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$\mathscr{H}_\infty$ Control Design for Time-Delay Linear Systems: a Rational Transfer Function Based Approach*

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***Abstract*— The aim of this paper is to cope with the $\mathscr{H}_\infty$ control synthesis for time-delay linear systems. We extend the use of a finite order LTI system, called *comparison system* to $\mathscr{H}_\infty$ analysis and design. Differently from what can be viewed as a common feature of other control design methods available in the literature to date, the one presented here treats time-delay systems control design with classical numeric routines based on Riccati equation and $\mathscr{H}_\infty$ theory. An illustrative example and a practical application involving a 3-DOF networked control system are presented.**

I. Introduction

Time delay in dynamic systems generally implies on poor performance, or even instability. For this reason, in the past decades, there has been a great effort for the development of efficient control design techniques to cope with time-delay. See the books [9] and [14], and the survey paper [18]. In this context the $\mathscr{H}_\infty$ controller plays a central role of maintaining the $\mathscr{H}_\infty$ norm of the transfer function between the external disturbance and the controlled output below a pre-specified level $\gamma > 0$ for a given value of time delay, keeping the closed-loop system stable [1].

In the literature, many works have dealt with state feedback design in the Riccati equation framework, as for example [12] and [20] for the delay independent case, whereas [6], [7] and [13] address the same design problem by means of Lyapunov-Krasoviskii functionals, obtaining delay dependent controllers. For the output feedback problem, [3] and [8] have proposed delay independent controllers obtained from the solution of Riccati equations, whereas [5] solved the same problem using Lyapunov-Krasoviskii functionals. In the context of delay dependent output feedback design problems, to the best of our knowledge, much less attention has been given, see [7].

Our goal is to present delay-dependent design procedure for output feedback control design. Towards this end we apply the Rekasius substitution [17] to replace the delay operator by a rational first order transfer function. The paper [15] has proposed an useful technique for stability analysis of time-delay system applying the Rekasius substitution and the Routh-Hurwitz criterion. In [11], the same problem has been addressed in the frequency domain, based on the celebrated Nyquist criterion. An important consequence of the Rekasius substitution in [11] is the definition of a finite order linear time invariant system, called *comparison system*, which provides a tight lower bound to the $\mathscr{H}_\infty$ norm of the time-delay system, allowing a simple and efficient filter synthesis algorithm. A *comparison system* for time-delay systems was first introduced in the paper [22], yielding one of the most important results for stability analysis and $\mathscr{H}_\infty$ norm calculation. Indeed, adopting such a comparison system approach, the well known Padé approximation is used to determine linear time invariant systems of increasing but finite order, allowing the direct determination of stability margin and bounds for the $\mathscr{H}_\infty$ norm performance of the time-delay system.

This paper follows the same stream as [11] and [22]. It is important to stress that comparing to [11], the present paper innovates in the following directions:

- The output feedback controller design needs a new parametrisation from the stabilizing solution of a Riccati equation (filter) and any feasible solution of a Riccati inequality (control). It assures the existence of a starting feasible solution that is essential for the development of our design method. Moreover, the design procedure, when compared to the ones already cited, is simpler to be implemented and provides more accurate results.
- A practical application concerning the control of a 3-DOF system of sixth order and four control inputs is presented. It puts in evidence that the proposed method is well adapted to deal with control-delayed signals arising in networked control systems.

The notation used throughout is standard. Exclusively, rational transfer functions of LTI systems are denoted as

$$C(sI-A)^{-1}B+D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad (1)$$

where all matrices are real and of compatible dimensions. The maximum singular value is denoted as $\sigma(\cdot)$.

II. Rational Comparison System

In this section we present the LTI comparison system introduced in [11] associated to the time-delay system minimal realization

$$\dot{x}(t) = A_0x(t)+A_1x(t-\tau)+E_0w(t) \quad (2)$$

$$z(t) = C_{z0}x(t)+C_{z1}x(t-\tau) \quad (3)$$

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The system state vector is $x \in \mathbb{R}^n$, $w \in \mathbb{R}^r$ is the exogenous input and $z \in \mathbb{R}^q$ is the output. It is assumed throughout that $x(t) = 0 \ \forall \ t \in [-\tau, 0]$, and the delay $\tau \geq 0$ is constant with respect to time. The basic idea stems from the Rekasius' substitution which implies that for $s = j\omega$ with $\omega \in \mathbb{R}$, the equality

$$e^{-s\tau} = \frac{1-\lambda^{-1}s}{1+\lambda^{-1}s} \tag{4}$$

holds for some $\lambda \in \mathbb{R}$ such that $\omega/\lambda = \tan(\omega\tau/2)$. It is important to notice that for a given pair (λ, ω) there exist many $\tau \geq 0$ satisfying this relation. Based on this we introduce a rational comparison system associated to (2)-(3):

$$\begin{aligned} H(\lambda, s) &= \left[\begin{array}{c|c} A_\lambda & E \\ \hline C_z & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ A_0 + A_1 & A_0 - A_1 - \lambda I & E_0 \\ \hline C_{z0} + C_{z1} & C_{z0} - C_{z1} & 0 \end{array}\right] \end{aligned} \tag{5}$$

Denoting by $T(\tau, s)$ the transfer function of the time-delay system from the input w to the output z, this LTI system has been determined in such a way that $H(\lambda, j\omega) = T(\tau, j\omega)$ holds whenever $\lambda \in \mathbb{R}$, $\tau \geq 0$ and the frequency $\omega \in \mathbb{R}$ are related by the relation $\omega/\lambda = \tan(\omega\tau/2)$.

