Dimension Reduction Near Periodic Orbits of Hybrid Systems

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Abstract— **When the Poincaré map associated with a periodic
orbit of a hybrid dynamical system has constant-rank iter-
ates, we demonstrate the existence of a constant-dimensional
invariant subsystem near the orbit which attracts all nearby
trajectories in finite time. This result shows that the long-
term behavior of a hybrid model with a large number of
degrees-of-freedom may be governed by a low-dimensional
smooth dynamical system. The appearance of such simplified
models enables the translation of analytical tools from smooth
systems—such as Floquet theory—to the hybrid setting and
provides a bridge between the efforts of biologists and engineers
studying legged locomotion.**

I. INTRODUCTION

Dynamic multi-legged locomotion presents a daunting
control task. A large number of degrees-of-freedom (DOF)
must be rapidly and precisely coordinated in the face of
state and environmental uncertainty. The ability of indi-
vidual limbs to exert forces on the body varies intermit-
tently with ground contact, body posture, and the efforts of
other appendages. Finally, the motion itself affects sensor
measurements, complicating pose estimation. In spite of
these difficulties, animals at all levels of complexity have
mastered the art of rapid legged locomotion over complex
terrain at speeds far exceeding those of comparable robotic
platforms [1], [2], [3], [4].

Numerous architectures have been proposed to explain
how animals control their limbs. For steady-state locomotion,
most posit a principle of coordination, synergy, symmetry or
synchronization, and there is a surfeit of neurophysiological
data to support these hypotheses [5], [6], [7], [8]. In effect,
the large number of DOF available to an animal are collapsed
during regular motion to a low-dimensional dynamical at-
tractor that may be captured by a *template* model embedded
within a higher-dimensional model *anchored* to the animal's
morphology [9], [10]. In this view, only a few parameters like
frequency and coupling strength are required to describe the
dynamics of any particular periodic gait over a broad range
of animal morphologies, offering a tantalizing target for ex-
perimental biologists. Were the dynamics of legged animals
smooth as a function of position and momentum, Floquet
theory [11] provides a canonical form for the structure of the
stability basin of a limit cycle [12], [13]. In such a canonical
form, the template may appear as an invariant attractor of
the linearized dynamics and be amenable to quantitative

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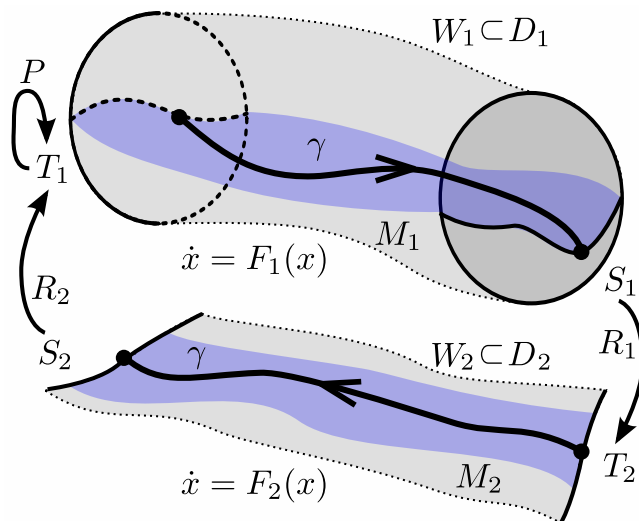`shrevzen@seas.upenn.edu`

Fig. 1. Illustration of the main result in a two-domain system: Whenever
iterates of the Poincaré map P associated with a periodic orbit γ of a hybrid
dynamical system are constant-rank, there is a constant-dimensional smooth
submanifold M_j in each domain D_j which is invariant under the flow of
the vector field F_j and which attracts all trajectories starting in an open set
$W_j \subset D_j$ containing γ in finite time.

measurement [14], [15]. A substantial motivation for the
present work has been to provide a theoretical framework for
applying this empirical approach to study legged locomotion.
The dynamics of legged locomotion are rarely smooth due
to intermittent contact of limbs with the substrate, so we
have generalized this approach to be aplicable to a class of
non-smooth systems called *hybrid dynamical systems*.

