics Applied to Building Energy Efficiency

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Abstract— **Temperature control in buildings is a dynamic resource allocation problem, which can be approached using nonlinear methods based on population dynamics (i.e., replicator dynamics). A mathematical model of the proposed control technique is shown, including a stability analysis using passivity concepts for an interconnection of a linear multivariable plant driven by a nonlinear control system. In order to illustrate our control strategy, some simulations are performed, and we compare our proposed technique with another control strategy in a model with a fixed structure.**

I. INTRODUCTION

One of the focuses of the current research is the proper use of energetic resources in control systems, where energy efficiency in buildings is one important issue. Recent studies have shown that in some countries, buildings use approximately 70% of total electricity usage (primarily for heating, cooling, and lighting) and emit approximately 40% of greenhouse gases [1], causing significant impact on the environment. These statistics plus the fact that people spend most of the time inside buildings, make the building energy efficiency a topic of growing importance.

The use of appropriate control and automation techniques in buildings is a promising approach that can lead to significant energy savings [2], and nowadays heating, ventilating, and air-conditioning (HVAC) systems have the most prevalent use of automation. However, to apply classical control techniques like on-off, and conventional PID to HVAC systems is not optimal [3], especially if the presence of constraints in the energy consumption. In this paper we focus on multizone building temperature control considering restrictions in the total amount of power of the heating system. Our objective is to reach the temperature setpoint specified in each of the N rooms that comprises the building. We address the problem from the perspective of dynamic resource allocation, where the task is to distribute a resource (heating power) in the rooms of the building, given a set of setpoints. Clearly, an appropriate heating power allocation leads to energy efficiency. Dynamic resource allocation methods have been proposed before for building temperature control. For example in [4] a resource allocation technique based on market mechanisms is used. In our work, we propose to use a technique based on evolutionary game theory, i.e., the replicator dynamics [5]. This methodology has been used for resource allocation in a variety of applications with successful results (e.g., [6], [7]). Replicator dynamics model an evolutionary game which is inspired by natural selection and use a simple population dynamics to show how the proportion of animals (players) in a habitat (game strategy) is affected according to the suitability perceived by each of the individuals. In the problem discussed in this work, the habitats correspond to each of the rooms in the building, and the proportion of individuals that is allocated is related to a share of the total available power for the actuators. Thus, the suitability perceived by the individuals is associated to the error between the temperature of each room and its setpoint. The connection between the control technique and the building thermal model is seen as a feedback interconnection, which allows us to use some nonlinear methods for our analysis [8]. For this specific problem, we show that this interconnection leads to an asymptotically stable equilibrium point, using passivity theory [9].

The organization of this paper is as follows: a building thermal model is established in Section II. Next, in Section III the control goals are set, and the problem is addressed from a dynamic resource allocation perspective. In Section IV a comparison with another control strategy is shown via simulations. Finally, in Sections V and VI, arguments and conclusions of the developed work are presented.

II. BUILDING THERMAL MODEL

In [10], the authors outline a general model that considers the two most important components that constitute a building: rooms and walls. We use these ideas in order to model the thermal performance of a building consisting of N rooms, where each of them is enclosed by a certain number of walls and can be arranged according to different topologies (i.e. the spatial location of rooms is, in general, arbitrary).

A. Thermal Model for a Wall

A wall can be divided into layers with uniform temperature. The temperature $T^w_{j,k}$ of the layer k within the wall j is described by

$$\begin{aligned}(\rho^w_{j,k} c^w_{j,k} V^w_{j,k})\dot{T}^w_{j,k} \quad &= \quad K_{j,k+1}(T^w_{j,k+1} - T^w_{j,k}) + \\ &\quad +K_{j,k}(T^w_{j,k-1} - T^w_{j,k}),\end{aligned} \tag{1}$$

where $\rho^w_{j,k}, c^w_{j,k}, V^w_{j,k}$ are, respectively, the density, specific heat, and volume of the layer k (the term $\rho^w_{j,k} c^w_{j,k} V^w_{j,k}$ is associated with the termal capacitance of the layer). The thermal conductance $K_{j,k}$, can be estimated by harmonic mean as $K_{j,k} = A_j / (L_{j,k-1}/2\lambda_{j,k-1} + L_{j,k}/2\lambda_{j,k})$, where A_j is the area of the j^{th} wall, $L_{j,k}$ denotes the thickness of the layer k, and $\lambda_{j,k}$ its thermal conductivity. Subscripts

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$k+1$ and $k-1$ are related to the adjacent layers. There are some boundary conditions for layers which are in direct contact with a room or with the external environment, these conditions can be described as follows: *i)* For a layer that is in contact with the i^{th} room, $K_{j,k}(T^w_{j,k-1} - T^w_{j,k}) = h^r_i(T^r_i - T^w_{j,k})$, where T^r_i represents the temperature of the i^{th} room, and h^r_i its coefficient of convective heat transfer. *ii)* For a layer that is in contact with the external environment, $K_{j,k}(T^w_{j,k-1} - T^w_{j,k}) = h_a(T_a - T^w_{j,k})$, where T_a represents the ambient temperature, and h_a the external coefficient of convective heat transfer.

