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# Strong Structural Controllability of Linear Systems Revisited

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***Abstract*— Recently, the concept of *strong structural controllability* has attracted renewed attention. In this context the existing literature to strong structural controllability has been revisited and some of the previous results have been found to be incorrect. Therefore, in this paper an overview of the previous results on strong structural controllability, counter–examples and a new graph–theoretic characterization of strong structural controllability are given.**

## I. Introduction

It is well known that many properties of linear state space models are *generic properties*, in the sense that these properties are typical properties of the associated structured system, e.g. [1]. A linear system is structured if each entry of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of the state space model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad (1)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (2)$$

is either a fixed zero or a free nonzero parameter. (Here, $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^p$ and $\mathbf{y}(t) \in \mathbb{R}^q$). That is, instead of numerically given matrices $\mathbf{M}$, associated *Boolean structure matrices* $[\mathbf{M}]$ are investigated. Moreover, many models of physical and technical systems depend on physical parameters and are structured originally [2].

From a practical point of view, the structured system approach is attractive since it requires neither knowledge of the exact values of parameters nor floating point operations, and hence, is not subject to any numerical errors [3]. Recently, this approach has also found interest in the field of networked control systems [4]–[7].

It has been known for some time that the generic controllability of structured systems is determined solely by the zero-nonzero structure of the pair ($\mathbf{A}$, $\mathbf{B}$) [8], i.e., "almost all systems"[1] of the same structure have identical controllability properties. However, due to numerical cancellation or interrelations between system parameters a system can be uncontrollable despite the fact that it is structurally controllable (see e.g. the PHEV–model in [9]). To avoid this kind of phenomenon, the concept of *strong structural controllability*, which requires that *all* systems with the same structure are controllable, has been introduced in [10].

Recently, the concept of *strong structural controllability* has attracted renewed attention in the course of the controllability analysis of a linear drivetrain model of a parallel hybrid electric vehicle [9]. This analysis has motivated the introduction of the new notion of *dimension of strongly structurally controllable subspaces*, which is a lower bound for the dimensions of the controllable subspaces of all systems with the same structure. In the context of this research the existing literature [2], [10]–[12] to strong structural controllability has been revisited and some of the previous results have been found to be incorrect.

In this paper, an overview of the previous results on strong structural controllability, counter–examples, and a new graph–theoretic characterization of strong structural controllability are given. The remainder of the paper is organized as follows: A brief introduction into the concepts of structural and strong structural controllability of linear systems is given in Section III and Section IV, and the main contribution of this paper is presented in Section V.

## II. Notations and Preliminaries

In the following we use the notations introduced in [2] and [11]. Let $\mathbf{Q}$ be a *square matrix* of order $m$. To $\mathbf{Q}$ one may assign a weighted *digraph* $G(\mathbf{Q})$ as follows:
$G(\mathbf{Q})$ has $m$ vertices $v_1, \ldots, v_m$. There is an edge directed from $v_j$ to $v_i$ associated with the weight $q_{ij}$ if the matrix entry $q_{ij}$ does not vanish.

The *determinant* $\det(\mathbf{Q})$ may be obtained from the collection of the spanning cycle families within $G(\mathbf{Q})$, e.g. [2, Appendix A2.1]. Here a *cycle family* of a digraph $G$ is a set of vertex disjoint cycles and a *spanning cycle family* is a cycle family which covers all the vertices of $G$.

The system (1) is numerically controllable if and only if

$$\text{rank}\,[\mathbf{A} - \lambda\mathbf{I}, \mathbf{B}] = n \quad (3)$$

for all eigenvalues $\lambda$ of the matrix $\mathbf{A}$, [13].

