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Positive feedback interconnection of Hamiltonian systems

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Abstract— **Recent results on counterclockwise input-output dynamics and negative-imaginary transfer matrices are interpreted from a geometric Hamiltonian systems point of view, providing additional insights and results.**

I. INTRODUCTION

This paper aims at further developing recent work on systems with 'counterclockwise input-output dynamics' [2], [3], [4], or having 'negative imaginary transfer matrices' [8], [10], motivated respectively by (multi-)stability considerations in biological systems or vibration control[1]. In this closely related work it was proved that the *positive feedback* interconnection of two such systems is stable provided a coupling condition on the dc-gains of the two systems holds. Furthermore, the tight connection with *passivity* of the system with time-differentiated output was emphasized. The current paper interprets these results from a Hamiltonian state space perspective[2], thereby providing additional insights and results, and allowing for further nonlinear generalizations.

In Section 2 we will start off with the Hamiltonian state space formulation of the results on linear systems obtained in [3], [8], [10], [19]. Indeed, we will show how the class of linear systems having 'negative imaginary transfer matrices' is a direct extension of the class of linear Hamiltonian input-output systems introduced and studied in e.g. [5], [12], [13], [14], and can be properly called linear 'input-output Hamiltonian systems with dissipation' (linear IOHD systems). Furthermore, we will show how the dc-gain coupling condition derived in [3], [8] has an immediate interpretation in terms of the Hamiltonian of the interconnected system, and in fact is equivalent to the condition that this Hamiltonian has a minimum at the origin, thus serving as a Lyapunov function (see also [3], Theorem 6).

In Section 3 we define nonlinear input-output Hamiltonian systems with dissipation, and derive similar results regarding stability of positive feedback interconnections of such systems, extending previous results in [3]. Furthermore, we interpret and further develop some of the results obtained in [4] on multi-stability to this setting.

In Section 4 we deal with a different facet of IOHD systems by showing how in the case that the Poisson structure corresponds to a symplectic structure Liouville's theorem of classical mechanics extends to IOHD systems. In particular we show how the volume form on the phase space of the system is connected to the volume form on the space of outputs and inputs; thus further explaining the terminology 'counterclockwise input-output dynamics' of [2], [3].

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[1]See also [9] for application of the notion of counterclockwise input-output dynamics to hysteretic models.

[2]See already [3] for showing that classical Hamiltonian systems with force inputs and position outputs have counterclockwise input-output dynamics.

II. THE LINEAR CASE

Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y &= Cx + Du, \quad y \in \mathbb{R}^m \end{aligned} \tag{1}$$

with transfer matrix $G(s) = C(Is - A)^{-1}B + D$. In [8], [10] $G(s)$ is called *negative imaginary*[3] if $D = D^T$ and the transfer matrix $H(s) := s(G(s) - D)$ is *positive real*. In [3] the same notion (mostly for the case $D = 0$) was coined as *counterclockwise input-output dynamics*.

From a state space point of view this means the following. A state space representation of $H(s) = s(G(s) - D)$ is given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= CAx + CBu \end{aligned} \tag{2}$$

with output[4] $z \in \mathbb{R}^m$. Throughout this section we make for convenience the following

Assumption 2.1: The linear system (1) is minimal (controllable and observable). Furthermore, the matrix A is invertible.

Under this assumption it is immediate that the state space system (2) is also minimal. Hence application of the Kalman-Yakubovich-Popov lemma to (2) yields

Proposition 2.2: The system (1) has negative imaginary transfer matrix if and only if $D = D^T$ and there exists an $n \times n$ symmetric matrix $Q > 0$ such that

$$\begin{bmatrix} A^TQ + QA & QB - (CA)^T \\ B^TQ - CA & -CB - (CB)^T \end{bmatrix} \leq 0 \tag{3}$$

In [8], [19] it is shown that the above characterization is equivalent to the following simplified statement:

Proposition 2.3: System (1) has negative imaginary transfer matrix if and only if $D = D^T$ and there exists an $n \times n$ symmetric matrix $Q > 0$ such that

$$A^TQ + QA \leq 0, \quad B = -AQ^{-1}C^T \tag{4}$$

The aim of this section is to interpret Propositions 2.2 and 2.3 from a Hamiltonian point of view; providing insightful interpretations and paving the way to nonlinear generalizations in the subsequent sections.

[3]The terminology 'negative imaginary', stems, similarly to 'positive real', from the Nyquist plot interpretation for single-input single-output systems. For the precise definition in the frequency domain we refer to [8], [3].