B. Thermal Model for a Room

In our model, we assume a uniform temperature in each room. Moreover, we consider that each of the rooms has a sensor and an actuator (heater). With these assumptions, the temperature T^r_i of the i^{th} room can be modeled as

$$(\rho_a c_a V^r_i)\dot{T}^r_i = \sum_{j \in \Omega^r_i} h^r_i A_j (T^w_{j,k} - T^r_i) + x_i + n_i, \qquad (2)$$

where Ω^r_i is the set of walls adjacent to room i, $T^w_{j,k}$ is the temperature of the layer within the wall j that is in direct contact with the room i; A_j is the area of wall j; ρ_a, c_a are, respectively, the density and the specific heat of the air; V^r_i is the volume of the i^{th} room, and h^r_i its coefficient of convective heat transfer; x_i is the heating power supplied by the actuator, and n_i is a thermal disturbance (e.g., because of the presence of people who are generating heat). In a similar way as in Equation (1), the terms $\rho_a c_a V^r_i$ and $h^r_i A_j$ in Equation (2) are associated with the thermal capacitance of the room i and with the thermal conductance of the junction between the room i and the wall j, respectively.

If we define the state vector

$$\mathbf{T} = [T^r_1, ..., T^r_N, T^w_{1,1}, ..., T^w_{1,m_1}, ..., T^w_{M,1}, ..., T^w_{M,m_M}]^\top, \qquad (3)$$

where N is the number or rooms of the building, M is the number of walls, and m_j is the number of layers of the j^{th} wall, Equations (1) and (2) can be unified into a single expression as follows

$$\theta_i \dot{T}_i = \sum_{j \in \Omega_i} \alpha_{ij}(T_j - T_i) + \alpha_{ia}(T_a - T_i) + s_i(x_i + n_i), \quad (4)$$

where T_i is the i^{th} element of the vector $\mathbf{T}$ (it corresponds to the temperature of a room or a layer), $\theta_i > 0$ is its thermal capacitance, Ω_i is the set of rooms/layers adjacent to the i^{th} element, T_a is the ambient temperature, $\alpha_{ij} > 0$ is the thermal conductance of the junction between elements i and j ($\alpha_{ij} = \alpha_{ji}$), and $\alpha_{ia} \geq 0$ is the thermal conductance of the junction between i^{th} element and the outside environment ($\alpha_{ia} = 0$ only if this junction does not exist). The variable s_i takes the value of 1 if $i = 1, \ldots, N$, or 0 otherwise. The model in Equation (4) yields a linear system of the form

$$\dot{\mathbf{T}} = \mathbf{AT} + \mathbf{Bu}, \quad \mathbf{y} = [T_1, \ldots, T_N]^\top, \qquad (5)$$

with $\mathbf{u} = [T_a, x_1, \ldots, x_N, n_1, \ldots, n_N]^\top$. Note that according to the state vector defined in Equation (3), the i^{th} output of the system corresponds to the temperature of the i^{th} room. For simplicity, in the rest of the document we assume that the ambient temperature (T_a) and the disturbances (n_i) are constant.

III. TEMPERATURE CONTROL USING REPLICATOR DYNAMICS

Our control objective is to maintain the temperature in each room close to a setpoint. It is well known that using a PI controller in each zone and without restriction of power in the actuators, the setpoints of each room can be reached without steady state error. In practice, in building temperature control, the power of control signals is not unlimited (real air conditioners and heaters can supply only a certain power). Moreover, it may have power limitations due to the requirement to save energy, which restrict the signal from the controller not to exceed a fixed value. In [11], it is shown that using independent PI controllers and considering the power constraints, it is not always possible to reach all the setpoints (even when the total power available is enough to this end). This adverse effect is due to the inefficient distribution of available power, and it can be solved by methods of dynamic resource allocation.

The replicator dynamics model [5] is the control strategy that we use to optimally allocate the available power. Replicator dynamics are a typical model of the population dynamics, they are based on an evolutionary game, where players can choose between N pure strategies. The populations of most successful players (compared with the average) tend to grow, while the least ones decline. The success of players who have chosen the i^{th} strategy is determined by a function f_i, which is a fitness function in behavioral ecology. Mathematically, replicator dynamics model is formulated as

$$\dot{x}_i = \beta x_i (f_i - \bar{f}), \qquad (6)$$

where x_i is the population playing the i^{th} strategy, $\bar{f} = \frac{1}{P}\sum_{j=1}^{N} x_j f_j$ is the average fitness, P is the total population ($P = \sum_{i=1}^{N} x_i$), and $\beta > 0$ is a parameter related to the population growth rate.

In order to apply the replicator dynamics model to the building temperature control, we relax the original assumptions such that the population playing the i^{th} strategy (i.e., x_i) is the power allocated to each room, P is the total power of the heating system, the strategies are the N rooms, and the fitness function is defined as the difference between the setpoint corresponding to the i^{th} zone and its current temperature plus a positive constant b, such that f_i is always positive. Then,

$$f_i = T_{si} - T_i + b, \text{ for } i = 1, \ldots, N. \qquad (7)$$

Note that f_i is greater when the temperature of the i^{th} room is further from its setpoint T_{si}, and viceversa. Moreover β can be viewed as a design parameter of the controller.

A. Limitation of Power using Replicator Dynamics

Given the chosen average fitness ($\bar{f}$), the replicator dynamics model possesses an important property: if $x(0) \in \Delta$, then $x(t) \in \Delta$ for all $t > 0$, where $x = [x_1, \ldots, x_N]^\top$, and

$\Delta = \{x : \sum_{i=1}^{N} x_i = P, x_i \geq 0\}$ [12]. This means that the power required is preserved over time.

