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A One-Step Procedure for Frequency Response Estimation based on a Switch-Mode Transfer Function Analyzer

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Abstract— **Frequency response analysis is a well established system identification method. In this paper, a simple and efficient method based on the classical Transfer Function Analyzer (TFA) technique is implemented for system identification. The novel algorithm has been called Switch-Mode-TFA. The originality of the proposed approach consists of: i) using a chirp TFA test input instead of a pure sine function, ii) using a variable sampling period instead of the traditional fixed sampling period and iii) a simplified implementation using switched electric circuits. The advantage of our approach is thus threefold: i) the frequency response of the system is obtained for the entire range of frequencies by means of only one identification test; ii) the number of samples per cycle remains constant, independent of the excited frequencies and iii) it can be easily implemented in hardware platforms. The results in both simulation and real-life examples indicate that the proposed Switch-Mode-TFA method can be reliably used for frequency response analysis in many other applications.**

I. Introduction

FREQUENCY domain identification techniques remain a subject which attracts a large number of researchers and engineers worldwide. The excitation of the process with periodic signals (e.g. sinusoids) is an efficient way of extracting accurate information upon the process dynamics from experiments. The analysis of frequency responses has gained an increasing interest in several application areas, e.g. the modeling of mechanical servo systems [1–3].

Frequency response analysis remained popular over other techniques simply because it is easy to apply, flexible and robust. The formulation of an identification criterion in the frequency domain can be useful, especially in those situations where the application of the model dictates a performance evaluation in terms of frequency domain properties. Frequency-domain identification can also provide a first insight into the system to be (later) identified by parametric models.

The common way of formulating an identification problem in the frequency domain is by assuming the availability of the exact frequency response of the (unknown) linear system, disturbed by some additive noise. For this situation a large number of identification methods exist, mostly dealing with least squares criteria [4], [5], [6]. Also subspace algorithms have been analyzed for frequency domain identification [7], and an overview on the various techniques can be found in [2]. A related approach to the problem based on the discrete Fourier transforms of input and output data in [1] shows a close similitude of the obtained results with the standard time-domain approach.

Apart from the theoretical advantage, frequency domain identification poses also a practical advantage; i.e. that of efficient hardware implementation. As such, the well-known transfer function analyzer (TFA) has been one of the first commercial devices for system identification purposes.

In this paper, a frequency domain identification algorithm based on the classical transfer function analyzer technique is implemented. The contribution of this study is to use switch-mode variable sampling period chirp signals instead of the traditional multipliers with fixed sampling period sinusoids. The advantage of this approach is that the same number of samples per cycle is ensured for any range of excited frequencies and that the approach can be easily implemented in hardware applications. Since the target is to obtain the frequency response of the system for the entire range of frequencies using one excitation signal, this approach is robust in terms of numerical complexity. The novel formulation is illustrated by means of simulated examples and real life tests on an electro-mechanical plant.

II. Transfer Function Analyzer

A. *The Classical Approach*

The sinusoidal output of a system in terms of its magnitude, phase and noise, can be written as:

$$y(t) = b\sin(\omega t + \varphi) + n(t) \tag{1}$$

where n is the noise, t is the time, b is the amplitude, ω is the angular frequency and φ is the phase shift, related to the time shift by $\varphi = \omega \cdot \Delta t$ (Fig. 1). The signal $n(t)$ is considered to be a correlated stochastic disturbance with zero average. Problems of non-linear distortion and noise corruption are overcome in the measurement scheme of Fig. 2, where the measured output $y(t)$ is first multiplied by sine and cosine respectively, and then integrated over the measurement period T_m:

$$\begin{aligned} y_s(T_m) &= \int_0^{T_m} y(t)\sin(\omega t)dt = \\ &\int_0^{T_m} b\sin(\omega t + \varphi)\sin(\omega t)dt + \int_0^{T_m} n(t)\sin(\omega t)dt \end{aligned} \tag{2}$$