We relegate a formal definition of the class of hybrid
systems under consideration to Section III. Informally, hybrid
dynamical systems are comprised of differential equations
written over disparate domains together with rules for switch-
ing between the domains. Of particular interest to us are
periodic orbits of such systems. From a modeling viewpoint,
a stable hybrid periodic orbit provides a natural abstraction
for the dynamics of steady-state legged locomotion. This
approach has been widely adopted, generating a variety of
models of bipedal [16], [17], [18] and multi-legged [19], [20]
locomotion as well as some general control-theoretic tech-
niques for composition [21], coordination [22], and stabiliza-
tion [23], [24], [25] of such models. In certain cases, it has
been possible to formally embed a low-dimensional abstrac-
tion in a higher-dimensional physically-realistic model [26],
[27].

This paper provides a conceptual link between formal
analysis of hybrid periodic orbits and the dramatic dimension

reduction observed empirically in successful legged loco-motors. Under the condition that iterates of the Poincaré map associated with a periodic orbit are constant rank, we demonstrate the existence of a constant-dimensional invariant subsystem which attracts all nearby trajectories in finite time. Analogous results for smooth dynamical systems typically impose stringent assumptions on the dynamics such as exact symmetries (cf. §8.9 in [28]) or timescale separation (cf. Chapter 4 in [13]). In contrast, the results of this paper imply that hybrid dynamical systems may exhibit dimension reduction near periodic orbits solely due to the interaction of the switching dynamics with the smooth flow.

Organization

The hybrid systems we consider are constructed using switching maps defined between boundaries of smooth dynamical systems. The behavior of such systems can be studied by alternately applying flows and maps. Thus, we begin in Section II by developing several results which provide canonical forms for the behavior of flows and maps near periodic orbits and fixed points, respectively. Then, we define hybrid systems in Section III and use these results to characterize the dynamics near their periodic orbits. Examples are presented in Section IV and implications of the results for the design and analysis of legged locomotors are explored in Section V. Due to space constraints, proofs of the results are relegated to an associated preprint [29].

II. SMOOTH DYNAMICAL SYSTEMS

This section contains three technical results used in the proof of the Theorem of Section III. The first two results concern smooth dynamical systems[1] and may be found in textbooks. The third establishes, under a non-degeneracy condition, a canonical form for the invariant set of a smooth map near a fixed point. A reader principally interested in the main result of this paper may proceed to Section III and refer back to this section as needed.

We assume familiarity with the tools and terminology of differential geometry. If any of the concepts we discuss are unfamiliar, we refer the reader to [28], [30] for more details, and to [29] for a terse overview of the notational conventions.

Definition 1. *A* smooth dynamical system *is a pair* (M, G):
  M is a smooth manifold with boundary ∂M;
  G is a smooth vector field on M, i.e. $G \in \mathcal{T}(M)$.

A. Flows Between Surfaces

We review the fact that the flow near a trajectory pass-ing transversally between two surfaces has a simple form (cf. Chapter 11.2 in [31]). In particular, nearby trajectories can be obtained from an embedding of a product manifold. This will be the prototype for the dynamics near a periodic orbit in one domain of a hybrid system.

Lemma 1. *Let (M, G) be a smooth dynamical system and $\phi : \mathcal{F} \to M$ its maximal flow. Suppose $T, S \subset M$ are smooth embedded submanifolds, S has codimension 1, $\phi(\alpha, \xi) \in S$ for some $\alpha > 0$ and $\xi \in T$ where G is transverse to T at ξ and S at $\phi(\alpha, \xi)$. Then we have the following consequences:*

(i) *there is a neighborhood $U \subset M$ containing ξ and a smooth map $\eta : U \to \mathbb{R}$ so that $\eta(\xi) = \alpha$ and for all $x \in U$, $\eta(x) > 0$ and $\phi(\eta(x), x) \in S$; η is called the* time-to-impact *map;*

(ii) *with $V := U \cap T$, the map $\psi : [0, 1] \times V \to M$ with*

$$\psi(\sigma, v) := \phi(\eta(v)\sigma, v)$$

is a smooth embedding into M whose image contains the trajectory $\gamma = \{\phi(t, \xi) : 0 \leq t \leq \alpha\}$.