Although, for building temperature control is not desirable that the power is maintained constant over time, it is required that the power used by the control strategy is limited. For this purpose, a “fictitious zone” $N+1$ is introduced in the model, which does not correspond to any room of the building. If we add the fictitious zone to the replicator dynamics model, it still satisfy the property $\sum_{i=1}^{N+1} x_i(t) = P$, where $x_{N+1}(t) \geq 0$ is the power allocated to the the fictitious zone, so it is a fictitious power. The real power (power allocated to the rooms of the building) is given by the expression $\sum_{i=1}^{N} x_i(t)$, which is clearly less than or equal to P. In conclusion, now we have the limitation of power required: $\sum_{i=1}^{N} x_i(t) \leq P, \quad x_i(t) \geq 0$.

The choice of the fitness function for the fictitious zone influences the achievement of the control objective, given that the replicator dynamics model tends to equalize the fitness function of all zones (an equilibrium point of replicator equation is reached when $f_i^* = \bar{f}^*$). Therefore, in the proposed controller, we choose the fitness function of the fictitious zone as $f_{N+1} = b$. In this way, if the temperatures of the rooms are bellow their respective setpoints, the corresponding fitness functions of the real zones are greater than b (according to Equation (7)) and therefore, the physical zones are more attractive to players (thermal power) than the fictitious zone. Instead, if the temperature of the rooms exceeds its corresponding setpoint, the fitness functions of the real zones become less than b. In this case, it is more profitable for players to choose the fictitious zone (whose fitness function remains at the constant value b). This fact causes that the excess of power is allocated to the fictitious zone, and thus, the desired temperature can be achieved in each room.

B. Equilibrium Points

In the thermal model of a building (Equation (4)) controlled via replicator dynamics (Equation (6), where $f_i = T_{si} - T_i + b$ for $i = 1, \ldots, N$, and $f_{N+1} = b$) an equilibrium point denoted by $(\mathbf{T}^*, \mathbf{x}^*)$, with $\mathbf{T}^* = [T_1^*, \ldots, T_{N+W}^*]^\top$ (W is the total number of layers within the walls that comprise the building), and $\mathbf{x}^* = [x_1^*, \ldots, x_{N+1}^*]^\top$, is achieved when in each room the temperature reaches its corresponding setpoint (i.e., $T_i^* = T_{si}$ for $i = 1, \ldots, N$). At this point, $f_i^* = f_j^* = \bar{f}^*$, where $\bar{f}^* = b$, which means that, in the steady state, each player in the game perceives the same benefit. At equilibrium, the temperatures in the layers of the walls (i.e., T_i^* for $i = N+1, \ldots, N+W$) are given by the solution of the linear equation $\mathbf{T}^w = -(\mathbf{A}_A^w)^{-1}(\mathbf{A}_B^w \mathbf{T}^r) + \mathbf{B}^w \mathbf{u}$, where $\mathbf{T}^w = [T_{N+1}^*, \ldots, T_{N+W}^*]^\top$, $\mathbf{T}^r = [T_1^*, \ldots, T_N^*]^\top$, $\mathbf{A}_A^w$ and $\mathbf{A}_B^w$ are submatrices formed, respectively, by the last W rows and columns, and by the last W rows and N first columns of the matrix $\mathbf{A}$ of the Equation (5), and $\mathbf{B}^w$ is a submatrix formed by the last W rows of matrix $\mathbf{B}$.

To complete the characterization of the equilibrium point of the system, it is necessary to establish the value of x_i^*. If $T_i^* = T_{si}$ for $i = 1, \ldots, N$, then from Equation (4), and because of $\sum_{i=1}^{N+1} x_i(t) = P$, it follows that

$$\begin{aligned} x_i^* &= -\textstyle\sum_{j \in \Omega_i} \alpha_{ij}(T_j^* - T_{si}) - \alpha_{ia}(T_a - T_{si}) - n_i \\ x_{N+1}^* &= P - \textstyle\sum_{i=1}^{N} x_i^*. \end{aligned} \tag{8}$$

The value of x_i^* for $i = 1, \ldots, N$ corresponds to the needed power from each actuator such that the rooms reach their respective setpoint. Moreover, the value of x_{N+1} is the excess of power which is allocated to the fictitious zone. According to the above if the available power is greater than or equal to the power required in steady state, the temperature in all rooms can reach the desired value.