In the context of "structural" analysis, *Boolean structure matrices* $[\mathbf{A}]$, $[\mathbf{B}]$ that represent the zero-nonzero structure of the matrices $\mathbf{A}$ and $\mathbf{B}$, rather than numerically given matrices $\mathbf{A}$ and $\mathbf{B}$, are investigated. A numerical matrix $\mathbf{M}$ is called an *admissible numerical realization* of the structure matrix $[\mathbf{M}]$, $\mathbf{M} \in [\mathbf{M}]$ for short, if $\mathbf{M}$ can be obtained by assigning nonzero numerical values to the non–vanishing entries of $[\mathbf{M}]$.

## III. Structural Controllability

A structured system is called *structurally controllable* if there exists at least one admissible numerical realization. In this case almost all admissible numerical realizations are controllable.

The concept of "Structural Controllability" has been introduced by Lin [8]. His results for single input systems are

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[1]For a detailed discussion of the term "almost all systems", see e.g. [1].

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based on a graph–theoretic and an algebraic approach. For multi–input systems, Shields and Pearson have established the following algebraic criteria by using two special forms of the matrix [**A**, **B**], [14]:

1) A structured system $([\mathbf{A}],[\mathbf{B}])$ is said to be reducible or to be in Form I if there exists a permutation matrix **P** such that

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{P}^T\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}, \quad (4)$$

where $\mathbf{A}_{ij}$ is an $n_i \times n_j$ matrix for $i,j = 1,2$ with $0 < n_1 \le n$ and $n_1 + n_2 = n$, $\mathbf{B}_2$ is a $n_2 \times p$ matrix, and $\mathbf{P}^T$ denotes the transpose of **P**. Otherwise, $([\mathbf{A}],[\mathbf{B}])$ is said to be irreducible.

2) A structured system $([\mathbf{A}],[\mathbf{B}])$ is said to be not of full row rank or to be in Form II, if the generic rank of $[\mathbf{A}\ \mathbf{B}]$ is less than $n$.

Here the generic rank is the maximal rank if the maximum is taken over all numerical realizations. The following Theorem has been established in [14].

*Theorem 1:* The structured system ([**A**],[**B**]) is structurally (or generically) controllable if and only if it is neither in Form I nor in Form II.

For a recent overview of further graph–theoretic criteria, see [1].

## IV. Strong Structural Controllability

The fundamental assumption of the structural approach is that the entries of the matrices **A** and **B** are either fixed zeros or mutually independent free parameters. Given that many practical systems do have interdependent entries [3], [10], that assumption constitutes a serious limitation of the structured approach. To avoid this kind of phenomenon, the concept of *strong structural controllability (strong s–controllability)*, which requires that *all* admissible numerical realizations are controllable, has been introduced in [10], and is formally defined as follows:

*Definition 1:* A class of systems (1) defined by the structured matrices $[\mathbf{A}],[\mathbf{B}]$ is said to be *strongly structurally controllable* if

$$\text{rank}\,[\mathbf{A} - \lambda\mathbf{I}, \mathbf{B}] = n$$

for all admissible numerical realizations $\mathbf{A} \in [\mathbf{A}], \mathbf{B} \in [\mathbf{B}]$.

The first graph–theoretic conditions for strong structural controllability of single–input linear systems have been given by Mayeda and Yamada in [10]. These conditions could easily be extended to the multi–input case in [15]. A straightforward computational implementation of these conditions can only be used for low order systems $(n < 15)$ due to the combinatorial complexity of $\mathcal{O}(2^n)$. This drawback seemed to be eliminated by a result given in [2, Corollary 14.1].

*Theorem 2:* A class of systems characterized by the structured matrix pair ([**A**], [**B**]) is strongly s–controllable if and only if the digraph $G(\mathbf{Q}_1)$ meets the following conditions:

(a) For each state–vertex in $G(\mathbf{Q}_1)$ there is at least one path from one of the $p$ input–vertices to the chosen state vertex.

(b) There is exactly one cycle family of width $n$ in $G(\mathbf{Q}_1)$.