Remark 1. *This lemma is applicable when $T, S \subset \partial M$, which will be relevant in the study of hybrid systems.*

B. Gluing Flows

In this section, we provide a method for gluing two smooth dynamical systems together along their boundaries to obtain a new smooth system; this construction uses basic results from differential topology (cf. Theorem 8.2.1 in [32]). We will use this construction in Section III to attach distinct hybrid domains to one another.

Lemma 2. *Suppose $(M_1, G_1), (M_2, G_2)$ are smooth n-dimensional dynamical systems, $\varphi : \partial M_1 \to \partial M_2$ is a diffeomorphism, G_1 is outward-pointing along ∂M_1 and G_2 is inward-pointing along ∂M_2. Then the topological quotient $M := \frac{M_1 \coprod M_2}{\partial M_1 \simeq \partial M_2}$ can be made into a smooth manifold for which (i) the inclusions $M_j \hookrightarrow M$ are smooth embeddings and (ii) there is a smooth vector field $G \in \mathcal{T}(M)$ that restricts to G_j on M_j, $j = 1, 2$.*

Remark 2. *The smooth structure described in Lemma 2 is unique. Further, if $\tilde{G}_1$ and $\tilde{G}_2$ are any other smooth vector fields on M_1 and M_2 which satisfy the hypotheses of the Lemma, the corresponding quotient $\tilde{M}$ is diffeomorphic to M (cf. Chaper 8 in [32]).*

C. Invariant Set of a Smooth Map Near a Fixed Point

In studying hybrid dynamical systems, we encounter smooth maps $f : M \to M$ which are not diffeomorphisms. Viewing iteration of f as a discrete dynamical system, we wish to study the behavior of these iterates near a fixed point $f(\xi) = \xi$. Note that if f has constant rank equal to $k \in \mathbb{N}$, then its image $f(M) \subset M$ is an embedded k-dimensional submanifold near ξ by the Rank Theorem (cf. Theorem 7.13 in [30]). With an eye toward dimension reduction, one might hope that the composition $(f \circ f) : M \to f(M)$ is also constant-rank, but this is not generally true[2]. If it is true that iterates of f are eventually constant-rank near the fixed point ξ, then one can study the behavior of these iterates by restricting the domain to a lower-dimensional submanifold.

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[1]For notational convenience, we work with objects which possess con-tinuous derivatives of all orders. However, the results in this paper are valid if we only assume continuous differentiability.

[2]Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) := (x^2, x)$.

Lemma 3. *Let M be a smooth manifold, $f : M \to M$ a smooth map with $f(\xi) = \xi$ for some $\xi \in M$, suppose the rank of f is bounded above by $n \in \mathbb{N}$, and suppose the composition of f with itself n times, f^n, has constant rank equal to $r \in \mathbb{N}$ on a neighborhood of ξ. Then $f^n(M)$ is an r-dimensional embedded submanifold near ξ and there are neighborhoods $U, V \subset f^n(M)$ containing ξ for which f maps U diffeomorphically onto V.*

III. HYBRID DYNAMICAL SYSTEMS

We describe a class of hybrid systems useful for modeling legged locomotion, then restrict our attention to the behavior of such systems near periodic orbits. It was shown in [33] that the Poincaré map of a hybrid system is generally not full rank. We explore the geometric consequences of this rank loss and demonstrate, under a non-degeneracy condition, the existence of a smooth invariant subsystem which attracts all nearby trajectories in finite time.

A. Hybrid Differential Geometry

For our purposes, it is expedient to define hybrid dynamical systems over disjoint unions of smooth manifolds.

Definition 2. *A* smooth hybrid manifold *is a finite disjoint union of connected smooth manifolds $M = \coprod_{j \in J} M_j$.*

Remark 3. *The dimensions of the constituent manifolds are not required to be equal.*

Differential geometric constructions which are confined to a single manifold have natural generalizations to such spaces, and we will prepend the modifier "hybrid" to make it clear when this generalization is being invoked. For instance, the *hybrid tangent bundle TM* is the disjoint union of the tangent bundles TM_j, the *hybrid boundary ∂M* is the disjoint union of the boundaries ∂M_j, and a *hybrid open set $U \subset M$* is obtained from a disjoint union of open sets $U_j \subset M_j$. Generalizing maps between manifolds requires more care, hence we provide explicit definitions.