C. Stability Analysis

We want to study the stability of the equilibrium point found in Section III-B using passivity theory for interconnected systems [8]. It can be established that the thermal system (Σ_1) and the system that corresponds to the replicator dynamics (Σ_2) are feedback interconnected as shown in Figure 1. Using Equations (4), (6), and (7), the total system

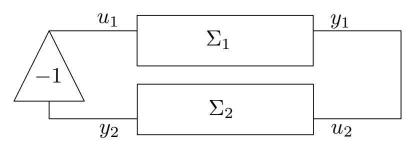

Fig. 1. Feedback interconnection of the thermal system and the controller.

is described by

$$\Sigma_1 : \begin{cases} \dot{e}_{T_i} = \displaystyle\sum_{j \in \Omega_i} \frac{\alpha_{ij}}{\theta_i}(e_{T_j} - e_{T_i}) - \frac{\alpha_{ia}}{\theta_i} e_{T_i} + \frac{s_i}{\theta_i} e_{x_i}, \\ \hfill \text{for } i = 1, \ldots, N+W \\ y_{1i} = e_{T_i}, \quad \text{for } i = 1, \ldots, N \end{cases}$$

$$\Sigma_2 : \begin{cases} \dot{e}_{x_i} = \beta(e_{x_i} + x_i^*) \left(-e_{T_i} + b + \displaystyle\sum_{j=1}^{N+1} \frac{e_{T_j} - b}{P}(e_{x_j} + x_j^*) \right) \\ \hfill \text{for } i = 1, \ldots, N+1 \\ y_{2i} = -e_{x_i}, \quad \text{for } i = 1, \ldots, N, \end{cases} \tag{9}$$

where $e_{T_i} = T_i - T_i^*$, and $e_{x_i} = x_i - x_i^*$ are the error coordinates for the thermal system and the controller, respectively, and $e_{T_{N+1}} = 0$ given that $f_{N+1} = b$.

Theorem 1: If $P \geq \sum_{i=1}^{N} x_i^*$ where $x_i^* \geq 0$ is given by Equation (8), the equilibrium point at the origin of the feedback interconnected system given by Equation (9) is asymptotically stable (AS).

Proof: In order to show that the origin of the feedback interconnected system is AS, first we need to prove that the multizone temperature system (Σ_1) with input $\mathbf{u_1} = [e_{x_1}, \ldots, e_{x_N}]^\top$ and output $\mathbf{y_1} = [e_{T_1}, \ldots, e_{T_N}]^\top$ is strictly passive. Then, we need to show that the replicator dynamics system (Σ_2) with input $\mathbf{u_2} = [e_{T_1}, \ldots, e_{T_N}]^\top$ and output $\mathbf{y_2} = [-e_{x_1}, \ldots, -e_{x_N}]^\top$ is lossless [8].

For Σ_1, we choose $V_1(e_T) = \frac{1}{2} \sum_{i=1}^{N+W} \theta_i e_{T_i}^2$ as a positive definite storage function. The derivative of $V_1(e_T)$ along the trajectories of Σ_1 is given by $\dot{V}_1 = \sum_{i=1}^{N+W} \sum_{j \in \Omega_i} \alpha_{ij}(e_{T_j} e_{T_i} - e_{T_i}^2) - \sum_{i=1}^{N+W} \alpha_{ia} e_{T_i}^2 + \sum_{i=1}^{N} e_{T_i} e_{x_i}$. In the thermal model of a building, if $j \in \Omega_i$ then $i \in \Omega_j$, so for each term $\alpha_{ij}(e_{T_j}e_{T_i} - e_{T_i}^2)$ there exists another one of the form $\alpha_{ji}(e_{T_i}e_{T_j} - e_{T_j}^2)$. Rewriting the equation, and given that $\alpha_{ij} = \alpha_{ji}$, we obtain

$$\dot{V}_1 = -\left(\sum_{i=1}^{N+W}\left(\alpha_{ia}e_{T_i}^2 + \frac{1}{2}\sum_{j\in\Omega_i}\alpha_{ij}(e_{T_i} - e_{T_j})^2\right)\right) + \sum_{i=1}^{N} e_{T_i}e_{x_i}$$

The expression in parentheses is a positive definite function $\psi(e_T)$ because for the walls that are in direct contact with the external environment, there is at least one $\alpha_{ia} \neq 0$. Moreover, $\sum_{i=1}^{N} e_{T_i}e_{x_i} = \mathbf{u}_1^\top \mathbf{y}_1$. In conclusion we have that $\dot{V}_1(e_T) = \mathbf{u}_1^\top \mathbf{y}_1 - \psi(e_T)$, and then the system Σ_1 described in Equation (9) is strictly passive.

For Σ_2, we choose a positive definite storage function (relative entropy function [13]), such as,

$$V_2(e_x) = -\frac{1}{\beta}\sum_{i=1}^{N+1} x_i^* \ln\left(\frac{e_{x_i} + x_i^*}{x_i^*}\right) \tag{10}$$

The derivative of $V_2(e_x)$ along the trajectories of Σ_2, is given by $V_2 = -\sum_{i=1}^{N+1} e_{T_i}e_{x_i} + b\sum_{i=1}^{N+1} e_{x_i}$. The last term in this expression is zero because $e_{x_i} = x_i - x_i^*$, and $\sum_{i=1}^{N+1} x_i = \sum_{i=1}^{N+1} x_i^* = P$ (according to the property of the replicator equation given in Section III-A). Then, for the system Σ_2, $\mathbf{u}_2^\top \mathbf{y}_2 = -\sum_{i=1}^{N} e_{T_i}e_{x_i}$. So we have that $\mathbf{u}_2^\top \mathbf{y}_2 = \dot{V}_2(e_x)$ given that $e_{T_{N+1}} = 0$, and then Σ_2 is lossless.

