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Exponential Stability Analysis of the Drilling System Described by a Switched Neutral Type Delay Equation with Nonlinear Perturbations

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Abstract— **This paper deals with the exponential stability analysis of switched neutral systems under certain state-dependent switching rules with nonlinear perturbations bounded in magnitude. The proposal of an energy functional allow us to investigate the asymptotic and exponential stability of switched neutral systems through the solution of linear matrix inequalities. The results are ilustrated with the exponential stability analysis of an oilwell drilling system allowing a significant reduction of the stick slip behavior.**

I. INTRODUCTION

Switched systems, as an important branch of hybrid control systems, have received great attention of researchers in recent years. A switched systems is a dynamic system that consists of a finite number of subsystems and a logical rule which orchestrates switching between these subsystems. Such systems are useful for modeling various real-world systems such as chemical processes, communication networks, traffic control, manufacturing system control and the oilwell drillstring system studied in this paper.

Switched systems with delay deserve attention because actuators, sensors and transmission lines may introduce time lags. In fact, many models involve not only time delay but also the derivative of the past state, due to the reduction of distributed parameter models into neutral type delay models. In recent years, some stability criteria of switched systems with time delay have been obtained (see for example [7] and [1]). The case of neutral type switched systems is addressed in [5], [9] and [2]. These articles investigate the stability of switched neutral type delay systems provided that all the neutral difference operators are stable, or that there exist Hurwitz linear convex combinations of state matrices, which reduce the scope of the obtained stability conditions.

In this paper we are interested in the stability analysis of switched neutral systems under state depending switching rules and nonlinear perturbations bounded in magnitude. The proposal of an energy functional and the property of strict completeness of matrices allow us to investigate the stability of this particular kind of systems through the solution of linear matrix inequalities. This approach avoids the use of convex combinations of system matrices, and reduces the number of variables.

Our motivation is the exponential stability analysis and stick slip control of an oilwell drilling system. Oilwell drillstrings are mechanisms that play a key role in the petroleum extraction industry. The drilling system is described by an hyperbolic partial differential equation with mixed boundary conditions. Through the D'Alembert method this model can be easily transformed into a formally stable neutral type delay system with autonomous switching which describes the behavior of the system at the ground level.

The paper is organized as follows: In Section II we present the distributed parameter model describing the drilling system and the equivalent neutral type delay model obtained trough the D'Alembert transformation. Section III concerns with the problem formulation, the definition of completeness of matrices is given. In Section IV we develop the strategy to analyze the asymptotic stability of switched neutral type delay systems with bounded nonlinear perturbations, a change of variable allow us to determine the exponential stability conditions. In Section V we present the numerical analysis of the drilling system. Conclusions are presented in the last section.

II. DRILLING SYSTEM MODEL

The main process during well drilling for oil is the creation of borehole by a rock-cutting tool called bit. The drillstring consists of the BHA (bottom hole assembly) and drillpipes screwed end to end to each other to form a long pipe. The BHA comprises the bit, stabilizers which prevent the drillstring from balancing, and a series of pipe sections which are relatively heavy known as drill collars. While the length of the BHA remains constant, the total length of the drill pipes increases as the borehole depth does. An important element of the process is the drilling mud or fluid which among others, has the function of cleaning, cooling and lubricating the bit. The drillstring is rotated from the surface by an electrical motor.

The drill pipe is considered as a beam in torsion. A lumped inertia I_B is chosen to represent the assembly at the bottom hole and a damping $\beta \geq 0$ which includes the viscous and structural damping, is assumed along the structure. The drillstring is rotated from the surface ($\xi = 0$) by an electrical motor, Ω is the angular velocity coming from the rotor that does not match the rotational speed of the load $\frac{\partial \theta}{\partial t}(0, t)$. This sliding speed results in the local torsion of the drillstring. The other extremity ($\xi = L$), is subject to a torque T, which is a function of the bit speed. The mechanical system is described