[4]Note that $z = \dot{y} - D\dot{u}$, and in particular $z = \dot{y}$ for $D = 0$.

Proposition 2.4: The system (1) has negative imaginary transfer matrix if and only if it can be written as

$$\begin{aligned} \dot{x} &= (J-R)(Qx - C^T u) \\ y &= Cx + Du, \quad D = D^T \end{aligned} \tag{5}$$

for some matrices Q, J, R of appropriate dimensions satisfying

$$Q = Q^T, J = -J^T, R = R^T \geq 0 \tag{6}$$

with $Q > 0$.

Proof. Start from Proposition 2.2 and decompose the matrix

$$\begin{bmatrix} A & B \\ -CA & -CB \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix}$$

into its skew-symmetric and symmetric part, i.e.,

$$\begin{bmatrix} A & B \\ -CA & -CB \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} J & G \\ -G^T & M \end{bmatrix} - \begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \tag{7}$$

for some skew-symmetric matrices J, M, symmetric matrices R, S, and appropriately dimensioned matrices G, P. Writing out (7) yields

$$A = (J-R)Q,\ B = G-P,\ CA = (G+P)^T Q,\ CB = S-M \tag{8}$$

Combination of the first and the third equality yields

$$C(J-R) = (G+P)^T \tag{9}$$

From Proposition 2.3 we know that

$$B = -AQ^{-1}C^T = -(J-R)QQ^{-1}C^T = -(J-R)C^T \tag{10}$$

Combining this with the second equality in (8), together with (10), yields

$$P = -RC^T \tag{11}$$

Furthermore, the fourth equality in (8) yields

$$\begin{aligned} &2S = CB + (CB)^T = \\ &-C(J-R)C^T - (C(J-R)C^T)T = 2CRC^T \end{aligned} \tag{12}$$

Using $P = -RC^T$ and $S = CRC^T$, the inequality (3) thus reduces to

$$\begin{bmatrix} R & P \\ P^T & S \end{bmatrix} = \begin{bmatrix} R & -RC^T \\ -CR & CRC^T \end{bmatrix} = \begin{bmatrix} I \\ -C \end{bmatrix} R \begin{bmatrix} I & -C^T \end{bmatrix} \geq 0, \tag{13}$$

which is equivalent to $R \geq 0$. ■

Motivated by the previous proposition we give the following definition.

Definition 2.5: A system (5) satisfying (6) is called an *input-output Hamiltonian system with dissipation* (IOHD). The storage function $\frac{1}{2}x^T Qx$ is called its *Hamiltonian* function. The matrix J defines a *Poisson structure matrix*, and R is called the *dissipation matrix*; see [16], [15], [11].

Hence the transfer matrix of an IOHD system with $Q > 0$ is negative imaginary.

A special case of Proposition 2.4 is the following

Corollary 2.6: A system (1) with CB skew-symmetric has negative imaginary transfer matrix if and only if it can be written as

$$\begin{aligned} \dot{x} &= (J-R)Qx - JC^T u \\ y &= Cx + Du, \quad D = D^T \end{aligned} \tag{14}$$

with

$$Q = Q^T > 0, J = -J^T, R = R^T \geq 0, CR = 0 \tag{15}$$

Proof. Skew-symmetry of CB is the same as $S = CRC^T = 0$, which is by $R = R^T$ equivalent to $CR = 0$. ■

The subclass of IOHD systems with CB skew-symmetric given by (14) will be denoted as IOHDss systems. Many systems with negative imaginary transfer matrices, such as mechanical systems with co-located position sensors and force actuators fall within this class.

Example 2.7: Linear mechanical systems with co-located position sensors and force actuators are represented in Hamiltonian state space form (with q denoting the position vector and p the momentum vector) as

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix} \begin{bmatrix} K & N \\ N^T & M^{-1} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ L^T \end{bmatrix} u \\ y &= Lq \end{aligned} \tag{16}$$

where usually $N = 0$ (no 'gyroscopic forces'). In this case the total energy is given as

$$H(q,p) = \frac{1}{2}q^T Kq + \frac{1}{2}p^T M^{-1}p, \tag{17}$$

where the first term is the total potential energy (with K the compliance matrix), and the second term is the kinetic energy (with M the mass matrix). Clearly (16) is an IOHDss system with $CB = 0$ and $D = 0$.