Assumption 1. *To simplify the exposition, we henceforth assume all manifolds and maps between manifolds are smooth.*

Definition 3. *A* hybrid map

$$f : \coprod_{j \in J} M_j \to \coprod_{\ell \in L} N_\ell$$

between hybrid manifolds restricts to a map $f|_{M_j} : M_j \to N_\ell$, some $\ell \in L$, for each $j \in J$. The hybrid map is called constant-rank, injective, *or* surjective *if each $f|_{M_j}$ is as well. It is called an* embedding *if each $f|_{M_j}$ is an embedding and f is a homeomorphism onto its image.*

Definition 4. *The* hybrid pushforward $f_* : TM \to TN$ *is the hybrid map defined piecewise as $f_*|_{TM_j} := (f|_{M_j})_*$.*

Definition 5. *A* hybrid vector field *on a hybrid manifold $M := \coprod_{j \in J} M_j$ is a hybrid map $G : M \to TM$ for which $G|_{M_j}$ is a vector field on M_j, i.e. $G|_{M_j} \in \mathcal{T}(M_j)$. We let $\mathcal{T}(M)$ denote the space of hybrid vector fields on M.*

To state the main result of this paper, we need to embed manifolds into hybrid manifolds. This can be achieved by first partitioning the smooth manifold to obtain a hybrid manifold, then embedding this hybrid manifold via the previous definitions.

Definition 6. *A* partition *of an n-dimensional manifold M is a finite set $\{M_j\}_{j \in J}$ of embedded n-dimensional submanifolds $M_j \subset M$ for which $\bigcup_{j \in J} M_j = M$ and if $i \neq j$ we have $\mathrm{Int}(M_j) \cap \mathrm{Int}(M_i) = \emptyset$.*

Definition 7. *A* hybrid embedding *of a manifold M into a hybrid manifold $N := \coprod_{j \in J} N_j$ is determined by a partition $\{M_j\}_{j \in J}$ of M and a hybrid embedding*

$$f : \coprod_{j \in J} M_j \to \coprod_{j \in J} N_j$$

for which $f_j : M_j \to N_j$ for each $j \in J$. Any $G \in \mathcal{T}(M)$ may be pushed forward to a unique $f_* G \in \mathcal{T}(f(M))$. The image of f is a hybrid embedded submanifold.

With these preliminaries established, we can define the class of hybrid systems considered in this paper.

Definition 8. *A* hybrid dynamical system *is specified by a triple $H := (D, F, R)$ where:*
  $D = \coprod_{j \in J} D_j$ *is a hybrid manifold;*
  $F \in \mathcal{T}(D)$ *is a hybrid vector field on D;*
  $R : S \to T$ *is a hybrid map, $S, T \subset \partial D$ are hybrid embedded submanifolds, and S has codimension 1.*
As in [34], we call R the reset map *and S the* guard.

Note that if F is tangent to S at $x \in D$, there is a possible ambiguity in determining a trajectory from x—one may either follow the flow of F on D or apply the reset map to obtain a new initial condition $y = R(x)$.

Assumption 2. *To ensure that trajectories are uniquely defined, we assume that F is outward-pointing on S.*

To the best of our knowledge, this definition of a hybrid dynamical system has not appeared before. However, in light of the constructions contained in this section, it may be seen as a mild generalization of a *simple hybrid system* (cf. §3.2 in [34]). Further, in Section III-C we will see that this definition supports powerful geometric analysis of the dynamics near a hybrid periodic orbit. Finally, this class of hybrid systems encompasses many closed-loop models of legged locomotion [16], [17], [18], [19], [20], [23], [24], [26], [27]. We contend that these facts justify the introduction of the novel definition.