Here the $(n+p)\times(n+p)$ matrix $\mathbf{Q}_1$ is defined by

$$\mathbf{Q}_1 = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{E} & \mathbf{0} \end{bmatrix},$$

where all entries of the $p \times n$ matrix **E** are free nonzero parameters. A cycle family of width $n$ is a cycle family that covers exactly $n$ state–vertices.

Unfortunately, the above result is wrong; Example 2 in [11] shows that (b) is not necessary for strong structural controllability, and the following example shows that (a), and (b) are not sufficient either: The eigenvalue 1 is not controllable if the matrices **A** and **B** of the system 1 are given by

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

However, there are edges connecting the input-vertex to the state vertex 1, and the latter, to any other state vertex. Moreover, it is easily seen that there is only one cycle family of width 4 here, which consists of the self cycles incident with vertices 3 and 4 and the cycle involving the input vertex and the state vertices 1 and 2. Hence, conditions (a) and (b) are fulfilled.

A new graph–theoretic criterion and a new simple algebraic criterion for a strong structural controllability test has been given in [11], together with an algorithm that checks the algebraic conditions within $\mathcal{O}(n^3)$ time. Similar algebraic conditions are given in [12] for the case that there is only one input vector which has exactly one nonzero entry. Here the authors use the notion *qualitative controllability* instead of *strong structural controllability*.

## V. Main results

In [11] the $(n+p)\times(n+p)$ matrix

$$\mathbf{Q}_0 = \begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (5)$$

and the corresponding digraph $G(\mathbf{Q}_0)$ are used to characterize graph–theoretically the property of strong structural controllability.

Here the state–vertices of $G(\mathbf{Q}_0)$ are denoted by $1, 2, \ldots, n$, and the input–vertices by $I1, \ldots, Ip$. Within the digraph $G(\mathbf{Q}_0)$, the diagonal entries $(a_{ii} - \lambda)$ of $\mathbf{Q}_0$ are represented by self cycles incident to the state-vertices $i$. In the following it is important to distinguish between two kinds of self-cycles. If $a_{ii} \neq 0$, the corresponding self cycle will be a "normal–line self cycle". In contrast, if $a_{ii} = 0$, the corresponding self cycle will be a "red bold–line self cycle."

The graph–theoretic characterization of strong s–controllability in [11] is based on an investigation of the minors of $[\mathbf{A} - \lambda\mathbf{I}, \mathbf{B}]$. By deleting $p$ columns of the $n \times (n+p)$ matrix $[\mathbf{A} - \lambda\mathbf{I}, \mathbf{B}]$, $\binom{n+p}{n}$ minors of order $n$ of $[\mathbf{A} - \lambda\mathbf{I}, \mathbf{B}]$ are obtained. The rank condition (3) is met iff at least one of these minors does not vanish.

The $\binom{n+p}{n}$ minors under consideration may be formed as follows:

one — of these minors is just $[\mathbf{A}-\lambda\mathbf{I}]$,

$p \cdot n$ — of these minors are obtained by deleting one column of $\mathbf{A}-\lambda\mathbf{I}$ and inserting one column of $\mathbf{B}$,

$\binom{p}{i}\binom{n}{i}$ — of these minors are obtained by deleting $i$ columns of $\mathbf{A}-\lambda\mathbf{I}$ and inserting $i$ columns of $\mathbf{B}$ $(0 \le i \le p[\le n])$.

This way, $\sum_{i=0}^{p}\binom{p}{i}\binom{n}{i} = \binom{n+p}{n}$ square matrices and the corresponding $n \times n$ minors are obtained.