A. Positive feedback interconnection of input-output Hamiltonian systems with dissipation

Consider two input-output Hamiltonian systems with dissipation

$$\begin{aligned} \dot{x}_i &= (J_i - R_i)(Q_i x_i - C_i^T u_i) \\ y_i &= C_i x_i + D_i u_i, \quad i = 1,2 \end{aligned} \tag{18}$$

with equal number of inputs and outputs. Their *positive feedback interconnection* is defined as

$$u_1 = y_2 + e_1, \quad u_2 = y_1 + e_2 \tag{19}$$

with e_1, e_2 two external inputs. This leads to

$$\begin{bmatrix} I & -D_2 \\ -D_1 & I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} C_2 x_2 \\ C_1 x_1 \end{bmatrix} + \begin{bmatrix} e_2 \\ e_1 \end{bmatrix} \tag{20}$$

In the rest of this section we make, following [8], the simplifying assumption

$$D_1 D_2 = 0 \tag{21}$$

In this case

$$\begin{bmatrix} I & -D_2 \\ -D_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & D_2 \\ D_1 & I \end{bmatrix},$$

and after some matrix computations it follows that the positive feedback interconnection of two IOHD systems (18) is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \left(\begin{bmatrix}J_1 & 0\\ 0 & J_2\end{bmatrix} - \begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix}\right)$$
$$\left(\begin{bmatrix}Q_1 - C_1^T D_2 C_1 & -C_1^T C_2\\ -C_2^T C_1 & Q_2 - C_2^T D_1 C_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} - \right.$$
$$\left. - \begin{bmatrix}C_1^T & C_1^T D_2\\ C_2^T D_1 & C_2^T\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}\right)$$
$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}C_1 & D_1 C_2\\ D_2 C_1 & C_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}D_1 & 0\\ 0 & D_2\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}, \tag{22}$$

which is again an IOHD system, with *interconnected Hamiltonian* given as

$$\begin{aligned} H_{\text{int}}(x_1, x_2) := \tfrac{1}{2} x_1^T (Q_1 - C_1^T D_2 C_1) x_1 + \\ + \tfrac{1}{2} x_2^T (Q_2 - C_2^T D_1 C_2) x_2 - x_1^T C_1^T C_2 x_2 \end{aligned} \tag{23}$$

Furthermore, it is easily seen that the positive feedback interconnection of two IOHDss is again an IOHDss.

Remark 2.8: Thus the positive feedback interconnection of two IOHD systems is again an IOHD system, similarly to the fact that the *negative feedback interconnection* of two *port-Hamiltonian systems* is again a port-Hamiltonian system [16], [15], [11]. Note that the Poisson structure matrix of the interconnected IOHD system (22) is the *direct sum* of the Poisson structure matrices of the two component IOHD systems. On the other hand, the interconnected Hamiltonian (23) is *more* than the sum of the component Hamiltonians and involves the output mappings as well. This is precisely opposite to the case of an interconnected port-Hamiltonian system, where the resulting Hamiltonian is the sum of the component Hamiltonians, while on the other hand the resulting Poisson structure is based on the component Poisson structure matrices *together* with the input- and output matrices [15].

Hence the stability of the interconnected system can be characterized in terms of the interconnected Hamiltonian (23):

Proposition 2.9: Consider two IOHD systems. The interconnected IOHD (22) is stable having no eigenvalue at zero if the interconnected Hamiltonian (23) has a strict minimum at the origin $(x_1, x_2) = (0, 0)$. Conversely, if (22) is asymptotically stable, then the interconnected Hamiltonian (23) has a strict minimum at the origin $(x_1, x_2) = (0, 0)$.

Proof. Define

$$\begin{aligned} \mathcal{Q} &:= \begin{bmatrix}Q_1 - C_1^T D_2 C_1 & -C_1^T C_2\\ -C_2^T C_1 & Q_2 - C_2^T D_1 C_2\end{bmatrix} \\ \mathcal{J} &:= \begin{bmatrix}J_1 & 0\\ 0 & J_2\end{bmatrix}, \quad \mathcal{R} := \begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix} \end{aligned} \tag{24}$$

Note that $\mathcal{J} - \mathcal{R}$ is invertible. Then $\dot{x} = \mathcal{A}x := (\mathcal{J} - \mathcal{R})\mathcal{Q}x$ is stable without eigenvalue at zero if $\mathcal{Q} > 0$. Conversely, $\mathcal{A}^T \mathcal{Q} + \mathcal{Q}\mathcal{A} \leq 0$, implying by asymptotic stability of $\mathcal{A}$ that $\mathcal{Q} \geq 0$ and by invertibility of $\mathcal{A}$ that $\mathcal{Q} > 0$. ■

Remark 2.10: Of course, if $(\mathcal{Q}\mathcal{R}\mathcal{Q}, (\mathcal{J} - \mathcal{R})\mathcal{Q})$ is *detectable* then $\mathcal{Q} > 0$ implies asymptotic stability of $\dot{x} = (\mathcal{J} - \mathcal{R})\mathcal{Q}x$.