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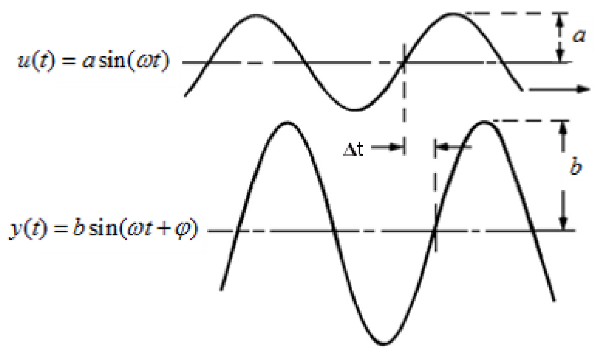

Fig. 1. Sinusoidal input u(t) and sinusoidal response y(t)

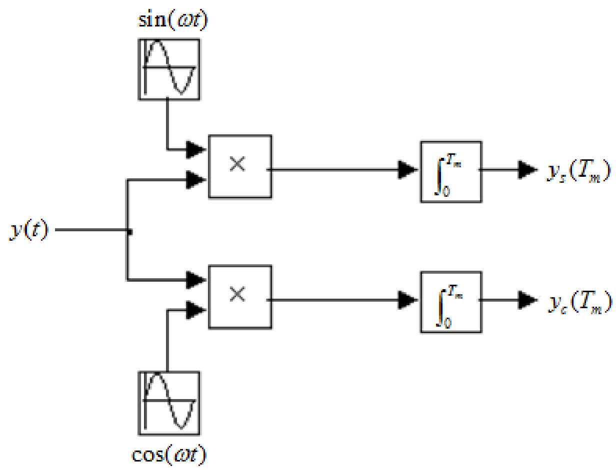

Fig. 2. Classical TFA implementation scheme

Using $\sin(\omega t+\varphi)\sin(\omega t)=\frac{1}{2}[\cos(\varphi)-\cos(2\omega t+\varphi)]$, we have:

$$y_s(T_m)=\frac{b}{2}T_m\cos\varphi-\\ -\frac{b}{2}\int_0^{T_m}\cos(2\omega t+\varphi)dt+\int_0^{T_m}n(t)\sin(\omega t)dt \tag{3}$$

Analogously, from

$$y_c(T_m)=\int_0^{T_m}b\sin(\omega t+\varphi)\cos(\omega t)dt+\int_0^{T_m}n(t)\cos(\omega t)dt \tag{4}$$

and using $\sin(\omega t+\varphi)\cos(\omega t)=\frac{1}{2}[\sin(\varphi)+\sin(2\omega t+\varphi)]$ in the first integral term, one obtains:

$$y_c(T_m)=\frac{b}{2}T_m\sin\varphi+\\ +\frac{b}{2}\int_0^{T_m}\sin(2\omega t+\varphi)dt+\int_0^{T_m}n(t)\cos(\omega t)dt \tag{5}$$

As the averaging time increases, the contribution of the 2nd and 3rd terms in (3) and (5) can be neglected compared to the 1st term, which is growing with T_m. If one integrates over a multiple of half the period for a certain frequency, one can observe that the second term in (3) and (5) will be zero. Also, if we consider the integration time to be long enough, the noise will be filtered out (i.e. zero average). Thus, we can write that:

$$y_s(T_m)\approx\frac{b}{2}T_m\cos\varphi,\quad y_c(T_m)\approx\frac{b}{2}T_m\sin\varphi \tag{6}$$

from where it follows that

$$b=\frac{2}{T_m}\sqrt{y_s^2(T_m)+y_c^2(T_m)}\quad\text{and}\quad\varphi=\arctan\frac{y_c(T_m)}{y_s(T_m)} \tag{7}$$

Plotting the b/a and φ values for several frequency points provides the Bode diagram for the observed system. An example of a system with non-rational transfer function is available in [8], demonstrating the merits of this method implemented in Matlab®Simulink environment.