Remark 4. *As defined above, hybrid dynamical systems possess unique executions or trajectories from every initial condition. This fact can be demonstrated algorithmically. For any $x \in D_j$, obtain the maximal integral curve of $F|_{D_j}$. This integral curve must either: a) continue for all time; b) exit D_j without intersecting ∂D_j (in which case execution terminates); or c) intersect the boundary at $y \in \partial D_j$. If $y \in S$, the map R is applied to obtain a new initial condition $R(y) \in T$, and otherwise execution terminates.*

The following definition enables us to embed smooth dynamical systems into hybrid dynamical systems in such a way that trajectories of the smooth system are preserved in the hybrid system. We illustrate the use of this construction by giving a terse description of trajectories for this class of hybrid systems. In the subsequent sections, we use this construction to state the main results of this paper.

Definition 9. *A* hybrid dynamical embedding *of a dynamical system* (M, G) *into a hybrid dynamical system* (D, F, R) *is a hybrid embedding* $f : M \to D$ *for which* $f_* G = F|_{f(M)}$ *and* $R|_{f(M) \cap S}$ *is a hybrid diffeomorphism from* $f(M) \cap S$ *onto* $f(M) \cap T$.

Remark 5. *A trajectory of a hybrid dynamical system* H *may be obtained from a hybrid dynamical embedding of the system* $\left(J, \frac{\partial}{\partial t}\right)$, *where* $J \subset \mathbb{R}$ *is a connected interval.*

Definition 10. *A* τ-periodic orbit *of a hybrid dynamical system is a hybrid dynamical embedding* γ *of the dynamical system* $(S^1, \frac{2\pi}{\tau} \frac{\partial}{\partial \varphi})$, *where* S^1 *is the unit circle.*

Remark 6. *We alternately refer to* γ *as a* periodic trajectory *and often write* γ *in place of the image* $\gamma(S^1)$.

B. Hybrid Poincaré map

To state the main result of this paper, we must construct the Poincaré map associated with a periodic orbit of a hybrid system. This has been developed before [24], [33]; the construction is more delicate than for smooth systems since trajectories of hybrid systems do not necessarily vary continuously with initial conditions. We directly demonstrate this continuous dependence in the construction of the map.

Let $H = (D, F, R)$ be a hybrid dynamical system and γ a periodic orbit of H with period τ. Then γ undergoes a finite number of transitions $k \in \mathbb{N}$, so we may index the corresponding sequence of domains as[3] $D_1, \ldots, D_k$. Without loss of generality, assume the D_j's are distinct[4]; let $\gamma_j := \gamma \cap T_j$ be the entry point of γ in D_j and let τ_j be the time spent by γ in D_j. We wish to construct the Poincaré map P associated with γ over a neighborhood of γ_j in T. To do this, we must ensure that each initial condition in that neighborhood has a well-defined non-zero first-return time to T; the following assumption guarantees this.

Assumption 3. *To ensure the Poincaré map is well-defined, we assume* F *is transverse to* T *and not outward-pointing.*

Now for $j = 1, \ldots, k$ and referring to Fig. 2 for an illustration of these objects, let:

- $\phi_j : \mathcal{F}_j \to D_j$ be the maximal flow of F on D_j;
- $T_j \subset T \cap D_j$ be a neighborhood of γ_j over which Lemma 1 may be applied between T and S on D_j;
- $\psi_j : [0, 1] \times T_j \to D_j$ be the embedding from Lemma 1;
- $S_j := \psi_j(1, T_j) \subset S \cap D_j$ be the image of T_j in S under the flow on D_j;

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[3] We regard subscripts modulo k so that $D_k \equiv D_0$.

[4] Otherwise we can find $\{B_j\}_{j=1}^k$ be such that $B_j \subset D_j$ is open, $\gamma \subset \bigcup_{j=1}^k B_j$, and $B_i \cap B_j = \emptyset$ if $i \neq j$, then proceed on $D := \coprod_{j=1}^k B_j$.

$R_j \ : S_j \to T$ denote the restriction $R_j := R|_{S_j}$;
$p_j \ : T_j \to T$ be defined by $p_j(u) := R_j(\psi_j(1, u))$.