In ([8], Theorem 5) it has been shown that a matrix of the form $\mathcal{Q}$, with $Q_1 > 0, Q_2 > 0$ and D_1, D_2 such that $D_1 D_2 = 0$ and at least one of them positive semi-definite, is positive definite if and only

$$\lambda_{\max}\left([-C_1 A_1^{-1} B_1 + D_1] \cdot [-C_2 A_2^{-1} B_2 + D_2]\right) < 1 \tag{25}$$

where $\lambda_{\max}(K)$ denotes the maximal eigenvalue of a symmetric matrix K. This allows for the following interpretation. The *dc-gain* of an IOHD system (5) is given by the expression

$$-CA^{-1}B + D = CQ^{-1}C^T + D, \tag{26}$$

Hence the interconnected IOHD (22) is stable having no eigenvalue at zero if and only if *the dc loop gain is less than unity.* This can be regarded as a rephrasing of a fundamental result concerning the stability of the positive feedback interconnection of two systems with negative imaginary transfer matrices, as obtained in [3] for the SISO case with $D = 0$ and in [8] for the general MIMO case.

Example 2.11 (Example 2.7 continued): The dc-gain of the IOHDss system (16) is given as $LK^{-1}L^T$, and thus only depends on the compliance matrix K (e.g., the spring constants) and the colocated sensor/actuator locations. Note that in this case positive feedback amounts to *positive position feedback*, while negative feedback of $z = \dot{y} = LM^{-1}p = L\dot{q}$ corresponds to negative velocity feedback; see also [10].

III. Nonlinear input-output Hamiltonian systems with dissipation

A. The affine nonlinear case

The definition of a linear IOHD system (5) is readily extended to the nonlinear case. We first consider the case without feedthrough terms and with affine dependence on u.

Definition 3.1: A system described in local coordinates $x = (x_1, \cdots, x_n)$ for some n-dimensional state space manifold $\mathcal{X}$ as[5]

$$\begin{aligned} \dot{x} &= (J(x) - R(x))[\tfrac{\partial H}{\partial x}(x) - \tfrac{\partial C^T}{\partial x}(x)u], \quad u \in \mathbb{R}^m \\ y &= C(x), \quad y \in \mathbb{R}^m \end{aligned} \tag{27}$$

where the $n \times n$ matrices $J(x), R(x)$ depend smoothly on x and satisfy

$$J(x) = -J^T(x), R(x) = R^T(x) \geq 0, \tag{28}$$

is called an affine nonlinear IOHD system, with Hamiltonian $H : \mathcal{X} \to \mathbb{R}$ and output mapping $C : \mathcal{X} \to \mathbb{R}^m$. If additionally

$$\left(\frac{\partial C^T}{\partial x}(x)\right) R(x) = 0 \tag{29}$$

then the system is called an affine nonlinear IOHDss.

[5]For a function $H : \mathbb{R}^n \to \mathbb{R}$ we denote by $\frac{\partial H}{\partial x}(x)$ the n-dimensional column vector of partial derivatives of H. For a mapping $C : \mathbb{R}^n \to \mathbb{R}^m$ we denote by $\frac{\partial C^T}{\partial x}(x)$ the $n \times m$ matrix whose j-th column consists of the partial derivatives of the j-th component function C_j.

Remark 3.2: The definition given above is a generalization of the definition of an affine input-output Hamiltonian system as originally proposed in [5] and studied in e.g. [12], [13], [14]. In fact, it reduces to this definition in case $R = 0$ and J defines a *symplectic form* (in particular, has full rank). The time-evolution of the Hamiltonian of an affine nonlinear IOHD system satisfies

$$\frac{d}{dt}H = \left(\frac{\partial H}{\partial x}(x)\right)^T (J(x) - R(x))[\frac{\partial H}{\partial x}(x) - \frac{\partial C^T}{\partial x}(x)u] =$$

$$-(\frac{\partial H}{\partial x}(x))^T R(x)(\frac{\partial H}{\partial x}(x) - \frac{\partial H}{\partial x}(x)(J(x) - R(x))\frac{\partial C^T}{\partial x}(x)u$$