A. *Stability Analysis*

An standard problem consists on the determination of a time delay $\tau^* > 0$ such that the system (2)-(3) remains asymptotically stable for all $\tau \in [0, \tau^*)$. Clearly, the determination of τ^* depends upon the poles of the transfer function $T(\tau, s)$, that are the roots of the characteristic equation

$$\Delta_T(\tau, s) = \det\big(sI - A_0 - A_1 e^{-s\tau}\big) \tag{6}$$

which is transcendental (whenever $\tau > 0$) and admits, generally, infinitely many roots. Most of the existing procedures are based on the detection of the crossings of poles through the imaginary axis since from the *root continuity argument* the poles vary continuously with respect to delay, so that any root crossing from the left to the right half-plane will need to pass through the imaginary axis. A comparison between algorithms to find the position where the roots cross the imaginary axis is provided by [21]. Another strategy to the solution of this problem is presented in [11]. Based on the celebrated Nyquist criterion, a simple graphical test is performed in order to decide if the system is asymptotic stable for a given value of the delay. Hence, by increasing $\tau \geq 0$, it is possible to determine the value τ^* corresponding to the first occurrence of an unstable pole, which defines the so-called stability margin of the time-delay system.

B. *$\mathscr{H}_\infty$ Norm Calculation*

In this section our purpose is to show how to calculate

$$\|T(\tau, s)\|_\infty = \sup_{\omega \in \mathbb{R}} \ \sigma(T(\tau, j\omega)) \tag{7}$$

for a given $\tau \in [0, \tau^*)$. A line search with respect to $\omega \geq 0$ together with a singular value decomposition of $T(\tau, j\omega)$ is an immediate, although certainly not the best, way to evaluate the supremum appearing in (7). The purpose of this section is to show that the rational transfer function $H(\lambda, s)$ can be successfully used for $\mathscr{H}_\infty$ norm calculation and does not present any inconvenient for linear control synthesis. In this context, we define the positive scalar $\lambda_o = \inf\big\{\lambda \mid A_\xi \text{ is Hurwitz } \forall \xi \in (\lambda, \infty)\big\}$, assuring that $\|H(\lambda, s)\|_\infty = \sup_{\omega \in \mathbb{R}} \ \sigma(H(\lambda, j\omega))$ holds and is bounded $\forall \lambda \in (\lambda_o, \infty)$.

Theorem 1: [11] Assume that $A_0 + A_1$ is Hurwitz. For each $\lambda \in (\lambda_o, \infty)$ define $\alpha \geq 0$ such that

$$\alpha = \arg \sup_{\omega \in \mathbb{R}} \ \sigma(H(\lambda, j\omega)) \tag{8}$$

and determine $\tau(\lambda)$ from $\alpha/\lambda = \tan(\alpha\tau/2)$. If $\tau(\lambda) \in [0, \tau^*)$ then the inequality $\|H(\lambda, s)\|_\infty \leq \|T(\tau(\lambda), s)\|_\infty$ holds.

At this point, it is relevant to analyze the previous result for $\lambda \to \infty$. The approximation $H(\lambda, s) \approx (C_{z0} + C_{z1})\big(sI - A_0 - A_1\big)^{-1} E_0$ is valid for all $|s|$ finite whenever λ goes to infinity. In this case, all poles of $H(\infty, s)$ are in the open left hand side of the complex plane and we get $\tau = 0$. As a consequence, the result provided by Theorem 1 turns out to be exact since the transfer functions $H(\infty, s)$ and $T(0, s)$ are asymptotically stable, equal and, consequently, $\|H(\infty, s)\|_\infty = \|T(0, s)\|_\infty < \infty$. Moreover, notice that we can not discard the possibility that for some $\lambda \in (\lambda_o, \infty)$ the value of the time delay calculated from (8) be such that $\tau(\lambda) \notin [0, \tau^*)$. In this case, the lower bound provided by Theorem 1 remains valid but only in a subset of the interval (λ_o, ∞).