The Poincaré map over the section T_j is obtained formally by iterating the p's around the cycle:

$$P_j := p_{j-1} \circ \cdots \circ p_1 \circ p_k \circ \cdots \circ p_j. \tag{1}$$

The neighborhood $\Sigma_j \subset T_j$ of γ_j over which this map is well-defined is determined by pulling T_j backward around the cycle,

$$\Sigma_j = \left(p_j^{-1} \circ \cdots \circ p_k^{-1} \circ p_1^{-1} \circ \cdots \circ p_{j-1}^{-1} \right) (T_j),$$
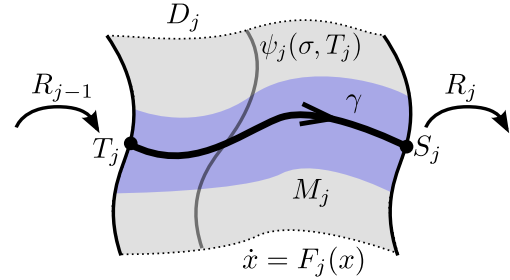
and similarly for any iterate of P_j.

Fig. 2. Illustration of a hybrid domain containing part of a periodic orbit. The periodic orbit γ enters the domain D_j inside the submanifold $T_j \subset \partial D_j$ via the reset map R_{j-1}. Initial conditions in T_j flow to the submanifold $S_j \subset \partial D_j$ via the vector field F_j. The map ψ_j smoothly reparameterizes these trajectories by the time required to pass from T_j to S_j, thus $\psi_j(\sigma, T_j)$ is an embedded submanifold for all $\sigma \in [0, 1]$ and $S_j = \psi_j(1, T_j)$. While in domain D_j, γ lies in the invariant submanifold M_j constructed in Theorem 1. By construction, M_j is an integral submanifold of F_j and $\dim M_j \leq \dim D_j$; see Fig. 1 for an illustration when $\dim M_j < \dim D_j$.

It is a standard result for smooth dynamical systems that *Floquet multipliers* (the eigenvalues of the linearized Poincaré map) do not depend on the choice of Poincaré section (cf. Section 1.5 in [13]). The following lemma generalizes this result to the hybrid setting by demonstrating that if a Poincaré map obtained from one domain has an attracting invariant submanifold via Lemma 3, then the map obtained in any other starting domain has a diffeomorphic attracting submanifold. As a consequence, non-zero Floquet multipliers are shared between the P_j's after a sufficient number of iterations.

Lemma 4. *Let* $j \in \{1, \ldots, k\}$ *and* $n \geq \min_\ell \dim D_\ell$. *If* P_j^n *has constant rank equal to* r *near* γ_j, *then* P_ℓ^{n+1} *has constant rank equal to* r *near* γ_ℓ *for all* $\ell \in \{1, \ldots, k\}$.

As a consequence, if the Poincaré map associated with *any* section for the periodic orbit γ satisfies the hypotheses of Lemma 3, then the Poincaré map associated with *any other* section also satisfies the hypotheses.

Remark 7. *It may be easier to evaluate the rank of the Poincaré map in some domains than others. In particular, if* P_j *is a diffeomorphism for some* $j \in \{1, \ldots, k\}$, *then all iterates are constant rank.*

C. Hybrid Invariant Subsystem

This section contains the main result of this paper: when iterates of the Poincaré map associated with a periodic orbit of a hybrid dynamical system have constant rank, trajectories starting near the orbit converge in finite time to an embedded smooth dynamical system.

Theorem 1. *Let $H = (D, F, R)$ be a hybrid dynamical system, γ a periodic orbit of H, and suppose the composition of any Poincaré map for γ with itself at least $\min_j \dim D_j$ times has constant rank equal to r on a neighborhood of its fixed point. Then there is an $(r+1)$-dimensional dynamical system (M, G), a hybrid dynamical embedding $f : M \to D$, and an open hybrid set $W \subset D$ so that $\gamma \subset f(M) \cap W$ and trajectories starting in W flow into M in finite time.*

Corollary 1. *γ is asymptotically stable for $H = (D, F, R)$ if and only if γ is asymptotically stable for (M, G).*

If each of the D_j's have the same dimension and $R : S \to T$ is a diffeomorphism, the rank condition of Theorem 1 is trivially satisfied, and we can globalize the construction using Lemma 2. This provides a smooth n-dimensional generalization of the construction in [35].