According to [8], if we have a strictly passive (Σ_1) subsystem and a lossless one (Σ_2), then the feedback interconnection of these subsystems (as shown in Figure 1) is passive and therefore its origin is stable. In order to prove that the origin of the feedback interconnection of Σ_1 and Σ_2 is asymptotically stable, we define $V_S = V_1 + V_2$ as a Lyapunov function candidate. The derivative of V_S along the trajectories of the system is $\dot{V}_S = \dot{V}_1 + \dot{V}_2$. But, we have that the subsystem Σ_1 is strictly passive, then $\dot{V}_1 = \mathbf{u}_1^\top \mathbf{y}_1 - \psi(e_T)$ with $\psi(e_T)$ positive definite. Also, the subsystem Σ_2 is lossless, then $\dot{V}_2 = \mathbf{u}_2^\top \mathbf{y}_2$. Therefore, $\dot{V}_S = -\psi(e_T)$. Since $\psi(e_T)$ is positive definite, $\dot{V}_S$ is negative semidefinite, given that $\dot{V}_S$ is also a function of e_x. Let us define the set $S = \{[e_T, e_x] : \dot{V}_S = 0\}$, then $S = \{[e_T, e_x] : e_T = 0\}$. But, from Σ_1 in Equation (9) $e_T \equiv 0$ if and only if $e_x = 0$. It follows that no solution can stay identically in S, other than the trivial solution, i.e., $[e_T, e_x] = 0$. Then, from the LaSalle's invariance principle, the origin of the feedback connected system is asymptotically stable (AS). ■

D. Introducing the Derivative of Error in the Fitness Function

Now, we relax the assumption that the fitness function is strictly positive, and consider the case of a fitness function that depends on the error and its derivative, that is

$$f_i = K_P e_{T_i} + K_D \dot{e}_{T_i}, \text{ for } i = 1, \ldots, N. \tag{11}$$

The inclusion of the derivative of error in the fitness function provides information to the controller about the rate of change of error, and therefore it allows to anticipate the appropriate control action (e.g., if the error is decreasing fast, it is necessary to weaken the control signal in order to avoid overshoots). For this case, we choose the fitness of the fictitious zone as $f_{N+1} = 0$.

For the stability analysis of the system when we use a fitness function as the one described in Equation (11), we assume that all elements that comprise the building have the same thermal capacity, i.e., $\theta_i = \theta$ for $i = 1, \ldots, N+W$ in Equation (4). Under this assumption, the building thermal system controlled via replicator dynamics can be viewed as the negative feedback interconnection of the following subsystems (described in error coordinates with respect to the equilibrium point $(\mathbf{T}^*, \mathbf{x}^*)$ given in Section III-B):

$$\Sigma_1 : \begin{cases} \dot{\mathbf{e}}_\mathbf{T} = \mathbf{A}\mathbf{e}_\mathbf{T} + \mathbf{B}\mathbf{e}_\mathbf{x} \\ \mathbf{y_1} = -(K_P\mathbf{I}' + K_D\mathbf{A}')\mathbf{e}_\mathbf{T} - K_D\mathbf{B}'\mathbf{e}_\mathbf{x} \end{cases}$$
$$\Sigma_2 : \begin{cases} \dot{e}_{x_i} = \beta(e_{x_i}+x_i^*)\left(f_i - \displaystyle\sum_{j=1}^{N+1}\frac{f_j}{P}(e_{x_j}+x_j^*)\right), \\ \qquad\qquad \text{for } i = 1, \ldots, N+1 \\ y_{2i} = -e_{x_i}, \quad \text{for } i = 1, \ldots, N, \end{cases} \tag{12}$$

where $\mathbf{e}_\mathbf{T} = [e_{T_1}, \ldots, e_{T_{N+W}}]^\top$, $\mathbf{e}_\mathbf{x} = [e_{x_1}, \ldots, e_{x_N}]^\top$, e_{T_i}, and e_{x_i} are the error coordinates for the thermal system and the control signal, respectively; $\mathbf{A}$ is the same as in Equation (5) but with $\theta_i = \theta$, for $i = 1, \ldots, N+W$. Therefore $\mathbf{A} = \mathbf{A}^\top < 0$ (in this particular case, $\mathbf{A}$ is negative definite because it is symmetric, and Hurwitz as we prove in Section III-C); $\mathbf{A}'$ is the submatrix formed by the N first rows of the matrix $\mathbf{A}$; $\mathbf{B} = [\frac{1}{\theta}\mathbf{I}_\mathrm{N}, \mathbf{0}_{\mathrm{N\times W}}]^\top$; $\mathbf{B}' = \frac{1}{\theta}\mathbf{I}_\mathrm{N}$; $\mathbf{I}' = [\mathbf{I}_\mathrm{N}, \mathbf{0}_{\mathrm{N\times W}}]$; $f_i = K_P e_{T_i} + K_D \dot{e}_{T_i}$, for $i = 1, \ldots, N$, and $f_{N+1} = 0$. $\mathbf{I}_\mathrm{N}$ denotes the $N \times N$ identity matrix. Note that $y_{1_i} = -f_i$ for $i = 1, \ldots, N$ (y_{1_i} is the i^{th} element of the vector $\mathbf{y_1}$).

Theorem 2: The equilibrium point at the origin of the feedback interconnected system given by Equation (12), with $\mathbf{A} = \mathbf{A}^\top < 0$, is asymptotically stable (AS) if $K_D \leq 0$, $K_P < \max_i\left\{\frac{\alpha_{ia}+2\sum_{j\in\Omega_i}\alpha_{ij}}{\theta}\right\}K_D$, and $P \geq \sum_{i=1}^{N} x_i^*$, where $x_i^* \geq 0$ is given by Equation (8).