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by the following partial differential equation:

$$GJ\frac{\partial^2\theta}{\partial\xi^2}(\xi,t) - I\frac{\partial^2\theta}{\partial t^2}(\xi,t) - \beta\frac{\partial\theta}{\partial t}(\xi,t) = 0, \qquad (1)$$
$$\xi \in (0,L),\ t > 0,$$

with boundary conditions

$$\begin{aligned} GJ\frac{\partial\theta}{\partial\xi}(0,t) &= c_a\left(\frac{\partial\theta}{\partial t}(0,t) - \Omega(t)\right); \\ GJ\frac{\partial\theta}{\partial\xi}(L,t) + I_B\frac{\partial^2\theta}{\partial t^2}(L,t) &= -T\left(\frac{\partial\theta}{\partial t}(L,t)\right), \end{aligned}$$

where $\theta(\xi,t)$ is the angle of rotation, I is the inertia, G is the shear modulus and J is the geometrical moment of inertia.

Considering that the damping β is negligible, the distributed parameter model (1) reduces to the unidimensional wave equation. Using the D'Alembert transformation we can describe the drilling behavior with the following neutral type delay equation:

$$\begin{aligned} \ddot{w}(t) - \Upsilon\ddot{w}(t-2\Gamma) + \Psi\dot{w}(t) + \Psi\Upsilon\dot{w}(t-2\Gamma) = \\ -\frac{1}{I_B}T\left(\dot{w}(t)\right) + \frac{1}{I_B}\Upsilon T\left(\dot{w}(t-2\Gamma)\right) \\ +2\Psi\left(\frac{c_a}{c_a+\sqrt{IGJ}}\right)\Omega(t-\Gamma), \end{aligned} \qquad (2)$$

where $\dot{w}(t)$ is the angular velocity at the bottom extremity,

$$\Upsilon = \frac{c_a - \sqrt{IGJ}}{c_a + \sqrt{IGJ}}, \Psi = \frac{\sqrt{IGJ}}{I_B}, \Gamma = \sqrt{\frac{I}{GJ}}L.$$

For the details of the transformation the reader is referred to [4].

Torsional drillstring vibrations appear due to downhole conditions, such as significant drag, tight hole, and formation characteristics. It can cause the bit to stall in the formation while the rotary table continues to rotate. When the trapped torsional energy (similar to a wound-up spring) reaches a level that the bit can no longer resist, the bit suddenly comes loose, rotating and whipping at very high speeds. This stick-slip behavior can generate a torsional wave that travels up the drillstring to the rotary top system. Because of the high inertia of the rotary table, it acts like a fixed end to the drillstring and reflects the torsional wave back down the drillstring to the bit. The bit may stall again, and the torsional wave cycle repeats as explained in [3]. The vibrations can originate problems such as drill pipe fatigue problems, drillstring components failures, wellbore instability. They contribute to drillpipe fatigue and are detrimental to bit life.

The following switched equation introduced in [3] approximates the physical phenomenon at the bottom hole

$$T\left(\dot{w}(t)\right) = c_b\dot{w}(t) + W_{ob}R_b\mu_b\left(\dot{w}(t)\right)sgn\left(\dot{w}(t)\right). \qquad (3)$$

The term $c_b\dot{w}(t)$ is a viscous damping torque at the bit which approximates the influence of the mud drilling and the term $W_{ob}R_b\mu_b sgn(\dot{w}(t))$ is a dry friction torque modelling the bit-rock contact. $R_b > 0$ is the bit radius and $W_{ob} > 0$ the weight on the bit. The bit dry friction coefficient $\mu_b(\dot{w}(t))$ is modeled as

$$\mu_b\left(\dot{w}(t)\right) = \mu_{cb} + (\mu_{sb} - \mu_{cb})e^{-\frac{\gamma_b}{v_f}\dot{w}(t)}, \qquad (4)$$

where $\mu_{sb}, \mu_{cb} \in (0,1)$ are the static and Coulomb friction coefficients and $0 < \gamma_b < 1$ is a constant defining the velocity decrease rate. The constant velocity $v_f > 0$ is introduced in order to have appropriate units.