Each of the $n \times n$ submatrices introduced above corresponds to a digraph which may be obtained by a slight modification of $G(\mathbf{Q}_0)$:

- $[\mathbf{A}-\lambda\mathbf{I}]$ corresponds to $G(\mathbf{A}-\lambda\mathbf{I})$.
- A submatrix formed by deleting the $k$-th column of $[\mathbf{A}-\lambda\mathbf{I}]$ and inserting the $l$-th column of $\mathbf{B}$ corresponds to a digraph obtained from $G(\mathbf{Q}_0)$ by the following modification:
  - delete all edges starting in the state–vertex $k$,
  - delete all input vertices except $Il$,
  - add a "feedback" edge from $k$ to $Il$.
- A submatrix formed by deleting the columns $k_1, \ldots, k_i$ of $[\mathbf{A}-\lambda\mathbf{I}]$ and inserting the columns $l_1, \ldots, l_i$ of $\mathbf{B}$ corresponds to a digraph obtained from $G(\mathbf{Q}_0)$ by the following modification:
  - delete all edges starting in the state–vertices $k_1, \ldots, k_i$,
  - delete all input–vertices different from $Il_1$,. . .,$Il_i$,
  - add $i$ "feedback" edges from $k_1$ to $Il_1, \ldots, k_i$ to $Il_i$, respectively.

These rules can easily be obtained from the following relation: An $n \times n$ minor of $[\mathbf{A}-\lambda\mathbf{I},\mathbf{B}]$ can be formed by deleting $p$ columns $k_1, k_2, ..., k_p$ of $[\mathbf{A}-\lambda\mathbf{I},\mathbf{B}]$. Its value corresponds to the determinant of $\mathbf{Q}_0$ if we have exactly one entry '1' in each of the last $p$ rows of $\mathbf{Q}_0$ in the columns $k_1, k_2, ..., k_p$.

These rules will be applied to a simple single input system in the following.

*Example 1:* We consider a pair $(\mathbf{A}, \mathbf{b})$ with the following structure:

$$\mathbf{A} = \begin{bmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & 0 & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ b_2 \\ 0 \\ 0 \end{bmatrix}. \tag{6}$$

Firstly, we build the compound matrix (5):

$$\mathbf{Q}_0 = \begin{bmatrix} -\lambda & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22}-\lambda & a_{23} & 0 & b_2 \\ 0 & a_{32} & -\lambda & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44}-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

The corresponding digraph $G(\mathbf{Q}_0)$ has 4 state–vertices and one input–vertex and is given in Figure 1.

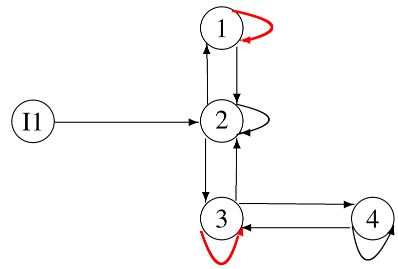


Fig. 1. Digraph $G(\mathbf{Q}_0)$ of Example 1

Here, the red–bold self cycles incident to the state–vertices one and three correspond to the diagonal entries of $[\mathbf{A}-\lambda\mathbf{I}]$ with $a_{ii} = 0$.

By applying the above introduced rules to the digraph in Figure 1 we get the digraphs in the Figures 2-6.

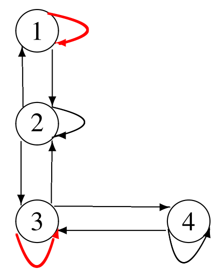


Fig. 2. Digraph $G(\mathbf{A}-\lambda\ \mathbf{I})$ of Example 1

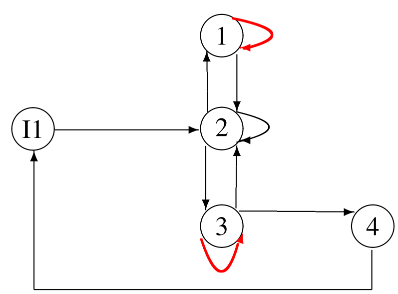


Fig. 3. Digraph of Example 1 after deleting column 4 of $(\mathbf{A}-\lambda\mathbf{I},\mathbf{b})$