Proof: In order to show that the origin of the feedback interconnected system is AS, first we need to prove that the multizone temperature system (Σ_1) with input $\mathbf{u_1} = [e_{x_1}, \ldots, e_{x_N}]^\top$ and output $\mathbf{y_1} = [-f_1, \ldots, -f_N]^\top$ is strictly passive. Then, we need to show that the replicator dynamics system (Σ_2) with input $\mathbf{u_2} = [-f_1, \ldots, -f_N]^\top$ and output $\mathbf{y_2} = [-e_{x_1}, \ldots, -e_{x_N}]^\top$ is lossless.

For Σ_1, we choose the positive definite storage function $V_1(e_T) = \frac{\theta}{2}\mathbf{e}_\mathbf{T}^\top(-K_P\mathbf{I}_{\mathrm{N+W}} - K_D\mathbf{A})\mathbf{e}_\mathbf{T}$, where $(-K_P\mathbf{I}_{\mathrm{N+W}} - K_D\mathbf{A})$ is a positive definite matrix if $K_D \leq 0$, and $K_P < \max_i\left\{\frac{\alpha_{ia}+2\sum_{j\in\Omega_i}\alpha_{ij}}{\theta}\right\}K_D$. This can be seen from the fact that, if these conditions hold,

$$\mathbf{e}_\mathbf{T}^\top(-K_P\mathbf{I}_{\mathrm{N+W}} - K_D\mathbf{A})\mathbf{e}_\mathbf{T} = \sum_{i=1}^{N+W}\sum_{j\in\Omega_i}\left(\frac{-K_D}{2\theta}\alpha_{ij}(e_{T_i} + e_{T_j})^2\right) + \\ + \sum_{i=1}^{N+W}\left(K_D\left(\frac{\alpha_{ia} + 2\sum_{j\in\Omega_i}\alpha_{ij}}{\theta}\right) - K_P\right)e_{T_i}^2$$

is positive when $\mathbf{e}_\mathbf{T} \neq 0$.

The derivative of $V_1(e_T)$ along the trajectories of Σ_1 is given by

$$\dot{V}_1 = \theta \mathbf{e_x}^\top \mathbf{B}^\top(-K_P\mathbf{I}_{\text{N+w}}-K_D\mathbf{A})\mathbf{e_T} - \theta \mathbf{e_T}^\top \mathbf{A}(K_P\mathbf{I}_{\text{N+w}}+K_D\mathbf{A})\mathbf{e_T}.$$

The rightmost term is a positive definite function $\psi_1(e_T)$ because of $\mathbf{A}(K_P\mathbf{I}_{\text{N+w}}+K_D\mathbf{A})$ is positive definite, and for the leftmost one we have $\theta\mathbf{e_x}^\top\mathbf{B}^\top(-K_P\mathbf{I}_{\text{N+w}}-K_D\mathbf{A})\mathbf{e_T} = \mathbf{e_x}^\top(-K_P\mathbf{I}'-K_D\mathbf{A}')\mathbf{e_T}$, so $\dot{V}_1 = \mathbf{e_x}^\top(-K_P\mathbf{I}'-K_D\mathbf{A}')\mathbf{e_T} - \psi_1(\mathbf{e_T})$. Moreover $\mathbf{u_1}^\top\mathbf{y_1} = \mathbf{e_x}^\top(-K_P\mathbf{I}'-K_D\mathbf{A}')\mathbf{e_T} - K_D\mathbf{e_x}^\top\mathbf{B}'\mathbf{e_x}$ with $K_D \leq 0$. Then $\dot{V}_1 \leq \mathbf{u_1}^\top\mathbf{y_1} - \psi_1(e_T)$, and Σ_1 is strictly passive. Let us remark that $\mathbf{u_1}^\top\mathbf{y_1} = -\sum_{i=1}^N f_i e_i$.

Now, let us prove that the controller based on replicator dynamics (Σ_2) is lossless if we take its input and output like $\mathbf{u_2} = [-f_1,\ldots,-f_N]$, and $\mathbf{y_2} = [-e_{x_1},\ldots,-e_{x_N}]^\top$, respectively. For do that we use the storage function defined in Equation (10). The derivative of V_2 along the trajectories of Σ_2 is $\dot{V}_2 = \sum_{i=1}^N f_i e_{x_i}$. Because of $\mathbf{u_2}^\top\mathbf{y_2} = \sum_{i=1}^N f_i e_{x_i}$, so we have that $\mathbf{u_2}^\top\mathbf{y_2} = \dot{V}_2$, and then Σ_2 is passive.

The feedback interconection shown in Figure 1 of a strictly passive subsystem (Σ_1) with a lossless subsystem (Σ_2) is passive, and its origin is stable. In order to proof that the origin is AS, we can use the LaSalle's invariance principle in the same way as in Section III-C. ■

We have shown that with the fitness function given in Equation (11), which includes the derivative of the error, it is possible to achieve the control objective by choosing appropriate K_P and K_D parameters.