Corollary 2. *Let $H = (D, F, R)$ be a hybrid dynamical system with $D = \coprod_{j \in J} D_j$, $R : S \to T$, and $\partial D = S \cup T$. If $\dim D_j = n$ for all $j \in J$ and R is a diffeomorphism, then there is a surjective hybrid dynamical embedding from an n-dimensional dynamical system (M, G) onto H.*

IV. EXAMPLES

A. Hybrid Floquet Coordinates

The following single-domain system clearly satisfies the hypotheses of Theorem 1, and demonstrates the canonical form for hybrid Floquet coordinates.

Example 1. *Let $H = (D, F, R)$ be a hybrid system over the single domain $D = [0, 1] \times \mathbb{R}^k \times \mathbb{R}^\ell$ with vector field $F(t, x, z) = \frac{\partial}{\partial t} + \sum_{j=1}^{k} f^j(t, x) \frac{\partial}{\partial x^j} + \sum_{i=1}^{\ell} g^i(t, z^i) \frac{\partial}{\partial z^i}$, reset map $R : \{1\} \times \mathbb{R}^k \times \mathbb{R}^\ell \to \{0\} \times \mathbb{R}^k \times \mathbb{R}^\ell$ defined by $R(1, x, z) = (0, x, Az)$ where $A \in \mathbb{R}^{\ell \times \ell}$ is nilpotent, $f(t, \xi) = 0$ for all $t \in [0, 1]$ and some $\xi \in \mathbb{R}^k$, and $g^i(t, 0) = 0$ for all i. Consider the Poincaré map*

$$P : \{0\} \times \mathbb{R}^k \times \mathbb{R}^\ell \to \{0\} \times \mathbb{R}^k \times \mathbb{R}^\ell.$$

It is clear that $P(0, \xi, 0) = (0, \xi, 0)$,

$$P^\ell(\{0\} \times \mathbb{R}^k \times \mathbb{R}^\ell) = \{0\} \times \mathbb{R}^k \times \{0\},$$

$$\mathrm{rank}\, R|_{\{1\} \times \mathbb{R}^k \times \{0\}} = k.$$

Therefore $\mathrm{rank}\, P^{k+\ell} = k$, whence we may apply Theorem 1. The resulting smooth invariant subsystem is diffeomorphic to $S^1 \times \mathbb{R}^k$.

For $(\phi, x, \dot{x}, y, \dot{y}) \in D_a$:

$$\dot{\phi} = \omega,$$
$$m\ddot{x} = -k\ell_0 - a \sin\phi$$
$$+ k(y - x) - b\dot{x} - gm,$$
$$M\ddot{y} = k\ell_0 + a \sin\phi$$
$$- k(y - x) - gM.$$

For $(\phi, y, \dot{y}) \in D_g$:

$$\dot{\phi} = \omega,$$
$$M\ddot{y} = k\ell_0 + a \sin\phi$$
$$- ky - gM.$$
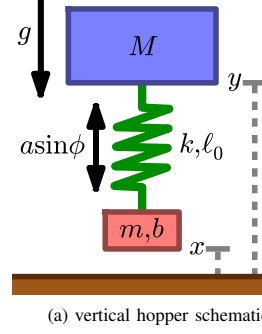
(a) vertical hopper schematic   (b) hopper dynamics

Fig. 3. Schematic and dynamics of vertical hopper. (a) Two masses m and M, constrained to move vertically above a ground plane in a gravitational field with strength g, are connected by a linear spring with stiffness k and nominal length ℓ_0. The spring is equipped with an actuator that exerts a phase-varying force $a \sin\phi$ where $\dot{\phi} = \omega$. The lower mass experiences viscous drag proportional to velocity with constant b when it is in the air, and impacts plastically with the ground (i.e. it is not permitted to penetrate the ground and its velocity is instantaneously set to zero whenever a collision occurs). (b) Hopper dynamics in aerial and ground domains.