Corollary 1: [11] Assume that $A_0 + A_1$ is Hurwitz. For any given positive parameter $\gamma > \|H(\infty, s)\|_\infty$ there exist $\lambda_\gamma \geq \lambda_o > 0$ and $0 \leq \tau_\gamma \leq \tau^*$ such that the inequality

$$\|H(\lambda, s)\|_\infty \leq \|T(\tau, s)\|_\infty < \gamma \tag{9}$$

holds $\forall \ \lambda \in (\lambda_\gamma, \infty)$ whenever the time-delay function $\tau(\lambda)$ given by Theorem 1 is continuous in the same interval.

From Corollary 1 we know that it is possible to determine a sub-interval of $\lambda > 0$ such that the lower and upper bounds $\|H(\lambda, s)\|_\infty \leq \|T(\tau, s)\|_\infty < \gamma$ hold. It is important to keep in mind that the determination of the sub-interval defined by λ_γ must be done with care due to the eventual occurrence of multiple solutions $\alpha(\lambda)$ to problem (8), which may cause discontinuities on the associated value of the time-delay $\tau(\lambda)$ extracted from the nonlinear relationship provided in Theorem 1. From Corollary 1, a numeric procedure to calculate these bounds is as follows: for each element of a strictly decreasing sequence $\lambda_k = \{\infty, \cdots, \lambda_o\}$ the time-delay value $\tau_k = \tau(\lambda_k)$ is computed. The index k is increased whenever $-2/\lambda^2 < d\tau(\lambda)/d\lambda < 0$ at $\lambda = \lambda_k$ and $\|T(\tau_k, s)\|_\infty < \gamma$. When this procedure stops we get $\lambda_\gamma = \lambda_{k-1}$ and $\tau_\gamma = \tau_{k-1}$. The existence of the derivative $d\tau(\lambda)/d\lambda < 0$ implies the continuity and monotonicity of $\tau(\lambda)$ and avoids its sudden variation with respect to the variation of λ. This allows us to identify any unboundedness tendency of $\|T(\tau(\lambda), s)\|_\infty$ and also the stability margin τ^* since this norm is continuous within the entire interval (λ_γ, ∞). The constraint $-2/\lambda^2 < d\tau(\lambda)/d\lambda < 0$ at $\lambda = \lambda_k$, inspired by inequality $0 \leq \tau(\lambda) < 2/\lambda$, is numerically implemented through the simple test $0 < \tau_k - \tau_{k-1} < 2(\lambda_{k-1} - \lambda_k)/\lambda_k^2$ whose accuracy is controlled by taking $|\lambda_{k+1} - \lambda_k|$ sufficiently small.

Remark 1: The first element of the sequence $\{\lambda_k\}$ can be chosen as $2/\varepsilon$, where $\varepsilon > 0$ is such that the norms of the finite order systems, namely $\|H(2/\varepsilon, s)\|_\infty$ and $\|T(0,s)\|_\infty$, are close to each other, which means that their distance is within some precision defined by the designer. Such an $\varepsilon > 0$ satisfying this condition always exists.

III. Output Feedback Design

Consider the minimal realization time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + B_0 u(t) + E_0 w(t) \quad (10)$$

$$y(t) = C_{y0} x(t) + C_{y1} x(t-\tau) + D_{yw} w(t) \quad (11)$$

$$z(t) = C_{z0} x(t) + C_{z1} x(t-\tau) + D_{zu} u(t) \quad (12)$$

where, in addition to the assumptions and the variables defined before, $y(t) \in \mathbb{R}^p$ is the measured signal. The aim of this section is to design a full order dynamic output feedback controller with the following structure

$$\dot{\hat{x}}(t) = \hat{A}_0 \hat{x}(t) + \hat{A}_1 \hat{x}(t-\tau) + \hat{B}_0 y(t) \quad (13)$$

$$u(t) = \hat{C}_0 \hat{x}(t) + \hat{C}_1 \hat{x}(t-\tau) \quad (14)$$

where $\hat{x}(t) \in \mathbb{R}^n$ and $\hat{x}(t) = 0 \ \forall \ t \in [-\tau, 0]$. When connected to (10)-(12) it yields the regulated output $z(t)$ as

$$\dot{\xi}(t) = F_0 \xi(t) + F_1 \xi(t-\tau) + G_0 w(t) \quad (15)$$

$$z(t) = J_0 \xi(t) + J_1 \xi(t-\tau) \quad (16)$$

where $\xi(t) = [x(t)' \ \hat{x}(t)']' \in \mathbb{R}^{2n}$ is the state and

$$F_0 = \begin{bmatrix} A_0 & B_0 \hat{C}_0 \\ \hat{B}_0 C_{y0} & \hat{A}_0 \end{bmatrix}, \ F_1 = \begin{bmatrix} A_1 & B_0 \hat{C}_1 \\ \hat{B}_0 C_{y1} & \hat{A}_1 \end{bmatrix} \quad (17)$$