B. *TFA Concept using a Chirp Signal*

A trivial solution to obtain the frequency response in a single experiment is to use a sequence of sine signals, sweeping from low to high frequencies. When a fixed sampling period is used, the drawback of this simple approach is that lower frequencies will be over-sampled, while higher frequencies will possibly be under-sampled. Ideally, the frequency of such a sinusoidal test signal should vary from a minimum frequency (f_1) until a maximum frequency (f_2) in a certain time (T), known as a chirp signal. In the newly proposed framework, the sampling period of the signal will not be fixed: it will vary, according to frequency, in order to maintain the same sampling resolution for all frequencies. Therefore, a fixed number of data points (N_s) to sample one period will be used, based on prior developments presented in [9].

The frequency sweeping can be done using either linearly-spaced, either logarithmically-spaced frequency points. The linear sweep in the chirp signal will change the frequency according to the formula:

$$f(t)=f_1+\frac{f_2-f_1}{T}t \tag{8}$$

with T the measurement time. Similarly, the logarithmic sweep is based on the formula:

$$\log f(t)=\log f_1+\frac{\log f_2-\log f_1}{T}t \tag{9}$$

with the frequency changing according to

$$f(t)=f_1\left(\frac{f_2}{f_1}\right)^{\frac{t}{T}} \tag{10}$$

The chirp signal is then given by

$$u(t)=\sin\left(2\pi\int_0^t f(x)dx\right)=\sin(\varphi(t)) \tag{11}$$

Notice that if $f(t)$ would be constant ($=f^*$) then

$$u(t)=\sin\left(2\pi\int_0^t f^*dx\right)=\sin(2\pi f^*t) \tag{12}$$

In this formulation, the linear sweep from (8) can be written as:

$$\frac{\varphi(t)}{2\pi}=\int_0^t\left(f_1+\frac{f_2-f_1}{T}x\right)dx=f_1t+\frac{f_2-f_1}{2T}t^2 \tag{13}$$

and using the relation $\int_0^t a^x dx = \frac{a^x}{\ln a}$, we obtain for the logarithmic sweep from (10) that:

$$\frac{\varphi(t)}{2\pi} = \int_0^t f_1 \left(\frac{f_2}{f_1}\right)^{\frac{x}{T}} dx = \frac{f_1 T}{log\frac{f_2}{f_1}}\left[\left(\frac{f_2}{f_1}\right)^{\frac{t}{T}} - 1\right] \qquad (14)$$

III. CHIRP-TFA FOR SAMPLED DATA

A variable sampling time $T_s(t)$ is used. This approach has the advantages that i) it runs much faster and ii) every period has the same number of samples N_s. An example of a chirp signal with fixed number of samples per period is given in Fig. 3.

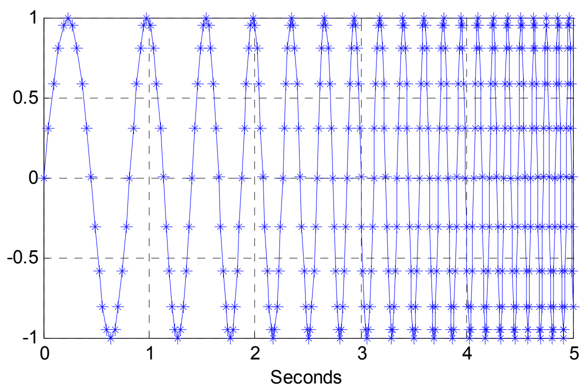

Fig. 3. Illustrative example of a chirp signal from 1 to 10 Hz in 5 seconds, 20 samples per period, logarithmic sweep.

The form of the chirp signal is given by $\sin(2\pi \cdot f \cdot t) = \sin(2\pi \cdot f(t) \cdot kT_s(t))$. Hence, at every time instant t, a variable sampling period $T_s(t)$ is calculated, such that one period contains exactly N_s samples, with k denoting the sample number 0,1,2,…. It is good practice to choose N_s as a multiple of 4, such that the top, bottom and the 2 zero crossings of the squeezed sine correspond exactly with a sample.

PROCEDURE: select $T_s(t)$ such that $f(t) \cdot T_s(t) = \frac{1}{N_s}$.