The following graph–theoretic characterization in terms of the digraphs corresponding to the $n \times n$ minors of $(\mathbf{A}-\lambda\mathbf{I},\mathbf{B})$ of strong s–controllability for multi–input systems is given in [11]:

*Theorem 3:*

1) A class of systems (1) cannot be strongly s–controllable if each of the $\binom{n+p}{n}$ digraphs introduced above contains more than one spanning cycle family.
2) A class of systems (1) is strongly s–controllable iff at least one of the following two conditions is satisfied:
   a) There is at least one among the $\binom{n+p}{n}$ digraphs that contains exactly one spanning cycle family without self cycles.

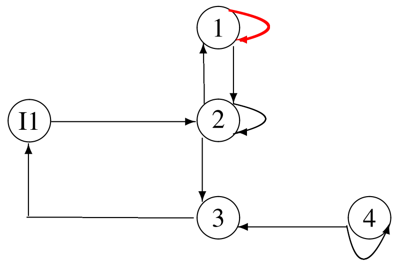


Fig. 4. Digraph of Example 1 after deleting column 3 of $(\mathbf{A}-\lambda\mathbf{I},\mathbf{b})$

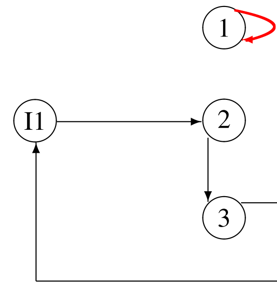


Fig. 7. Unique spanning cycle family of the Digraph of Fig. 3

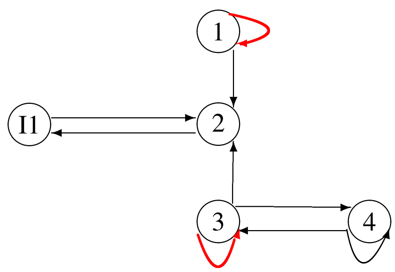


Fig. 5. Digraph of Example 1 after deleting column 2 of $(\mathbf{A}-\lambda\mathbf{I},\mathbf{b})$

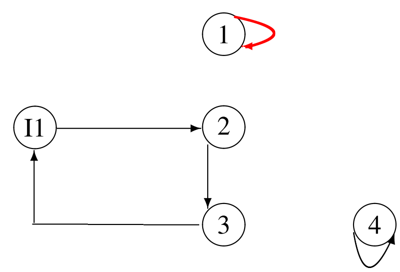


Fig. 8. Unique spanning cycle family of the Digraph of Fig. 4

b) There are two among the $\binom{n+p}{n}$ digraphs which contain exactly one spanning cycle family each, where one family does not involve red bold–line self cycles, and the other does not involve normal–line self cycles.

Now, Theorem 3 is applied to Example 1. The digraph shown in Figure 3 has exactly one spanning cycle family with one red–bold self cycle, which is illustrated in Figure 7.

Hence, condition 1) of Theorem 3 is not fulfilled, and the condition 2) has to be checked. The sufficient condition 2a) of Theorem 3 requires the digraph to contain exactly one spanning cycle family without self cycles. Obviously, the spanning cycle family in Figure 7 does not meet this condition.

The digraph in Figure 4 contains exactly one spanning cycle family which, however, contains one normal self cycle and one red–bold self cycle (see Figure 8).

It is easy to see that the digraphs in Figures 5 and 6 contain two spanning cycle families each. Hence, none of

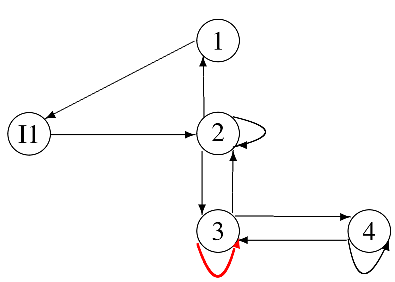


Fig. 6. Digraph of Example 1 after deleting column 1 of $(\mathbf{A}-\lambda\mathbf{I},\mathbf{b})$

the conditions of the second part of Theorem 3 is fulfilled, and by that theorem, Example 1 should not be strongly s–controllable.