$G_0' = [E_0', \ D_{yw}' \hat{B}_0']$, $J_0 = [C_{z0}, \ D_{zu} \hat{C}_0]$ and $J_1 = [C_{z1}, \ D_{zu} \hat{C}_1]$. The transfer function from the external input $w(t)$ to the controlled output $z(t)$ is given by

$$T_C(\tau, s) = \left(J_0 + J_1 e^{-\tau s} \right) \left(sI - F_0 - F_1 e^{-\tau s} \right)^{-1} G_0 \quad (18)$$

where the subindex "C" indicates its dependence on a controller of the form (13)-(14). Hence, for a given $\gamma > 0$, the goal is to design a controller such that $\|T_C(\tau, s)\|_\infty < \gamma$ based on the $4n$-th order rational comparison system

$$H_C(\lambda, s) = \left[\begin{array}{c|c} F_\lambda & G \\ \hline J & 0 \end{array} \right] = \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ F_0 + F_1 & F_0 - F_1 - \lambda I & G_0 \\ \hline J_0 + J_1 & J_0 - J_1 & 0 \end{array} \right] \quad (19)$$

For each $\lambda > 0$, the $\mathscr{H}_\infty$ output feedback design problem is solved and the corresponding time-delay $\tau(\lambda)$ is extracted as indicated in Corollary 1. This is possible because the state space realization (19) admits an important property that is the key for output feedback control design. Indeed, applying the similarity transformation

$$S = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \quad (20)$$

one can rewrite (19) in the equivalent form

$$H_C(\lambda, s) = \left[\begin{array}{c|c} S F_\lambda S^{-1} & SG \\ \hline J S^{-1} & 0 \end{array} \right] = \left[\begin{array}{cc|c} A_\lambda & B\hat{C} & E \\ \hat{B} C_y & \hat{A}_\lambda & \hat{B} D_{yw} \\ \hline C_z & D_{zu} \hat{C} & 0 \end{array} \right] \quad (21)$$

where the matrices (A_λ, E, C_z) have been defined in (5), $B' = [0, \ B_0']$, $C_y = [C_{y0} + C_{y1}, \ C_{y0} - C_{y1}]$,

$$\hat{A}_\lambda = \begin{bmatrix} 0 & \lambda I \\ \hat{A}_0 + \hat{A}_1 & \hat{A}_0 - \hat{A}_1 - \lambda I \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0 \\ \hat{B}_0 \end{bmatrix} \quad (22)$$

and $\hat{C} = [\hat{C}_0 + \hat{C}_1, \ \hat{C}_0 - \hat{C}_1]$. Hence, the controller (13)-(14) whenever connected to the time-delay system (10)-(12) produces an LTI comparison system associated to the regulated output (15)-(16) whose transfer function can be alternatively determined from the connection of the LTI comparison system of the system (10)-(12) and the LTI comparison system of the controller (13)-(14).

At first glance (21) leads to the conclusion that the state space realization of $H_C(\lambda, s)$ has the classical structure of the regulated output. This is true, however, matrices $\hat{A}_\lambda$ and $\hat{B}$ must be constrained to have the particular structures given in (22). To circumvent this difficult we firstly propose to design an LTI full order output feedback controller replacing the matrices variables $(\hat{A}_\lambda, \hat{B}, \hat{C})$ in (21) by general matrices variables (A_C, B_C, C_C) and solve $\|H_C(\lambda, s)\|_\infty < \gamma$ for given $\lambda > 0$ and $\gamma > 0$, which is a classical problem in $\mathscr{H}_\infty$ theory. The second step is to determine a non-singular matrix $V \in \mathbb{R}^{2n \times 2n}$ such that $(\hat{A}_\lambda, \hat{B}, \hat{C}) = (V A_C V^{-1}, V B_C, C_C V^{-1})$, which naturally implies that the regulated output transfer function of the comparison system remains unchanged. Once we have the controller at hand, it is simple to verify whether $\|T_C(\tau(\lambda), s)\|_\infty < \gamma$ holds.

For a given $\gamma > 0$, under the usual assumptions $C_z' D_{zu} = 0$, $E D_{yw}' = 0$, $D_{yw} D_{yw}' = I$ and $D_{zu}' D_{zu} = I$, imposed just for simplicity, it is a well known that the existence of a stabilizing matrix $P = P' > 0$ and $\Pi = \Pi' > 0$ satisfying

$$A_\lambda \Pi + \Pi A_\lambda' + E E' - \Pi \left(C_y' C_y - \gamma^{-2} C_z' C_z \right) \Pi = 0 \quad (23)$$

$$A_\lambda' P + P A_\lambda + C_z' C_z - P \left(B B' - \gamma^{-2} E E' \right) P < 0 \quad (24)$$

and the spectral radius constraint $r_s(P\Pi) < \gamma^2$ is a necessary and sufficient condition for the existence of a full order LTI controller (depending on λ) such that $\|H_C(\lambda, s)\|_\infty < \gamma$. In the affirmative case, the desired controller has the state space realization defined by matrices in (41) and (42) (see, for instance, [4] and [25] for more details).