The N_s samples are then given by $\sin(\frac{2\pi k}{N_s})$, with k=0, 1… N_s-1. Finding $T_s(t)$ requires an iterative algorithm:

i) put t=*t_old* (with *t_old* the time of the last calculated sample; initial value *t_old*=0)

ii) calculate $f(t)$ with either linear or logarithmic formulae [(8) resp. (10)]

iii) calculate $T_s(t) = \frac{1}{N_s f(t)}$,

iv) put t=*t_old* + $T_s(t)$

v) REPEAT steps ii-iii-iv until t remains constant (in practice, it takes 2-3 iterations).

The result is then a new sample k at time t corresponding to the frequency f and with value $\sin(\frac{2\pi k}{N_s})$. Hence, each period will contain an exact number of N_s samples, which will result in a properly sampled chirp signal.

IV. SWITCH-MODE TFA WITH CHIRP SIGNAL

Instead of the traditional TFA algorithm of section IIA, a switch-mode approach is used. Fig. 4 illustrates the concept. It does not require multipliers as in Fig. 2. The signal $n(t)$ is considered to be a correlated stochastic disturbance with zero average.

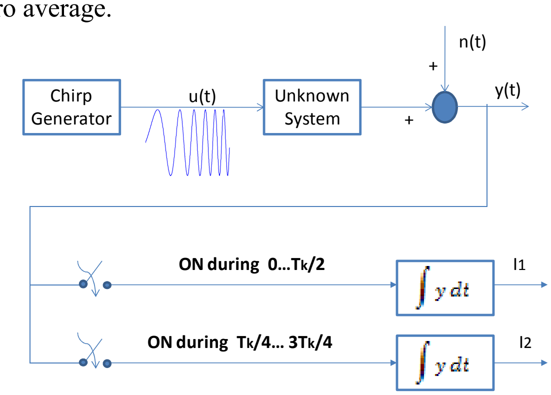

Fig. 4. Schematic overview of the switch-mode chirp TFA.

Every period k of the chirp signal is approximately considered to be a sine signal of the form:

$$u(t) = A\sin(\omega_k t) \qquad (15)$$

with the measured output given by

$$y(t) = B\sin(\omega_k t + \varphi) + n(t) \qquad (16)$$

For each period, two values are calculated, as indicated in Fig. 4. The first one is given by the integral:

$$I_1 = \int_0^{\frac{T_k}{2}} y(t)\ dt =$$

$$\frac{1}{\omega_k}\int_0^{\pi} Bsin(\omega_k t + \varphi) d(\omega_k t) + \int_0^{\frac{T_k}{2}} n(t)dt \qquad (17)$$

which is then

$$I_1 = \frac{B}{\omega_k}(-\cos(\omega_k t + \varphi))|_0^{\pi} + (\sim 0) = \frac{2B}{\omega_k}\cos\varphi \qquad (18)$$

Similarly, the second value is given by the integral:

$$I_2 = \int_{\frac{T_k}{4}}^{\frac{3T_k}{4}} y(t)dt = \frac{-2B}{\omega_k}\sin\varphi \qquad (19)$$

From these relations, we find the modulus and phase of the system:

$$M = \frac{B}{A} = \frac{\pi}{A\,T_k}\sqrt{I_1^2 + I_2^2};\ \ \varphi = arctan\left(\frac{-I_2}{I_1}\right) \qquad (20)$$

which are evaluated for each period of the squeezed sine, using the averaged frequency (resp. averaged period) within each sine. A good design of the parameters T (sweep duration) and N_s (number of samples per period) is crucial for obtaining the correct results. This is further illustrated by a simulation example and a real-life example.

V. Simulated Theoretical Study

A. *Good Design Parameters – no noise*

Consider the transfer function:

$$P(s) = \frac{1}{1+s} \quad (21)$$

For this system, using the design parameters $N_s = 400$, $f_1 = \frac{0.1}{2\pi}$, $f_2 = \frac{10}{2\pi}$ and T=1000, with a logarithmic chirp as an input, we obtain the results given by Fig. 5. As observed, the estimated modulus and phase are practically the same as the real values. Recall here that $N_s = 400$ is the number of samples per period (of each squeezed sine in the chirp signal) and T is the total measurement time. Since the lowest frequency is 0.0159 Hz, corresponding to a period of 62.8 seconds, the measurement time of 1000 seconds is reasonable.