Unfortunately, Example 1 is strongly s–controllable and Theorem 3 is incorrect. This can easily be deduced from the digraphs corresponding to the $n \times n$ minors of $[\mathbf{A}-\lambda\mathbf{I},\mathbf{b}]$ in Figures 2 - 6. Figure 3 shows the digraph of $[\mathbf{A}-\lambda\mathbf{I},\mathbf{b}]$ with column 4 removed. This digraph contains exactly one spanning cycle family, which is shown in Figure 7. The corresponding $n \times n$ minor has only one term, which is the product of the edge weights of the family:

$$|\mathbf{A}-\lambda\mathbf{I},\mathbf{b}|_{1,2,3,5} \quad = \quad b_2 \cdot (-\lambda) \cdot a_{32} \cdot a_{43} \ . \tag{8}$$

This minor is nonzero for $\lambda \neq 0$, so it ensures the controllability of all nonzero eigenvalues. In addition, if the red–bold self cycle is removed, the digraph in Figure 6 has exactly one spanning cycle family (see Figure 9). The corresponding

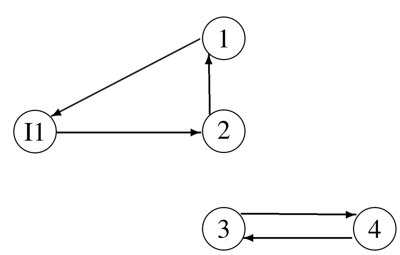


Fig. 9. Unique spanning cycle family of the Digraph of Fig. 6 after deleting the red–bold self cycle

$n \times n$ minor has exactly one term, which is given by the product of the edge weights of the cycle family,

$$|\mathbf{A},\mathbf{b}|_{2,3,4,5} \quad = \quad b_2 \cdot a_{12} \cdot a_{43} \cdot a_{34} \ . \tag{9}$$

This ensures the Hautus condition (3) is fulfilled for $\lambda = 0$ and any admissible realization. Thus, Example 1 is strongly s–controllable.

The same result can be obtained by using the algebraic criteria given in [11]. For this algebraic characterization of strong s–controllability a further special form of the matrix [**A**,**B**] is required:

*Definition 2:* A structured pair ([**A**],[**B**]) is said to be in Form III if there exist two permutation matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ such that

$$\mathbf{P}_1[\mathbf{A},\mathbf{B}]\mathbf{P}_2 \ = \ \begin{bmatrix} \otimes & \dots & \otimes & \times & & & \mathbf{0} \\ \vdots & & \vdots & \ddots & \times & & \\ \vdots & & \vdots & & \ddots & \ddots & \\ \otimes & \dots & \otimes & \dots & \dots & \otimes & \times \end{bmatrix} \tag{10}$$

where the $\times$-entries must be nonzero. The $\otimes$-entries can be either zero or nonzero.

Then the following algebraic characterization of strong s–controllability is given in [11] as a corollary to Theorem 3 (which we have just demonstrated to be incorrect).

*Theorem 4:* The structured pair ([**A**],[**B**]) is strongly s–controllable if and only if

1) the matrix [**A**,**B**] is of Form III, and

2) the matrix $[\mathbf{A}-\lambda\mathbf{I},\mathbf{B}]$ can be transformed into Form III in such a way that the $\times$-entries do not correspond to terms $(a_{ii}-\lambda)$ with $a_{ii} \neq 0$.

It is easy to see that the rank of [**A**,**B**] is equal to $n$ for all admissible realizations of the structured pair ([**A**],[**B**]) if condition 1) of Theorem 4 is fulfilled. It follows that 0 is not an uncontrollable eigenvalue of any admissible realization. On the other hand, condition 2) ensures that admissible realizations do not have nonzero uncontrollable eigenvalues. Thus, ([**A**],[**B**]) is strongly s–controllable. The necessity of the conditions in the above theorem follows from the results in [16], [17].