Lemma 1: For $\lambda > 0$ large enough, the stabilizing positive definite solution of (23) and any positive definite feasible solution of (24) exhibit the structures

$$\Pi = \begin{bmatrix} Z & \lambda^{-1} Q \\ \lambda^{-1} Q' & \lambda^{-1} W \end{bmatrix}, P^{-1} = \begin{bmatrix} Y + R & -R \\ -R & R \end{bmatrix} \quad (25)$$

where $Z > 0$, $W > 0$, Q, $Y > 0$, $R > 0$ are $n \times n$ matrices.

Proof: See the Appendix. ■

Theorem 2: Consider $\gamma > \min_C \|H_C(\infty,s)\|_\infty$. For $\lambda > 0$ large enough the relations $\|H_C(\infty,s)\|_\infty = \|T_C(0,s)\|_\infty < \gamma$ hold.

Proof: It follows from (25) and the controller parametrisation given in the Appendix. ■

An important point concerning the controller design is how to obtain a suitable similarity transformation $V \in \mathbb{R}^{2n\times 2n}$. Notice that V must put A_C and B_C in the appropriate form. Moreover it might guarantee that the closed-loop system is stable since V does not define a similarity transformation for the time-delay system.

Lemma 2: [11] Assume that $\dim(y) = p \leq n = \dim(x)$, $\lambda > 0$ and the matrix

$$V = \begin{bmatrix} N \\ \lambda^{-1} N A_C \end{bmatrix} \in \mathbb{R}^{2n\times 2n} \tag{26}$$

where $N' \in \mathbb{R}^{2n\times n}$ belongs to the null space of B'_C is non-singular. Then $(\hat{A}_\lambda, \hat{B}, \hat{C}) = (VA_CV^{-1}, VB_C, C_CV^{-1})$ holds.

An important fact is the influence of the similarity transformation V in (26) as $\lambda \to \infty$. Partitioning matrix $N = [N_1\ N_2]$, where N_1 is assumed to be non-singular, making use of Lemma 1 the similarity transformation (26) yields

$$C_\infty(s) = \left[\begin{array}{c|c} N_1 A_{C0} N_1^{-1} & N_1 B_{C0} \\ \hline C_{C0} N_1^{-1} & 0 \end{array}\right] \tag{27}$$

where (A_{C0}, B_{C0}, C_{C0}) is the central controller associated to $\tau = 0$. As expected, the similarity transformation does not affect the controller transfer function when $\tau = 0$, and consequently $\|H_C(\infty,s)\|_\infty = \|T_C(0,s)\|_\infty$ for any nonsingular $V \in \mathbb{R}^{2n\times 2n}$.

We are able to extend the previous algorithm to obtain intervals $\lambda \in (\lambda_\gamma, \infty)$ and $\tau(\lambda) \in [0, \tau_\gamma)$ assuring the existence of a controller (13)-(14) for each pair $(\lambda, \tau(\lambda))$ such that $\|H_C(\lambda,s)\|_\infty \leq \|T_C(\tau(\lambda),s)\|_\infty < \gamma$. At each iteration k we must calculate the time delay $\tau_k = \tau(\lambda_k)$, the central controller (A_{Ck}, B_{Ck}, C_{Ck}) and the similarity transformation matrix V_k. In order to assure the continuity of $\|T_C(\tau(\lambda),s)\|_\infty$, what enables us to detect any unboundedness tendency, we must compute matrix N_k in (26) with care. As in [11] we have chosen N'_k as the first n column vectors provided by the Matlab null space routine applied to B'_{Ck} and we verify continuity by evaluating the norm condition $\|N_k - N_{k-1}\| \leq \varepsilon$, with $\varepsilon > 0$ sufficiently small. In fact, the problem of generate a continuous null space basis for matrix B_C depending on the parameter λ is not a simple task, see [2].

IV. Illustrative Example

Let us consider an example borrowed from [7]. The time-delay system (10)-(12) matrices are defined as follows[1]:

$$\left[\, A_0 \mid A_1 \mid E_0 \mid B_0 \,\right] = \left[\begin{array}{cc|cc|cc|c} 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -0.9 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\, C_{y0} \mid C_{y1} \mid D_{yw} \,\right] = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0.1 \end{array}\right]$$

[1]In this case, a simple change of variables must be performed in order to get a model satisfying $D'_{yw}D_{yw} = I$ and $D_{zu}D'_{zu} = I$.

$$\left[\, C_{z0} \mid C_{z1} \mid D_{zu} \,\right] = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 \end{array}\right]$$

The proposed algorithm generates a sequence of stabilizing controllers for each pair $(\lambda_k, \tau(\lambda_k))$ such that $\lambda_k \in (\lambda_\gamma, \infty)$ and $\tau(\lambda_k) \in [0, \tau_\gamma)$. With $\gamma = 1$ we have determined $\lambda_\gamma = 1.0100$ and $\tau_\gamma = 1.2477$ [s]. For $\tau(\lambda) \in [0, 0.5)$ the values of the lower bound $\|H_C(\lambda,s)\|_\infty$ and the true values of $\|T_C(\tau(\lambda),s)\|_\infty$ are identical and, in the remaining interval, the maximum difference between them is about 3.2%.