Furthermore, the time-differentiated output of an affine nonlinear IOHD system is given as

$$z := \dot{y} = (\frac{\partial C^T}{\partial x}(x))^T (J(x) - R(x))[\frac{\partial H}{\partial x}(x) - (\frac{\partial C^T}{\partial x}(x))u], \tag{30}$$

which reduces for an IOHDss system to

$$z = (\frac{\partial C^T}{\partial x}(x))^T J(x)[\frac{\partial H}{\partial x}(x) - \frac{\partial C^T}{\partial x}(x)u]$$

Using $u^T(\frac{\partial C^T}{\partial x}(x))^T J(x)\frac{\partial C^T}{\partial x}(x)u = 0$ by skew-symmetry of $J(x)$, it follows that (leaving out arguments x)

$$\frac{d}{dt}H = u^T z -$$
$$\begin{bmatrix}(\frac{\partial H}{\partial x})^T & u^T\end{bmatrix} \begin{bmatrix} R & -R\frac{\partial C^T}{\partial x} \\ -(\frac{\partial C^T}{\partial x})^T R & (\frac{\partial C^T}{\partial x})^T R \frac{\partial C^T}{\partial x}\end{bmatrix} \begin{bmatrix}\frac{\partial H}{\partial x} \\ u\end{bmatrix} \leq u^T z \tag{31}$$

thus proving

Proposition 3.3: The affine nonlinear IOHD system (27) with differentiated output $z = \dot{y}$ is passive with storage function H, and defines a port-Hamiltonian system [16], [15].

Similar to the linear case it is seen that the positive feedback interconnection of two affine nonlinear IOHD systems indexed by $i = 1, 2$, is the affine nonlinear IOHD system

$$\begin{aligned}
\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} &= \left(\begin{bmatrix}J_1(x_1) & 0 \\ 0 & J_2(x_2)\end{bmatrix} - \begin{bmatrix}R_1(x_1) & 0 \\ 0 & R_2(x_2)\end{bmatrix}\right) \\
&\left(\begin{bmatrix}\frac{\partial H_{\text{int}}}{\partial x_1}(x_1, x_2) \\ \frac{\partial H_{\text{int}}}{\partial x_2}(x_1, x_2)\end{bmatrix} - \begin{bmatrix}\frac{\partial C_1^T}{\partial x_1}(x_1) & 0 \\ 0 & \frac{\partial C_2^T}{\partial x_2}(x_2)\end{bmatrix}\begin{bmatrix}e_1 \\ e_2\end{bmatrix}\right) \\
\begin{bmatrix}y_1 \\ y_2\end{bmatrix} &= \begin{bmatrix}C_1(x_1) \\ C_2(x_2)\end{bmatrix},
\end{aligned} \tag{32}$$

with interconnected Hamiltonian H_{int} given by

$$H_{\text{int}}(x_1, x_2) := H_1(x_1) + H_2(x_2) - C_1^T(x_1)C_2(x_2) \tag{33}$$

(compare with [3], Theorem 6)). Furthermore, the positive feedback interconnection of two affine nonlinear IOHDss systems is an affine nonlinear IOHDss system. Like in the linear case, the stability properties of the interconnected system are determined by H_{int}.

Remark 3.4: As in ([3], Theorem 6) the interconnected Hamiltonian $H_{\text{int}}(x_1, x_2)$ can be also used for showing *boundedness* of solutions of the interconnected system; this is e.g. guaranteed if $H_{\text{int}}(x_1, x_2)$ is radially unbounded.

B. General nonlinear IOHD systems

The preceding definitions of *linear* and *affine nonlinear* IOHD systems suggest the following generalization.

Definition 3.5: A *general nonlinear IOHD system* is defined as a system of the form

$$\begin{aligned}
\dot{x} &= (J(x) - R(x))\frac{\partial H}{\partial x}(x, u), \quad u \in \mathbb{R}^m \\
y &= -\frac{\partial H}{\partial u}(x, u), \quad y \in \mathbb{R}^m
\end{aligned} \tag{34}$$

for some function $H(x, u)$, with $R(x), J(x)$ satisfying (28). (This definition reduces to Definition 3.1 by taking $H(x, u) = H(x) - u^T C(x)$.) For $R = 0$ and J defining a *symplectic form* the definition of a general IOHD system amounts to the definition of an input-output Hamiltonian system given in [5] and explored in e.g. [12].

Remark 3.6: The notion of 'internally stored energy', as well as of passivity with respect to the input u and differentiated output $\dot{y}$, is problematic for a general function $H(x, u)$; see e.g. [12] for further discussion.