The friction torque (3)-(4) leads to a decreasing torque-on-bit with increasing bit angular velocity for low velocities which acts as a negative damping (Stribeck effect) and is the cause of stick-slip self-excited vibrations. The exponential decaying behavior of T coincides with experimental torque values.

Due to the stick-slip phenomenon, the angular velocity at the bottom extremity varies between zero and positive values. The sgn function in the model of the torque on the bit leads to represent the neutral type system (2) as a particular class of switched systems.

Setting

$$\begin{aligned} x_1 &= w, \quad x_2 = \dot{w}, \quad x = (x_1\ \ x_2)^T, \\ u(t) &= \Omega(t), \quad \tau_1 = 2\Gamma \quad \tau_2 = \Gamma, \end{aligned}$$

we obtain the following equation which describes the behavior of the oilwell drilling system at the bottom extremity:

$$\begin{aligned} \dot{x}(t) - C\dot{x}(t-\tau_1) = Ax(t) + Bx(t-\tau_1) \qquad (5) \\ +Du(t-\tau_2) + f_{1\sigma}(t,x_2(t)) + f_{2\sigma}(t,x_2(t-\tau_1)),\ \sigma = 1,2, \end{aligned}$$

$x_1(t), x_2(t)$ are the angular position and velocity of the drillstring at the bottom end respectively. The constant matrices A, B, C and D are given by:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\Psi - \frac{c_b}{I_B} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\Upsilon c_b}{I_B} - \Upsilon\Psi \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 0 \\ 0 & \Upsilon \end{pmatrix}, \qquad D = \begin{pmatrix} 0 \\ \Pi \end{pmatrix},$$

with $\Upsilon = \frac{c_a - \sqrt{IGJ}}{c_a + \sqrt{IGJ}}$, $\Psi = \frac{\sqrt{IGJ}}{I_B}$, $\Pi = \frac{2\Psi c_a}{c_a + \sqrt{IGJ}}$ and $\tau_2 = \sqrt{\frac{I}{GJ}}L$, $\tau_1 = 2\tau_2$.

The system (5) is considered as a switched system since the functions $f_{1\sigma}(t,x_2(t))$, $f_{2\sigma}(t,x_2(t-\tau_1))$, $\sigma = 1,2$ are switched according to the following autonomous state-dependent rule,

$$\begin{cases} \text{for} \quad x_2 = 0: \\ f_{11}(t,x_2(t)) = f_{21}(t,x_2(t-\tau_1)) = 0 \\ \\ \text{for} \quad x_2(t) > 0: \\ f_{12}(t,x_2(t)) = -c_1 - c_2e^{-\frac{\gamma_b}{v_f}x_2(t)} \\ f_{22}(t,x_2(t-\tau_1)) = c_1\Upsilon + c_2\Upsilon e^{-\frac{\gamma_b}{v_f}x_2(t-\tau_1)} \end{cases} \qquad (6)$$

with $c_1 = \frac{W_{ob}R_b}{I_B}\mu_{cb}$, and $c_2 = \frac{W_{ob}R_b}{I_B}(\mu_{sb} - \mu_{cb})$.

In the following section we present the strategy to investigate the stability of switched neutral systems with nonlinear perturbations.

III. PROBLEM STATEMENT

Consider the following neutral system with state-dependent switching and nonlinear perturbations:

$$\begin{aligned}\dot{x}(t) - C_\sigma \dot{x}(t-\tau_1) &= A_\sigma x(t) + B_\sigma x(t-\tau_1) \\ &+ D_\sigma u(t) + f_{1\sigma}(t, x(t)) + f_{2\sigma}(t, x(t-\tau_1)) \\ x(t_0+\theta) &= \varphi(\theta), \quad \forall \theta \in [-\tau_1, 0]\end{aligned} \tag{7}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control input, τ_1 is a positive constant time delay, φ is a continuously differentiable initial function. $\sigma \in \{1, 2, ..., N\}$ is a piece-wise constant switching signal. The matrices $(A_\sigma, B_\sigma, C_\sigma)$ are allowed to take values, at an arbitrary time, in the finite set $(A_\sigma, B_\sigma, C_\sigma) \in \{(A_1, B_1, C_1), ..., (A_N, B_N, C_N)\}$. For simplicity only, we consider one delay, however, the results of this paper can be easily extended to the case of multiple constant delays.