E. System Performance to Strong Power Limitation

When the power available is strongly limited (i.e., when $P < \sum_{i=1}^N x_i^*$, where x_i^* is defined as in Equation (8)), it is not possible to achieve the temperature setpoints of the rooms. If that happens, it is important that all occupants of the building's rooms have a similar welfare. For this reason, some authors suggest to use the variance of the error between the temperatures of the rooms and their respective setpoints as an index that measures the efficiency of the control algorithm used [3]. The smaller the variance, the better the performance of the algorithm.

When the controller based on replicator dynamics is used with the restriction described above, another asymptotically stable equilibrium point emerges. The main characteristics of this equilibrium point can be expressed as follows: *i)* all the available power is allocated to the real zones (i.e., $x_{N+1}^* = 0$); and *ii)* the fitness functions corresponding to each room have the same value (i.e., $f_i^* = f_j^*$ for $i,j = 1,\ldots,N$). Therefore, if we use a fitness function as the one defined in Equation (7) (that corresponds to the error between the temperature of the room and its setpoint), at equilibrium, all errors would be equal and the variance is reduced to zero. In order to prove that an equilibrium point satisfying these characteristics is asymptotically stable, we proceed in the same way as in Section III-C. The only difference is that now, the replicator dynamics system (Σ_2) in the new error coordinates is expressed as follows

$$\begin{aligned} \dot{e}_{x_i} &= \beta(e_{x_i}+x_i^*)\left(-e_{T_i}+E_i+b+\sum_{j=1}^{N+1}\frac{e_{T_j}-E_j-b}{P}(e_{x_j}+x_j^*)\right) \\ &\qquad \text{for } i=1,\ldots,N+1 \\ y_{2i} &= -e_{x_i}, \quad \text{for } i=1,\ldots,N, \end{aligned} \tag{13}$$

where $E_i = T_{si} - T_i^*$ for $i=1,\ldots,N$, and $E_{N+1} = 0$

However, the proof uses the same ideas as before because this system is also passive. In order to show that, we use the storage function defined in Equation (10). The derivative of this storage function along the trajectories of the system Σ_2 described by the Equation (13) is given by

$$\dot{V}_2 = -\sum_{i=1}^{N+1} x_i^*\left(-e_{T_i}+E_i+b+\frac{1}{P}\sum_{j=1}^{N+1}(e_{T_j}-E_j-b)(e_{x_j}+x_j^*)\right),$$

but E_i for $i=1,\ldots,N$ is a constant, because $f_i^* = f_j^*$, so replacing $e_{x_i} = x_i - x_i^*$, we have

$$\dot{V}_2 = -\sum_{i=1}^{N+1} e_{T_i}e_{x_i} + E\left(\underbrace{\sum_{i=1}^N x_i}_{\leq P} - \underbrace{\sum_{i=1}^N x_i^*}_{P}\right) + b\underbrace{\sum_{i=1}^{N+1} e_{x_i}}_{0}.$$

It is natural to suppose that E is positive because, due to the lack of power, the steady state temperature in each room is below its respective setpoint. Moreover, as the input to the system Σ_2 is $\mathbf{u}_2 = [e_{T_1},\ldots,e_{T_N}]^\top$ and its output is $\mathbf{y}_2 = [-e_{x_1},\ldots,-e_{x_N}]^\top$, we have that $\dot{V}_2(e_x) \leq \mathbf{u}_2^\top\mathbf{y}_2$ given that $e_{T_{N+1}} = 0$, and then Σ_2 is passive. So, we have proven that using a building temperature controller based on replicator dynamics, the variance of the error between the temperature in each room and its respective setpoint is reduced to a value of zero.

IV. SIMULATION RESULTS AND COMPARISON

We simulate the response of a building thermal system when the controller based on replicator dynamics is applied with the fitness function described in Equation (11). The simulation conditions are described as follows: the building consists of four rooms which are arranged in a row. Moreover, we have a heating system of 8 kW, which will provide thermal power for the four rooms. The ambient temperature is 10$^{\text{o}}$C, and the setpoints for each zone are respectively 23$^{\text{o}}$, 22$^{\text{o}}$, 21$^{\text{o}}$ and 20$^{\text{o}}$C. The results based on these conditions are presented in Figure 2.*a*. Notice that when the replicator dynamics are used, it is possible to reach the reference temperature in all rooms, even with power restrictions. In this simulation a disturbance is introduced at $t = 117$ minutes, which corresponds to a heat loss (e.g., by opening a window). It can be noticed that the system responds to the disturbance and achieves a satisfactory recovery.

In order to compare the performance of the control strategy proposed in this paper, we study the response of the system when a model predictive control (MPC) [14] is applied (this technique is widely used in the literature for building

TABLE I

PERFORMANCE INDICES FOR EACH CONTROL TECHNIQUE

Control Technique	ISE ($\times 10^5$)	ITSE ($\times 10^7$)	Energy Spent (kWh)
Replicator dynamics	1.046	1.772	22.41
MPC	1.178	2.080	22.92

temperature control). For tuning the controller, we seek to reduce the settling time, and the overshoot of the closed loop system response. Moreover, we consider the power constraint. The simulation conditions are the same, and the response of the controlled system is presented in Figure 2.*b*. It may be noted that using this control technique, the temperature in each room reaches its corresponding setpoint with a similar settling time. Moreover, the control signal satisfies the established restriction (i.e., $P \leq 8$kW). The performance of the controllers can be quantified by means of the indices ISE (integral of squared error), ITSE (integral of time multiply squared error), and the energy used by the actuators. Table I summarizes the values of these indices when using each control technique. We remark that the replicator dynamics based controller has the best ISE and ITSE indices, and at the same time, it consumes less energy than the MPC controller.