Remark 3.7: In [2] it has been shown that any *static* nonlinearity of the form $y = -\frac{\partial H}{\partial u}(u)$ has counterclockwise input-output dynamics. The definition of general nonlinear IOHD systems (34) can be regarded to be a dynamic extension of this property.

The positive feedback interconnection of two general nonlinear IOHD systems with Hamiltonians $H_i(x_i, u_i)$ is (under regularity assumptions) again a nonlinear IOHD system, where the interconnected Hamiltonian $H_{\text{int}}(x_1, x_2)$ is constructed as follows. The functions $H_i(x_i, u_i)$ are generating functions for two Lagrangian submanifolds [1], [18], [12] defined as

$$z_i = \frac{\partial H_i}{\partial x_i}(x_i, u_i), \quad y_i = -\frac{\partial H_i}{\partial u_i}(x_i, u_i), \quad i = 1, 2$$

The composition of these two Lagrangian submanifolds through the positive feedback interconnection $u_1 = y_2$, $u_2 = y_1$ defines a subset in the x_1, x_2, z_1, z_2 variables, which is under a transversality condition [6] again a submanifold. Furthermore, it follows [7], [6] that it is again a Lagrangian submanifold. Assuming additionally that it can be parametrized by the x_1, x_2 variables (this corresponds to well-posedness of the interconnection), it thus possesses (at least locally) a generating function $H_{\text{int}}(x_1, x_2)$. Like in the linear case, this interconnected Hamiltonian $H_{\text{int}}(x_1, x_2)$ determines the stability properties of the interconnected system.

The notion of *dc-gain* of a linear IOHD system generalizes to a general nonlinear IOHD system as follows; specializing the approach of [4] to the Hamiltonian case. Consider a general nonlinear IOHD system (34) with Hamiltonian $H(x, u)$. Assume that for any constant input $\bar{u}$ there exists a unique $\bar{x}$ such that

$$\frac{\partial H}{\partial x}(\bar{x}, \bar{u}) = 0 \tag{35}$$

It follows that $\bar{x}$ is an equilibrium of the system for $u = \bar{u}$. Define

$$\bar{y} = \frac{\partial H}{\partial u}(\bar{x}, \bar{u}) \tag{36}$$

Then, see e.g. [18], (35,36) define a Lagrangian submanifold in the space of outputs and inputs $(\bar{y},\bar{u}) \in \mathcal{Y} \times \mathcal{U}$. Assuming additionally that this Lagrangian submanifold can be parametrized by the $\bar{u}$ variables, then there exists (locally) a generating function K such that the relation between $\bar{u}$ and $\bar{y}$ is described as

$$\bar{y} = \frac{\partial K}{\partial \bar{u}}(\bar{u}) \tag{37}$$

We call this relation the *static input-output response* of the IOHD system.

Remark 3.8: Note that for a linear IOHD system (37) reduces to the symmetric linear map $\bar{y} = \left(CQ^{-1}C^T + D\right)\bar{u}$, i.e., to the linear dc-gain (26).

C. A bifurcation perspective and multi-stability

Consider two nonlinear IOHD systems with equilibria x_1^*, x_2^* corresponding to *strict global minima* of $H_1(x_1)$, respectively $H_2(x_2)$. Then the parametrized positive feedback

$$\begin{array}{rcl} u_1 & = & ky_2 \\ u_2 & = & ky_1 \end{array} \tag{38}$$

for $k \geq 0$ results in an interconnected Hamiltonian $H^k_{\text{int}}(x_1,x_2)$, which for k small will have (by continuity) a strict minimum at (x_1^*, x_2^*), corresponding to a stable equilibrium. By increasing k the shape of $H^k_{\text{int}}(x_1,x_2)$ is generally going to change, possibly resulting in multiple local minima, and thus multiple stable equilibria. In a general, non-Hamiltonian, setting this has been studied in [4], where conditions were derived for *multi-stability* of the resulting interconnected system, meaning that for generic initial conditions the system trajectories will always converge to one of those stable equilibria. A main ingredient in these conditions are the static input-output responses as identified before; see [4] for various interesting results.

IV. Divergence of IOHD systems

In this section we will study a particular (but commonly appearing) type of nonlinear IOHD systems where the Poisson structure matrix $J(x)$ corresponds to a *symplectic form*[6] ω. Furthermore, we will first make the additional assumption that $R(x) = 0$; that is, we will study input-output Hamiltonian systems in the sense of [5], [12]. In coordinate-free notation such systems are described as follows.