Now, Theorem 4 is applied to Example 1. A permutation of the rows 2,4 and 2,3 followed by a permutation of the columns 3 and 4 transforms the matrix $[\mathbf{A},\mathbf{b}]$ into Form III,

$$\mathbf{P}_1[\mathbf{A},\mathbf{b}]\mathbf{P_2} = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ 0 & a_{32} & a_{34} & 0 & 0 \\ 0 & 0 & a_{44} & a_{43} & 0 \\ a_{21} & a_{22} & 0 & a_{23} & b_2 \end{bmatrix} . \tag{11}$$

Therefore, condition 1) of Theorem 4 is fulfilled. Note that the diagonal entries of (11) correspond to the weights of the spanning cycle family in Figure 9.

Application of the row permutation [ 4 3 1 2 ] and the column permutation [ 4 3 2 1 5 ] to the matrix

$$[\mathbf{A}-\lambda\mathbf{I},\mathbf{b}] = \begin{bmatrix} -\lambda & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22}-\lambda & a_{23} & 0 & b_2 \\ 0 & a_{32} & -\lambda & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44}-\lambda & 0 \end{bmatrix} \tag{12}$$

yields

$$\begin{bmatrix} a_{44}-\lambda & a_{43} & 0 & 0 & 0 \\ a_{34} & -\lambda & a_{32} & 0 & 0 \\ 0 & 0 & a_{12} & -\lambda & 0 \\ 0 & a_{23} & a_{22}-\lambda & a_{21} & b_2 \end{bmatrix} . \tag{13}$$

Here, the diagonal entries of (13) correspond to the weights of the spanning cycle family in Figure 7.

The permuted matrix (13) is in Form III (10) without terms $(a_{ii}-\lambda)$ with $a_{ii} \neq 0$ on the diagonal. Therefore, condition 2) of Theorem 4 is also fulfilled, and the structured pair ([**A**],[**B**]) of Example 1 is strongly s–controllable.

Hence, there is only a problem with the graph–theoretic conditions of strong s–controllability in [11], and in particular, with the condition 2 b). In Theorem 5 two new graph–theoretic conditions are given in part 2 b):

*Theorem 5:*

1) A class of systems (1) cannot be strongly s–controllable if each of the $\binom{n+m}{n}$ digraphs introduced above contains more than one spanning cycle family.
2) A class of systems (1) is strongly s–controllable iff at least one of the following two conditions is satisfied:
   a) There is at least one of the $\binom{n+m}{n}$ digraphs that contains exactly one spanning cycle family without self cycles.
   b)
   - There is at least one of the $\binom{n+m}{n}$digraphs that contains exactly one spanning cycle family after the red–bold self cycles have been deleted.
   - There is at least one of the $\binom{n+m}{n}$ digraphs that contains exactly one spanning cycle family that involves no normal–line self cycles.

*Proof:*
2 a): The digraphs introduced above correspond to the $n \times n$ minors of $[\mathbf{A}-\lambda\mathbf{I},\mathbf{B}]$. If one of these digraphs has exactly one spanning cycle family without self cycles, then the value of the corresponding minor is a nonzero constant and independent of $\lambda$. Hence, the controllability condition (3) is fulfilled for all admissible numerical realizations $\mathbf{A} \in [\mathbf{A}], \mathbf{B} \in [\mathbf{B}]$.