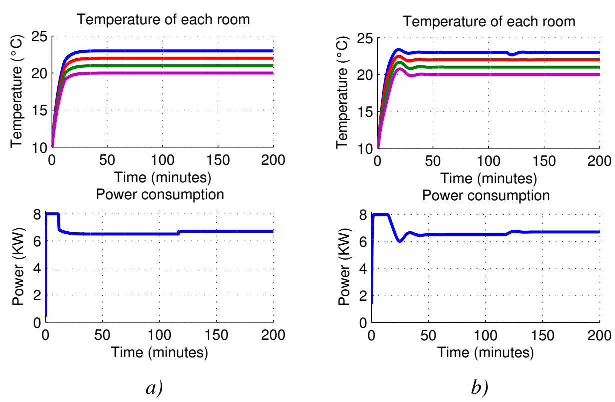

Fig. 2. Response of the system and power required. *a)* controlled via replicator dynamics, $f_i = K_P e + K_D \dot{e}$. *b)* controlled via MPC.

V. DISCUSSION

Due to the limitations of power, building temperature control cannot be approached from conventional control techniques. To achieve the control goals, it is necessary to use optimal resource allocation techniques such as replicator dynamics.

Using replicator dynamics (Figure 2.*a*) the control objective is achieved, and it is also important to emphasize the robustness of this strategy against disturbances, in addition to the simplicity of the model and the possible easy implementation. The replicator dynamics model is formulated through a set of first order ordinary differential equations, which implies that its implementation does not require, in principle, the use of advanced processors. In this aspect, this control technique has an important advantage compared to other resource allocation methods (e.g., model predictive control (MPC), which has to perform an optimization process at each iteration). It is also important to note that the algorithm based on replicator dynamics is easy to extend to complex cases (e.g., for temperature control in large buildings with a lot of rooms).

VI. CONCLUSION

The building temperature control is a dynamic resource allocation problem, since the heaters and air conditioner systems have limited power. For this reason, the use of conventional controllers does not provide satisfactory results. Many of the dynamic resource allocation techniques are heuristics, and thus they lack a proper mathematical basis to ensure stability or achieving the control objective. Contrary to this, the replicator dynamics model proposed in this work, allows us to use some nonlinear mathematical tools to show that under some conditions an asymptotically stable equilibrium point is reached. Moreover, the simulations show that the resource allocation method implemented has a good noise rejection and its performance is similar to the one obtained using another advanced control technique. This paper does not consider the dynamics of the actuators, the presence of noise in the measurements provided by sensors, or the delay in the information transfer used by the controller. As future work we propose to study the robustness of the replicator dynamics algorithm under these scenarios.

REFERENCES

[1] J. Kleissl and Y. Agarwal, "Cyber-physical energy systems: Focus on smart buildings," in *2010 47th ACM/IEEE Design Automation Conference (DAC)*, pp. 749–754, 2010.

[2] N. Gershenfeld, S. Samouhos, and B. Nordman, "Intelligent infrastructure for energy efficiency," *Science*, vol. 327, no. 5969, pp. 1086–1088, 2010.

[3] F. Ygge and H. Akkermans, "Decentralized markets versus central control: A comparative study," *Journal of Artificial Intelligence Research*, vol. 11, pp. 301–333, 1999.

[4] S. Clearwater, R. Costanza, M. Dixon, and B. Schroeder, "Saving energy using market-based control," *Market-Based Control: A Paradigm for Distributed Resource Allocation*, pp. 253–273, 1996.

[5] P. Taylor and L. Jonker, "Evolutionarily stable strategies and game dynamics," *Mathematical Biosciences*, vol. 40, no. 2, pp. 145–56, 1978.

[6] A. Pantoja, N. Quijano, and S. Leirens, "A bioinspired approach for a multizone temperature control system," *Bioinspiration & Biomimetics*, vol. 6, no. 1, p. 016007, 2011.

[7] E. Ramirez-Llanos and N. Quijano, "A population dynamics approach for the water distribution problem," *International Journal of Control*, vol. 83, no. 9, pp. 1947–1964, 2010.

[8] H. Khalil, *Nonlinear systems*. Pearson Educational, 2001.

[9] J. Willems, "Dissipative dynamical systems part I: General theory," *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 321–351, 1972.

[10] N. Mendes, G. Oliveira, H. Araújo, and L. Coelho, "A Matlab-based simulation tool for building thermal performance analysis," in *Proc. of the Eighth International Conference on Building Performance Simulation, Eindhoven The Netherlands*, pp. 855–862, 2003.

[11] G. Obando, A. Pantoja, and N. Quijano, "Evolutionary game theory applied to building temperature control," *Proceedings of the Nolcos 2010*, pp. 1140–1145, 2010.

[12] J. Hofbauer and K. Sigmund, "The theory of evolution and dynamical systems. Mathematical aspects of selection.," *Cambridge University Press, New York, NY(USA).*, 1988.

[13] J. Weibull, *Evolutionary game theory*. The MIT press, 1997.

[14] J. Rossiter, *Model-based predictive control: a practical approach*. CRC Press, 2003.