For the time delay $\tau = 0.9990$ [s], obtained from $\lambda = 1.40438$ the norm $\|T_C(\tau,s)\|_\infty = 0.2731$ is 68% smaller than the $\mathscr{H}_\infty$ norm obtained in [7]. Moreover, the lower bound $\|H_C(\lambda,s)\|_\infty = 0.2681$ is only 1.81% smaller than the true norm value. As it can be verified, the time-delay controller

$$\left[\, \hat{A}_0 \mid \hat{A}_1 \,\right] = \left[\begin{array}{cc|cc} -28.6072 & 1.4110 & 3.6807 & -2.4378 \\ -76.1020 & 3.8891 & 11.2365 & -7.4419 \end{array}\right]$$

$$\left[\, \hat{B}_0 \mid \hat{C}'_0 \mid \hat{C}'_1 \,\right] = \left[\begin{array}{c|c|c} 15.0420 & -10.5733 & 2.2117 \\ 36.8268 & 0.4678 & -0.9181 \end{array}\right]$$

makes the closed-loop system asymptotically stable with the transfer function $\mathscr{H}_\infty$ norm previously calculated.

V. Practical Application - Networked Control

Networked control systems (NCS) have received a great amount of attention in recent years. The main feature of NCS is that measurement and control actions are supported by a communication network. As a result, the control design has to take into account some phenomena that can deteriorate the final performance and reduce the stability margin. Among them, it is worth mention bandwidth limitations, packet dropout, and delay. See the survey paper [10] and the papers [23], [24] for interesting and useful discussions on this topic. Presently, we focus our attention to the effect of network-induced delay in NCS control, exclusively, see [24]. In addition, since depending on the medium access control (MAC) protocol of the control network, network-induced delay can be constant, time varying, or even random, [24] we also make the assumption that the time-delay is deterministic and constant. Our main purpose is to design a dynamic output feedback controller with good stability margin and small performance deterioration in terms of maximum delay and $\mathscr{H}_\infty$-norm cost.

In this framework, we consider a system with 4 propellers mounted on a three degrees of freedom (3-DOF) pivot, [16]. Each pair of diametrically opposed propellers generate lift forces that control the pitch p and roll r angles, while the total torque causes a yaw y to the body as well, see [16]. The lift forces are proportional to the voltages applied to the motors that command the propellers and the angular displacements are measured by encoders placed in the three rotation axis of the body. To change the measurement frame to the body axis, instead of the encoder axis, it must be performed a basis transformation by means of a nonsingular matrix T_{meas} defined in [16]. We define the state vector $x = [p\ r\ y\ \dot{p}\ \dot{r}\ \dot{y}]'$ and the control input vector $u = [V_f\ V_b\ V_r\ V_l]'$, depending upon the voltages applied to the propellers. Moreover, the

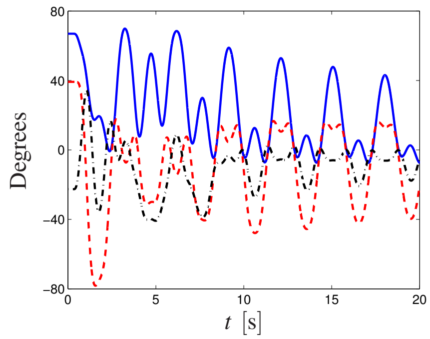

Fig. 1. Time response for $\tau = 0.34$ [s] and the classical $\mathscr{H}_\infty$ controller.

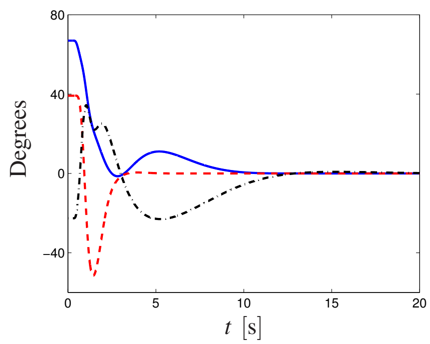

Fig. 2. Time response for $\tau = 0.34$ [s] and the proposed $\mathscr{H}_\infty$ controller.