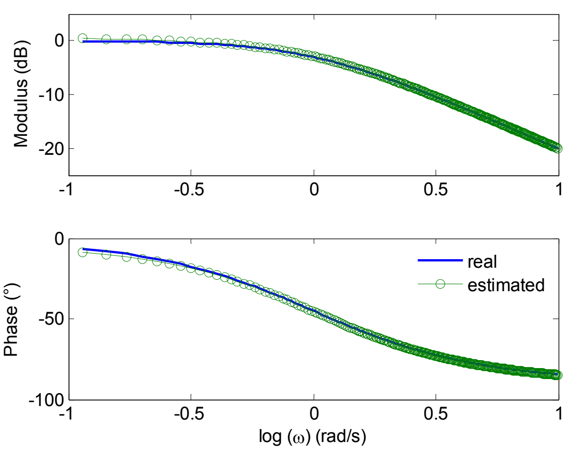

Fig. 5. Perfect estimation of the frequency response of (21) in the absence of noise and optimal choice of the design parameters

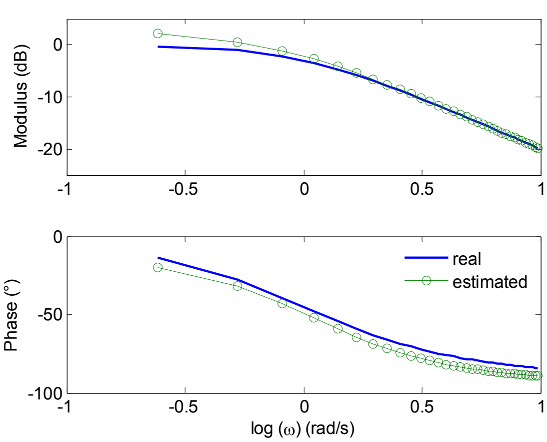

Fig. 6. Poorer result for frequency response estimation of (21) due to sub-optimal design of the tuning parameters

B. *Effect of the Design Parameters – no noise*

The importance of the choice of the design parameters N_s and T can be illustrated by means of a counter-example, namely a poor design. For the same system (21), we apply now a logarithmic chirp signal with $N_s = 40$, $f_1 = \frac{0.1}{2\pi}$, $f_2 = \frac{10}{2\pi}$ and T=100. The results are in Fig. 6.

In this case, we have an overestimation in the modulus at low frequencies due to the fact that the measurement period T=100 is now rather short. This will then result in fast changes in frequency within each period of the squeezed sine, as illustrated in Fig. 7 below. The first periods in the chirp signal are thus far from being close to a pure sinusoid. This then results in the integrals I_1 (18) and I_2 (19) and the modulus M (20) being over-estimated. Notice that it would be possible to alleviate this error by integrating also over [$T_k/2$→T_k] for I_1 and over [0→$T_k/4$ and $3T_k/4$→T_k] for I_2. This would then require more calculations, with the advantage of a shorter measurement time.