We consider that the nonlinear perturbations are bounded in magnitude, i.e. there exist positive constants $\alpha_{1\sigma}$, $\alpha_{2\sigma}$ such that

$$\begin{aligned}\|f_{1\sigma}(t, x(t))\| &\leq \alpha_{1\sigma} \|x(t)\| \qquad \forall t \geq 0, \\ \|f_{2\sigma}(t, x(t-\tau_1))\| &\leq \alpha_{2\sigma} \|x(t-\tau_1)\| \;\; \sigma \in \{1, ..., N\}.\end{aligned} \tag{8}$$

Let $u(t)$ be a state-feedback controller in the form $u(t) = Kx(t-\tau_1)$. Substituting this control law into (7), we obtain the following closed loop system:

$$\begin{aligned}\dot{x}(t) - C_\sigma \dot{x}(t-\tau_1) &= A_\sigma x(t) + \bar{B}_\sigma x(t-\tau_1) \\ &+ f_{1\sigma}(t, x_2(t)) + f_{2\sigma}(t, x_2(t-\tau_1)) \\ x(t_0+\theta) &= \varphi(\theta), \quad \forall \theta \in [-\tau, 0]\end{aligned} \tag{9}$$

where $\bar{B}_\sigma = B_\sigma + D_\sigma K$.

Definition 1: [6] The system of matrices $\{\Psi_i\}$, $i = 1, 2, ..., N$, is said to be strictly complete if for every $x \in R^n \backslash \{0\}$ there is $i \in \{1, 2, ..., N\}$ such that $x^T \Psi_i x < 0$. Let us define $\Omega_i = \{x \in R^n : x^T \Psi_i x < 0\}$, $i = 1, 2, ..., N$. It is easy to show that the system $\{\Psi_i\}$, $i = 1, 2, ..., N$, is strictly complete if and only if

$${}_{i=1}^{N}\Omega_i = R^n \backslash \{0\}. \tag{10}$$

Remark 2: A sufficient condition for the strict completeness of system $\{\Psi_i\}$ is that there exist $\xi_i \geq 0$, $i = 1, 2, ..., N$, such that $\sum_{i=1}^{N} \xi_i > 0$ and $\sum_{i=1}^{N} \xi_i \Psi_i < 0$. If $N = 2$, then the above condition is also necessary for the strict completeness [6].

IV. MAIN RESULTS

In [8] they analyze the asymptotic stability of switched neutral systems, following these ideas we extend the result to a more general class of neutral systems: the switched neutral systems with bounded nonlinear perturbations. Next, we derive conditions for the exponential stability of such systems.

A. Asymptotic stability of the closed-loop system

Theorem 3 (Asymptotic stability): Given a gain matrix K, the switched neutral system (9) with the nonlinear perturbations bounded as in (8) is asymptotically stable if there are symmetric positive definite matrices P, Q_1, Q_2, R_1 such that the set of matrices $\Psi_i, i = 1, N$ is strictly complete, where

$$\Psi_i = \begin{pmatrix} \Psi_{i11} & \sqrt{2}P & \sqrt{2}\alpha_{1i}W & \Psi_{i14} & 0 & \Psi_{i16} \\ * & -I & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & \Psi_{i44} & \alpha_{2i}W & \Psi_{i46} \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & \Psi_{i66} \end{pmatrix}, \tag{11}$$