vector of external disturbances w belongs to $\mathbb{R}^7$, whose first 4 elements correspond to control signal noise, and its last 3 components are related to the measurement noise. Hence, with the data provided in [16] the state space realization (10)-(12) follows. The angular positions p, r and y are measured and, inspired in [19], they are sent to the controller through a network which adds a total delay $\tau > 0$ to the signals. Under this context, our goal is to control the angular positions p, r and y subjected to a $\mathscr{H}_\infty$ norm level $\gamma = 5$. Thus, we define the output matrices

$$C_{y0} = 0,\ C_{y1} = \left[\begin{array}{cc} I_3 & 0 \end{array}\right],\ D_{yw} = \left[\begin{array}{cc} 0 & 0.1I_3 \end{array}\right]$$

$$C_{z0} = \left[\begin{array}{cc} I_3 & 0 \\ 0 & 0 \end{array}\right],\ C_{z1} = 0,\ D_{zu} = \left[\begin{array}{c} 0 \\ I_4 \end{array}\right]$$

To evaluate the robustness of the controller designed for $\tau = 0$ we consider the autonomous system ($w = 0$) with initial condition $x(0) = [\pi/6\ \pi/4\ \pi/3\ 0\ 0\ 0]'$ [rad] measured in the encoder axis frame. Fig. 1 depicts the time simulation for a time delay value $\tau = 0.34$ [s]. The continuous line denotes the pitch angle p, the dashed line denotes the roll angle r and the dot-dashed line denotes the yaw angle y measured, in degrees, in the body axis frame. The time delay $\tau = 0.34$ [s] corresponds to the stability threshold, and it can be verified that although the stability is guaranteed, the $\mathscr{H}_\infty$ norm level is not preserved. On the other hand, Fig. 2 shows again the time simulation for the position angles behavior based on the same assumptions as before, but for a controller properly designed to cope with the time delay $\tau = 0.34$ [s], assuring the pre-specified norm level. It is clear that the performance of the second controller is much better than in the first one, and this fact agrees with the discussion that in a networked control framework, neglecting the existence of time-delays may lead to poor performance and, even worse, lead the closed-loop system to instability. Applying the algorithm proposed in Section III, we have $\lambda_\gamma = 3.3229$ and $\tau_\gamma = 0.5960$ [s], which is an adequate value for actual long distance network delay.

VI. Conclusions

In this paper we have proposed a new procedure for time-delay control design. It is based on what we call comparison system which is an LTI system with order twice the number of state variables of the time-delay system. Moreover, under some weak limitations, it is well adapted not only for stability analysis but also for linear control design of time-delay dynamic systems. The most important feature of the comparison system is that it makes possible the control design by manipulating finite order LTI systems, exclusively. As a consequence, the classical routines for control synthesis can be applied, opening the possibility to handle time-delay systems with high number of state variables.

Appendix

Consider the $2n$th order LTI system

$$\begin{aligned} \dot{x}(t) &= A_\lambda x(t) + Bu(t) + Ew(t) && (28) \\ y(t) &= C_y x(t) + D_{yw} w(t) && (29) \\ z(t) &= C_z x(t) + D_{zu} u(t) && (30) \end{aligned}$$

and the full order output feedback controller to be designed

$$\begin{aligned} \dot{x}_c(t) &= A_C x(t) + B_C y(t) && (31) \\ u(t) &= C_C x_c(t) && (32) \end{aligned}$$

such that the closed-loop system, with state vector $\xi(t) = [x(t)'\ x_c'(t)]' \in \mathbb{R}^{4n}$, has its dynamics governed by

$$\begin{aligned} \dot{\xi}(t) &= \tilde{A}\xi(t) + \tilde{B}w(t) && (33) \\ z(t) &= \tilde{C}\xi(t) && (34) \end{aligned}$$

where

$$\left[\begin{array}{c|c|c} \tilde{A} & \tilde{B} & \tilde{C}' \end{array}\right] = \left[\begin{array}{cc|c|c} A_\lambda & BC_c & E & C_z' \\ B_c C_y & A_c & B_c D_{yw} & C_c' D_{zu}' \end{array}\right] \quad (35)$$

The closed-loop system satisfies an $\mathscr{H}_\infty$ performance level $\gamma > 0$ if there exist $\tilde{P} > 0 \in \mathbb{R}^{2n}$ such that

$$\tilde{A}'\tilde{P} + \tilde{P}\tilde{A} + \tilde{C}'\tilde{C} + \gamma^{-2}\tilde{P}\tilde{B}\tilde{B}'\tilde{P} < 0 \quad (36)$$