2 b): If the sufficient condition 2 a) is not fulfilled, the first part of 2 b) ensures that $\lambda = 0$ can not be an uncontrollable eigenvalue of the pair $(\mathbf{A},\mathbf{B})$: After deleting the red–bold self cycles in the digraphs that correspond to the $n \times n$ minors of $[\mathbf{A}-\lambda\mathbf{I},\mathbf{B}]$, these digraphs correspond to the $n \times n$ minors of $[\mathbf{A},\mathbf{B}]$. If one of these digraphs has exactly one spanning cycle family then the corresponding minor of $[\mathbf{A},\mathbf{B}]$ is a nonzero constant for all admissible numerical realizations. Then all admissible numerical realizations of the structured pair ([**A**], [**B**]) fulfill (3) for $\lambda = 0$.

If this condition is fulfilled, a digraph that meets the second part of 2 b) corresponds to an $n \times n$ minor of $[\mathbf{A}-\lambda\mathbf{I},\mathbf{B}]$ that is a polynomial of the form $p_\nu \cdot \lambda^\nu$, with $\nu$ the number of red–bold self cycles contained in the spanning cycle family. Obviously, the corresponding minor is nonzero for all $\lambda \neq 0$ and ensures that all nonzero eigenvalues are controllable. ∎

Finally, a further statement in [11] regarding the algebraic strong s–controllability conditions is incorrect. The statement was, that a system is strongly s–controllable if the matrix [**A**,**B**] is of From III and the $\times$-entries in that same form contain no $a_{ii}$-entry. In this case verification of condition 2) of Theorem 4 would not be necessary.

Obviously, Form III (11) of the strongly s–controllable Example 1 fulfills this condition. Now, when we introduce in Example 1 a further nonzero entry $a_{31}$ the transformed matrix (11) of [**A**,**b**] is still in Form III:

$$\mathbf{P}_1[\mathbf{A},\mathbf{b}]\mathbf{P_2} = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{34} & 0 & 0 \\ 0 & 0 & a_{44} & a_{43} & 0 \\ a_{21} & a_{22} & 0 & a_{23} & b_2 \end{bmatrix}. \quad (14)$$

However, this system is not strongly s–controllable since the condition 2) of Theorem 4 cannot be met. This can be verified by including the nonzero entry $a_{31}$ in the matrix (13):

$$\begin{bmatrix} a_{44}-\lambda & a_{43} & 0 & 0 & 0 \\ a_{34} & -\lambda & a_{32} & a_{31} & 0 \\ 0 & 0 & a_{12} & -\lambda & 0 \\ 0 & a_{23} & a_{22}-\lambda & a_{21} & b_2 \end{bmatrix}. \quad (15)$$

Hence, we have the following result for the modified Example 1:

- None of the admissible realizations of the structured pair ([**A**],[**b**]) can have uncontrollable eigenvalues at $\lambda = 0$ since [**A**,**b**] can be transformed into Form III.
- Admissible realizations of the structured pair ([**A**],[**b**]) with uncontrollable nonzero eigenvalues do exist since the condition 2) of Theorem 4 is not fulfilled.

The modified Example 1 shows also that condition a) of Theorem 2 is only a necessary condition for the nonzero eigenvalues of all admissible realizations of a structured pair ([**A**],[**B**]) to be controllable.

The problem with the modified Example is caused by entry $a_{33} = 0$ in the upper right zero part of (11). Hence, it is not enough to avoid diagonal entries of **A** on the diagonal of the Form III of [**A**, **B**]. The statement of [11] should be corrected to read:

*Corollary 1:* If the $\times$-entries of the Form III of [**A**,**B**] contain no $a_{ii}$-entry and no zero diagonal entries of **A** are in the upper right zero part of Form III of [**A**,**B**], then the system is strongly structurally controllable.

*Proof:* Obviously, the condition 2) of Theorem 4 is fulfilled if Form III of [**A**,**B**] meets the Corollary. ■

## VI. Conclusion

In this paper, the concept of *strong structural controllability* has been revisited. Additionally, counter–examples to previous published graph–theoretic characterizations of strong structural controllability are given. Moreover, we have discussed a new graph–theoretic criterion for strong structural controllability of structured linear systems.

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