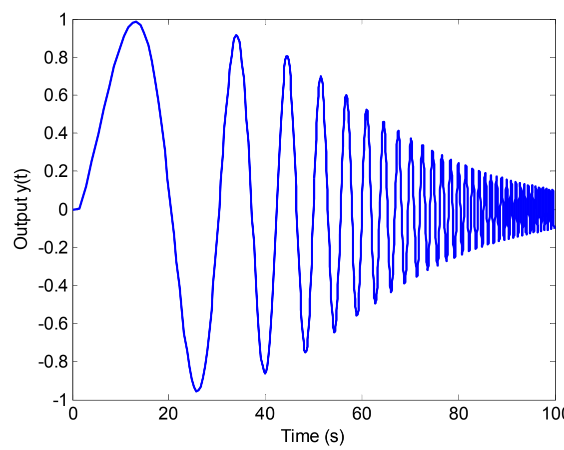

Fig. 7. The output of the system as a result of short measurement time

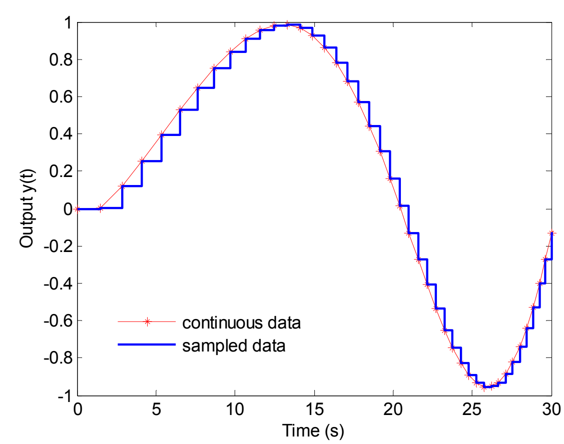

Fig. 8. Illustrative example of the artificial delay (phase lag) introduced by using a zero-order-hold in a sampled-data system.

Also, a bias in the phase is clearly visible in Fig. 6. It is introduced by the fact that the number of samples in one period is too short: $N_s = 40$. It is well-known that the zero-order-hold principle results in an artificial time-delay, corresponding to a supplementary phase lag. This is clearly visible in Fig. 8, where the continuous data and its zero-order-hold equivalent (sampled data) are depicted. From this explanation of the source of the error, it is also obvious how the phase error can be reduced: use more samples per period.

C. *Effect of the noise*

Finally, the no-noise situation described above is of course ideal and it does not apply in practice, because stochastic disturbances usually corrupt the output measurements. To illustrate the efficiency of the method in the presence of significant disturbances with high-amplitude, we introduce a stochastic disturbance, superimposed on the output signal, as shown in detail in Fig. 9. The results of our method are illustrated in Fig. 10, using the design parameters $N_s = 400$, $f_1 = \frac{0.1}{2\pi}$, $f_2 = \frac{10}{2\pi}$ and T=1000, with a logarithmic chirp.

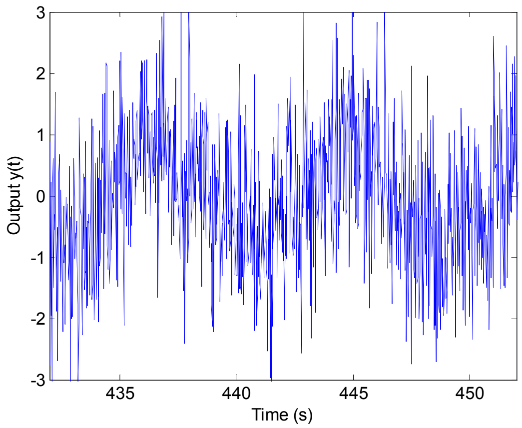

Fig. 9. Illustrative example of an output signal with significant noise superimposed on the output of the system

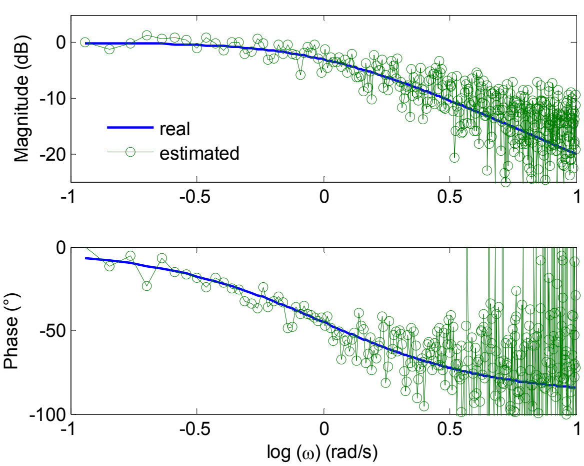

Fig. 10. The noisy estimation of the frequency response of system (21)

In order to improve the noisy results from Fig. 10, a simple averaging over 9 successive frequency samples leads to the frequency response depicted in Fig. 11. It is obvious that acceptable results are being obtained in the pass-band of the system. By averaging over more samples, it would even be possible to obtain also better results in the high frequency band (where indeed more samples are available, due to the logarithmic nature of the sweep).