Considering the classical partitioning

$$\tilde{P} = \left[\begin{array}{cc} X & U \\ U' & \hat{X} \end{array}\right],\ \tilde{P}^{-1} = \left[\begin{array}{cc} Y & V \\ V' & \hat{Y} \end{array}\right],\ \tilde{\Gamma} = \left[\begin{array}{cc} Y & I \\ I & 0 \end{array}\right] \quad (37)$$

multiplying (36) by $\tilde{\Gamma}'$ on the left and by its transpose on the right, we obtain the following inequalities

$$\begin{aligned} A_\lambda Y + YA_\lambda' + Y\tilde{C}_z'\tilde{C}_z Y + \gamma^{-2}EE' + & \\ +BL + L'B' + L'L &< 0 \quad (38) \end{aligned}$$

$$XA_\lambda + A_\lambda' X + \gamma^{-2} X\tilde{E}\tilde{E}'X + C_z'C_z + \\ + FC_y + C_y'F' + \gamma^{-2}FF' \quad < \quad 0 \qquad (39)$$

where $L = V'C_C$, $F = UB_C$, and $\tilde{C}_z$ and $\tilde{E}$ have been defined in Section III. Moreover, in order to (38) and (39) be equivalent to (36) the following equality must also hold

$$XA_\lambda Y + FC_yY + XBL + UA_cV' + A_\lambda' + \\ + C_z'C_zY + \gamma^{-2}XEE' \quad = \quad 0 \qquad (40)$$

From (38) and (39) we determine the gains $L = -B'$ and $F = -\gamma^2 C_y'$, corresponding to the central controller parameterisation. Considering these values, we multiply (38) and (39) on both sides by Y^{-1} and $\gamma^2 X^{-1}$, respectively, and define $P = Y^{-1}$ and $\Pi = \gamma^2 X^{-1}$. As an immediate consequence, (24) is obtained from (38). Since the equality $XY + UV' = I$ must hold, we can fix $U = X = \gamma^2\Pi^{-1}$ and calculate

$$\begin{array}{rcl} B_C & = & -\Pi C_y' \\ C_C & = & B'P(I - \gamma^{-2}\Pi P)^{-1} \end{array} \qquad (41)$$

Assuming that $\tilde{E}_0\tilde{E}_0' > 0$, which can be assured with no loss of generality by a slight perturbation in this matrix if necessary, a positive definite solution X^{-1} lying on the border of inequality (39) is determined by replacing it by the correspondent Riccati equation. This procedure together with the proposed change of variables yield (23) and then, from (40) and (23), we can recover after some tedious calculations the controller matrix

$$A_C = A_\lambda + \gamma^{-2}\Pi C_z'C_z + B_CC_y - BC_C \qquad (42)$$

Moreover, the condition $\tilde{P} > 0$ is equivalent to $X > Y^{-1} > 0$, which from the above change of variables reduces to $\Pi > 0$, $P > 0$ and $\gamma^2\Pi^{-1} > P > 0$.

Now, we determine the behavior of positive definite matrices Π and P stated in Lemma 1, for $\lambda \to \infty$. First, to evaluate the behavior of Π with respect to λ, let us consider the matrix function

$$\Phi_\lambda = A_\lambda\Pi + \Pi A_\lambda' + \tilde{E}\tilde{E}' - \Pi\big(C_y'C_y - \gamma^{-2}C_z'C_z\big)\Pi \qquad (43)$$

where Π is given in (25). Calculating $\lim_{\lambda\to\infty}\Phi_\lambda = \Phi_\infty$ we obtain the following matrix blocks identities:

$$\begin{array}{rcl} \Phi_\infty^{11} & = & Q + Q' - Z(C_{y0}+C_{y1})'(C_{y0}+C_{y1})Z + \\ & & +\gamma^{-2}Z(C_{z0}+C_{z1})'(C_{z0}+C_{z1})Z \\ \Phi_\infty^{12} & = & W + Z(A_0+A_1)' - Q \\ \Phi_\infty^{22} & = & -2W + \tilde{E}_0\tilde{E}_0' \end{array} \qquad (44)$$

Assuming $\tilde{E}_0\tilde{E}_0' > 0$, otherwise perturb it slightly, the condition $\Phi_\infty = 0$ is satisfied whenever $Z > 0$ is the stabilizing solution of the Riccati equation

$$(A_0+A_1)Z + Z(A_0+A_1)' + \tilde{E}_0\tilde{E}_0' - Z\big((C_{y0}+C_{y1})' \times \\ \times (C_{y0}+C_{y1}) - \gamma^{-2}(C_{z0}+C_{z1})'(C_{z0}+C_{z1})\big)Z = 0 \qquad (45)$$

because the condition $\Pi > 0$ for $\lambda \to \infty$ is equivalent to $Z > 0$ and $W > 0$. Hence, the first part of Lemma 1 follows.

For the second part, considering any $P > 0$ feasible to inequality (24), multiplying it by P^{-1} on both sides we can apply the same procedure as in Theorem 2 of [11], which yields the conclusion that (24) is satisfied for $\lambda \to \infty$ if and only if P^{-1} has the structure given in (25) with $Y > 0$ satisfying

$$(A_0+A_1)'Y^{-1} + Y^{-1}(A_0+A_1) - Y^{-1}(BB' + \\ -\gamma^{-2}EE')Y^{-1} + (\tilde{C}_{z0}+\tilde{C}_{z1})'(\tilde{C}_{z0}+\tilde{C}_{z1}) \ < \ 0 \qquad (46)$$

and $R > 0$ arbitrary.

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