B. Vertical Hopper

We apply Theorem 1 to demonstrate the existence of low-dimensional invariant dynamics in the model for forced vertical hopping illustrated in Fig. 3a. The state space in the aerial phase is $D_a := S^1 \times T\mathbb{R}_{\geq 0} \times T\mathbb{R}$. Writing $(\phi, x, \dot{x}, y, \dot{y}) \in D_a$, the aerial dynamics are given in Fig. 3b. When the lower mass rests on the ground, the state space resides in $D_g := S^1 \times T\mathbb{R}$ and the dynamics of the upper mass are obtained by restricting to the submanifold $\{(\phi, x, \dot{x}, y, \dot{y}) : x = \dot{x} = 0\}$ as in Fig. 3b. Transition from the aerial to the ground domain occurs when the lower mass collides with the ground, and the state is reset according to $(\phi, 0, \dot{x}, y, \dot{y}) \mapsto (\phi, y, \dot{y})$. The lower mass lifts off when the normal force required to keep it from penetrating the ground plane becomes zero, i.e. when $mg = -k\ell_0 - a \sin\phi + ky$, and the state is reset via $(\phi, y, \dot{y}) \mapsto (\phi, 0, 0, y, \dot{y})$.

Numerical simulations[5] indicate that with parameters $(m, M, k, b, \ell_0, a, \omega, g) = (1, 2, 10, 5, 2, 20, 2\pi, 2)$, the hybrid system possesses a stable periodic orbit, γ. Choosing a Poincaré section in domain D_g at $\phi = 0$, we find that γ intersects this section at the point $(y, \dot{y}) = (1.96, 1.88)$ and that the eigenvalues of the linearized Poincaré map are $-0.25 \pm 0.70j$. Both eigenvalues lie inside the unit disc, corroborating the observed stability of the orbit. Further, since neither eigenvalue is close to zero, we conclude the Poincaré map has full rank equal to 2 near its fixed point. Therefore by Remark 7 the hypotheses of Theorem 1 are satisfied, and we conclude the system's dynamics collapse to a smooth 3-dimensional subsystem after one hop.

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[5]Note that simulation of hybrid dynamical systems is non-trivial. We make use of a recently-developed algorithm with desirable convergence properties [36]. In particular, we use Euler step size $h = 1 \times 10^{-3}$ and relaxation parameter $\epsilon = 1 \times 10^{-12}$. As a note to practitioners, we found that numerical linearization of the Poincaré map via finite differenes was sensitive to the coordinate displacement when using large values for the relaxation parameter. The sourcecode for this simulation is available online at http://purl.org/sburden/cdc2011

V. Discussion

We demonstrated the existence of a locally attracting constant-dimensional invariant subsystem near a hybrid periodic orbit whenever iterates of the associated Poincaré map have constant rank. Under a genericity condition, near a periodic orbit of a smooth dynamical system there exist *Floquet coordinates* in which the dynamics decouple into a constant-frequency phase variable and a time-invariant transverse linear system [11], [12], [13]. Under the additional rank hypothesis of Theorem 1, we obtain a canonical form for the Floquet structure of a hybrid periodic orbit. Indeed, the smooth subsystem (M, G) foliates the dynamics near the periodic orbit in each domain. Thus the behavior of the hybrid system near the orbit is a trivial extension of the behavior of a smooth system—portions of the smooth dynamics are "stacked" in transverse coordinates and annihilated within a finite number of cycles via a nilpotent linear operator. On the smooth subsystem, the standard construction of Floquet coordinates may be applied, generalizing the class of systems which may be analyzed using the empirical approach developed in [14], [15].

In addition to providing a canonical form for the dynamics near such non-degenerate periodic orbits, the results of this paper suggest a mechanism by which a many-legged locomotor may formally collapse a large number of degrees-of-freedom to produce a low-dimensional coordinated gait. This provides a link between currently disparate lines of research, namely the formal analysis of hybrid periodic orbits, the design of robots for locomotion and manipulation tasks, and the scientific probing of neuromechanical control architectures in organisms. It shows that hybrid models naturally exhibit dimension reduction, that this reduction may be deliberately designed into an engineered system, and that evolution may have exploited this reduction in developing its spectacular locomotors.

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