Let $\mathcal{X}$ denote the state space (necessarily even-dimensional) endowed with a symplectic form ω, that is, a non-degenerate 2-form, satisfying $d\omega = 0$. The total prolongation of ω to the tangent bundle $T\mathcal{X}$ defines a symplectic form on $T\mathcal{X}$, denoted as $\dot{\omega}$, see e.g. [12]. By Darboux's theorem [1] there exist local *canonical* coordinates $(q,p) = (q^1,\cdots,q^n,p^1.\cdots,p^n)$ for $\mathcal{X}$ such that $\omega = \sum_{i=1}^n dp^i \wedge dq^i$. With $(q,p,\dot{q},\dot{p})$ denoting the corresponding natural coordinates for $T\mathcal{X}$ it follows that

$$\dot{\omega} = \sum_{i=1}^n d\dot{p}^i \wedge dq^i + dp^i \wedge d\dot{q}^i$$

[6]This is equivalent to requiring that $J(x)$ has full rank everywhere, and moreover is satisfying the *Jacobi-identity*.

Finally, define the symplectic form $\omega^e := \sum_{j=1}^m du^j \wedge dy^j$ on the product $\mathcal{Y} \times \mathcal{U} = \mathbb{R}^m \times \mathbb{R}^m$ of the output and the input spaces[7]. Then [12] a general input-output Hamiltonian system (34) with J the Poisson structure matrix corresponding to the symplectic form ω and $R = 0$ is defined as a *Lagrangian submanifold* of the symplectic space $T\mathcal{X} \times \mathcal{Y} \times \mathcal{U}$ with the symplectic form $\dot{\omega} - \omega^e$, while the generating function of this Lagrangian submanifold is the Hamiltonian $H(x,u)$.

Considering now two general input-output Hamiltonian systems indexed by $i = 1,2$, it is immediate why the positive feedback interconnection $u_1 = y_2, u_2 = y_1$ will result in a Hamiltonian system. Indeed, the sum of the two external symplectic forms

$$\sum_{j=1}^m du_1^j \wedge dy_1^j + \sum_{j=1}^m du_2^j \wedge dy_2^j \tag{39}$$

will be zero *restricted* to the subspace of $\mathcal{Y}_1 \times \mathcal{U}_1 \times \mathcal{Y}_2 \times \mathcal{U}_2$ defined by the interconnection constraints $u_1 = y_2, u_2 = y_1$. Hence the interconnected system will define a Lagrangian submanifold of the product tangent bundle $T\mathcal{X}_1 \times T\mathcal{X}_2$ with symplectic form $\dot{\omega}_1 + \dot{\omega}_2$, and thus a Hamiltonian vector field.

A. Liouville's theorem

Let us concentrate on *affine* input-output Hamiltonian systems corresponding to a symplectic form ω and $R = 0$. In this case the Hamiltonian $H(x,u)$ is of the form $H(x,u) = H(x) - \sum_{j=1}^m u^j C^j(x)$, and the system is represented in coordinate-free notation as

$$\begin{array}{rcl} \dot{x} & = & X_H(x) - \sum_{i=1}^m u^j X_{C^j}(x) \\ y^j & = & C^j(x), \quad j = 1,\cdots,m \end{array} \tag{40}$$

where X_H is the Hamiltonian vector field defined by the symplectic form ω and the Hamiltonian H; that is $\omega(X_H,\cdot) = -dH$, and similarly for X_{C^j}. In canonical coordinates the Hamiltonian vector field X_H takes the classical form

$$\begin{array}{rcl} \dot{q}^i & = & \frac{\partial H}{\partial p^i}(q,p) \\ \dot{p}^i & = & -\frac{\partial H}{\partial q^i}(q,p) \end{array}, \quad i = 1,\cdots n \tag{41}$$

and similarly for X_{C^j}.

Hamiltonian vector fields have the property that they are volume-preserving with respect to the *volume form* determined by the symplectic form. Define on the $2n$-dimensional phase space $\mathcal{X}$ with symplectic form ω the $2n$-form [1]

$$\Omega := \frac{(-1)^{[n/2]}}{n}\omega^n \tag{42}$$

where $[n/2]$ is the largest integer $\leq \frac{n}{2}$. Since ω as a symplectic form is non-degenerate it follows that Ω is non-degenerate, and thus defines a volume-form. In canonical coordinates, that is $\omega = \sum_{i=1}^n dp^i \wedge dq^i$, Ω equals the standard volume form on the phase space

$$\Omega = dp^1 \wedge dp^2 \wedge \cdots \wedge dp^n \wedge dq^1 \wedge dq^2 \wedge \cdots \wedge dq^n$$

[7]Coordinate-free, and for $\mathcal{Y}$ being a manifold, this can be replaced by the co-tangent bundle $T^*\mathcal{Y}$.