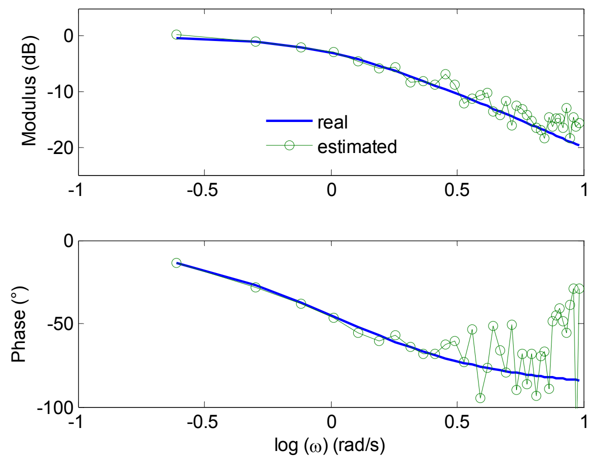

Fig. 11. An improved frequency response estimation for system (21) in the presence of considerable noise, by averaging over 9 samples.

VI. Real-Life Test: A Mass-Spring-Damper System

In this real-life example, the Chirp-Switch-Mode TFA identification method was applied to the electromechanical system illustrated in Figure 12.

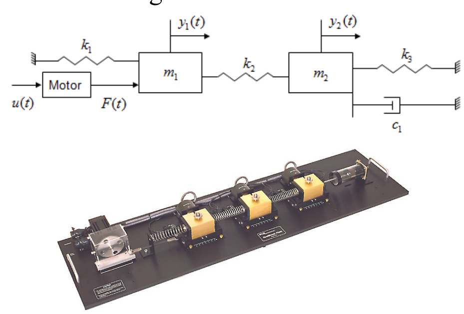

Fig. 12. A scheme of the 2 masses electro-mechanical pilot system and a photo of the generic setup (in our setup, the 3rd mass has been kept fixed and the damper has been connected to the 2nd mass)

The input of the system is the voltage to the motor $u(t)$ and the outputs are the mass displacements $y_1(t)$ and $y_2(t)$ expressed in centimeters. Therefore a complete model of the electromechanical plant should describe the dynamics from $u(t)$ to $y_1(t)$ and from $u(t)$ to $y_2(t)$. The (fast) dynamics of the electrical motor can be neglected; hence, the motor can be represented by a pure static gain $F(t) = K \cdot u(t)$, with $F(t)$ the force on the 1st mass. The parameters of the set-up are: m_1=1.7 Kg, m_2=1.2 Kg, $k_1 = k_2$=800 N/m, k_3=450 N/m, c_1=9 N/(m/s). The two differential equations (Newton's second law) describing the dynamics of the plant are:

$$F(t) = m_1\ddot{y}_1 + k_2(y_1 - y_2) + k_1 y_1 \qquad (22)$$

$$0 = m_2\ddot{y}_2 + c_1\dot{y}_2 + k_3 y_2 - k_2(y_1 - y_2) \qquad (23)$$

Laplace transformation of equations (22) and (23) allows us to derive the transfer function models for this plant:

$$\frac{Y_1(s)}{U(s)} = \frac{\left[\frac{1}{m_1}s^2 + \frac{c_1}{m_1 m_2}s + \frac{(k_2+k_3)}{m_1 m_2}\right]K}{Den} \qquad \frac{Y_2(s)}{U(s)} = \frac{\left[\frac{k_2}{m_1 m_2}\right]K}{Den} \quad (24)$$

$$Den = s^4 + \frac{c_1}{m_2}s^3 + \left[\frac{(k_1+k_2)}{m_1} + \frac{(k_2+k_3)}{m_2}\right]s^2 + \frac{c_1(k_1+k_2)}{m_1 m_2}s + \left[\frac{(k_1+k_2)(k_2+k_3)-k_2^2}{m_1 m_2}\right]$$

It is clear that *Den* represents a fourth order system; hence by taking into account the characteristic equation, we can calculate the resonant frequencies based on the mathematical model. The mathematical model (24) can be rewritten in the form:

$$\frac{Y_i(s)}{U(s)} = K\frac{Num_i}{(s^2 + 2\xi_1\omega_{n1}s + \omega_{n1}^2)(s^2 + 2\xi_2\omega_{n2}s + \omega_{n2}^2)} \quad (25)$$

with i=1,2 and the resonant frequencies calculated as: $\omega_{pi} = \omega_{ni}\sqrt{1-2\xi_i^2}$, for $\xi_i \le 0.707$. Due to the very low damping of the system, we can approximate the resonant frequencies ω_{pi} by the natural frequencies ω_{ni}. Consequently, for the numerical parameter values which were given earlier, the two pairs of complex conjugated poles are $-1.5829 \pm 19.75j$ and $-1.7504 \pm 37.203j$ and the corresponding natural frequencies are: $\omega_{n1} = 19.813$ rad/s, and $\omega_{n2} = 37.244$ rad/sec. The model of the second mass presents two peaks in the frequency response, located at 3.15 Hz and 5.92 Hz. Hence, the frequency range for the chirp signal was chosen from 1.59 Hz to 15.9 Hz in T=20 seconds measurement time with $N_s = 40$ samples in each period. The chirp signal test input and the corresponding system output are illustrated in Fig. 13. The frequency response estimated with our Chirp-Switch-Mode TFA method is shown in Fig. 14 (notice that even the phase-estimation at high frequency is allright taking into account that 0°=-360°).

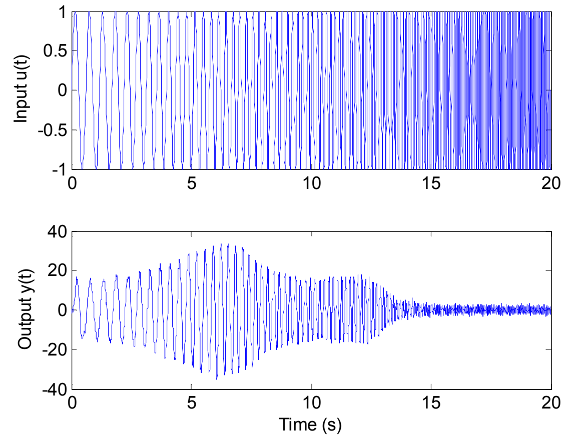

Fig. 13. Input chirp and output signal from the electromechanical system

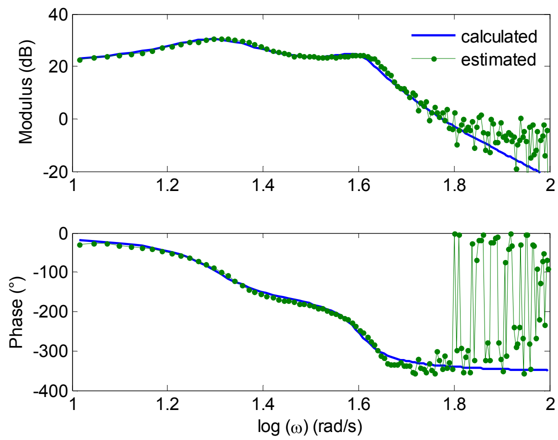

Fig. 14. Calculated and estimated Bode characteristic of the electromechanical system

VII. Conclusions

A novel frequency domain identification method based on the original Transfer Function Analysis (TFA) technique has been presented. It has been illustrated on a simulated system and tested on a real-life system. The new method has been called Chirp-Switch-Mode TFA. The novelty of the method is threefold: i) it uses a chirp test input instead of a pure sine function, ii) it uses a variable sampling period instead of the traditional fixed sampling period and iii) it offers a simple implementation using a switch (instead of the traditional multiplier). The switch implementation would allow for an easy hardware implementation (compared to multipliers).

The use of a variable sampling period - such that a fixed number of samples is ensured in every period for the whole range of frequencies in the chirp signal - would lead to a minimum of computational effort in a software implementation.

It has been shown that the Bode characteristics estimated with the novel Chirp-Switch-Mode TFA method are quite close to the real ones.

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