Any Hamiltonian vector field X_H has the property that the Lie-derivative of ω along X_H is zero, since by Cartan's formula

$$L_{X_H}\omega = di_{X_H}\omega + i_{X_H}d\omega = -d(dH) = 0\,, \qquad (43)$$

because $d\omega = 0$. From here it follows that

$$L_{X_H}\Omega = 0, \qquad (44)$$

that is, the *divergence* of X_H with respect to the volume form Ω is zero (commonly known as *Liouville's theorem*). Since the same holds for the divergence of the Hamiltonian input vector fields X_{C^j} with respect to Ω it follows that for any *constant* input function $u : \mathbb{R} \to \mathbb{R}^m$ the system $\dot{x} = X_H(x) - \sum_{i=1}^m u^j X_{C^j}(x)$ leaves the volume-form Ω invariant; that is, is divergence-free. The same holds for any fixed *time-function* $u : \mathbb{R} \to \mathbb{R}^m$.

Remark 4.1: In the presence of a *dissipation term* the volume form Ω is not preserved anymore. For example, the divergence of the vector field

$$\begin{array}{rcl} \dot{q}^i & = & \frac{\partial H}{\partial p^i}(q,p) \\ \dot{p}^i & = & -\frac{\partial H}{\partial q^i}(q,p) - \frac{\partial D}{\partial v^i}(\frac{\partial H}{\partial p}(q,p)) \end{array}, \quad i = 1, \cdots n, \qquad (45)$$

for some Rayleigh function $D(v_1, \cdots, v_n)$ modeling dissipation, is computed as

$$-\ \text{trace}\ (\frac{\partial^2 D}{\partial v^2} \cdot \frac{\partial^2 H}{\partial p^2}) \leq 0 \qquad (46)$$

B. An identity for closed-loop systems

Let us now consider the case of any arbitrary (differentiable) *feedback* $u = \alpha(x)$. Then the closed-loop system $\dot{x} = X_{cl}(x) := X_H(x) - \sum_{j=1}^m \alpha^j(x) X_{C^j}(x)$ satisfies

$$\begin{aligned} L_{X_{cl}}\omega &= L_{X_H - \sum_{j=1}^m \alpha^j X_{C^j}}\omega = \\ &= \sum_{j=1}^m d(\alpha^j dC^j) = \sum_{j=1}^m d\alpha^j \wedge dC^j \end{aligned}$$

This is summarized in the following

Proposition 4.2: For any differentiable feedback $u = \alpha(x)$ the closed-loop system $\dot{x} = X_{cl}(x)$ satisfies

$$L_{X_{cl}}\omega = (C, \alpha)^* \omega_e \qquad (47)$$

where $(C, \alpha) : \mathcal{X} \to \mathcal{Y} \times \mathcal{U}$.

This formula suggests a further connection between the volume in the space $\mathcal{Y} \times \mathcal{U}$ of outputs and inputs and the (change of) volume in the state space. It follows that X_{cl} leaves the symplectic form ω invariant if and only if $(C, \alpha)^*\omega_e = 0$, or equivalently, if the symplectic form ω_e on the space of outputs and inputs is zero restricted to the image of the map $(C, \alpha) : \mathcal{X} \to \mathcal{Y} \times \mathcal{U}$. In particular, if we assume that the rank of the mapping $C : \mathcal{X} \to \mathcal{Y}$ is equal to m, this is the case if and only if the image of (C, α) is a *Lagrangian submanifold*, or equivalently, if there exists (locally) a function $P : \mathcal{Y} \to \mathbb{R}$ such that

$$\alpha(x) = \frac{\partial P}{\partial y}(C(x)), \qquad (48)$$

which corresponds to the addition of an extra energy function $P(C(x))$ to the Hamiltonian $H(x)$.

V. Conclusions

In this paper we have interpreted and extended various results in [2], [3], [4], [8], [10] from a Hamiltonian systems point of view, thereby making a direct relation to the geometrically defined class of systems already introduced and studied in [5], [12], [13], [14].

Current investigations are concerned with the further development of the preliminary results sketched in Section IV, as well as with applications of this theory to nonlinear vibration control (see e.g. [10] for linear vibration control) and metabolic reaction networks (using their port-Hamiltonian formulation as presented in [17]).

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