$$\begin{aligned}\Psi_{i11} &= PA_i + A_i^T P + Q_1 + A_i^T W A_i - R_1 + \alpha_{1i}^2 I \\ &\quad + \alpha_{1i}^2 W + 2A_i^T A_i \\ \Psi_{i14} &= P\bar{B}_i + A_i^T W \bar{B}_i + R_1 + 2A_i^T \bar{B}_i \\ \Psi_{i16} &= PC_i + A_i^T W C_i + 2A_i^T C_i \\ \Psi_{i44} &= -Q_1 + \bar{B}_i^T W \bar{B}_i - R_1 + 2\alpha_{2i}^2 I \\ &\quad + \alpha_{2i}^2 W + 2\bar{B}_i^T \bar{B}_i \\ \Psi_{i46} &= \bar{B}_i^T W C_i + 2\bar{B}_i^T C_i \\ \Psi_{i66} &= -Q_2 + C_i^T W C_i + 2C_i^T C_i \\ \bar{B}_i &= B_i + D_i K \\ W &= Q_2 + \tau_1^2 R_1.\end{aligned}$$

Proof: As in [8], we consider the energy functional

$$\begin{aligned}V(x_t) &= x^T(t)Px(t) + \int_{t-\tau_1}^{t} x^T(s)Q_1 x(s)ds \\ &+ \int_{t-\tau_1}^{t} \dot{x}^T(s)Q_2 \dot{x}(s)ds \\ &+ \tau_1 \int_{-\tau_1}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)R_1 \dot{x}(s)dsd\theta.\end{aligned}$$

Taking the derivative of $V(x_t)$ along the trajectories of any subsystem ith of (9), we have

$$\begin{aligned}\dot{V}(x_t) &= 2x^T(t)P\dot{x}(t) - x^T(t-\tau_1)Q_1 x(t-\tau_1) \\ &+ x^T(t)Q_1 x(t) - \dot{x}^T(t-\tau_1)Q_2 \dot{x}(t-\tau_1) \\ &+ \dot{x}^T(t)\left(Q_2 + \tau_1^2 R_1\right)\dot{x}(t) \\ &- \tau_1 \int_{t-\tau_1}^{t} \dot{x}^T(s)R_1 \dot{x}(s)ds\end{aligned} \tag{12}$$

using the Jensen's inequality we can see that

$$\begin{aligned}-\tau_1 \int_{t-\tau_1}^{t} \dot{x}^T(s)R_1 \dot{x}(s)ds &\leq -{}_{t-\tau_1}^{t}\dot{x}^T(s)ds R_1 {}_{t-\tau_1}^{t}\dot{x}(s)ds \\ &= -(x(t) - x(t-\tau_1))^T R_1 (x(t) - x(t-\tau_1))\end{aligned} \tag{13}$$

Then, substituting (13) into (12) gives

$$\begin{aligned}\dot{V}(x_t) \leq\ & 2x^T(t)P\left[C_i\dot{x}(t-\tau_1)+A_ix(t)+\bar{B}_ix(t-\tau_1)\right] \\ & -x^T(t-\tau_1)Q_1x(t-\tau_1)+x^T(t)Q_1x(t) \\ & -\dot{x}^T(t-\tau_1)Q_2\dot{x}(t-\tau_1) \\ & +\left[C_i\dot{x}(t-\tau_1)+A_ix(t)+\bar{B}_ix(t-\tau_1)\right]^T\cdot W\cdot \\ & \cdot\left[C_i\dot{x}(t-\tau_1)+A_ix(t)+\bar{B}_ix(t-\tau_1)\right] \\ & -(x(t)-x(t-\tau_1))^TR_1(x(t)-x(t-\tau_1))+F_i\end{aligned}$$

where $W := Q_2+\tau_1^2R_1$, and

$$\begin{aligned}F_i =\ & F_i(x_t,f_i) := 2x^T(t)P\left[f_{1i}(\cdot)+f_{2i}(\cdot)\right] \\ & +G_i^TW\left[f_{1i}(\cdot)+f_{2i}(\cdot)\right] \\ & +\left[f_{1i}(\cdot)+f_{2i}(\cdot)\right]^TWG_i \\ & +\left[f_{1i}(\cdot)+f_{2i}(\cdot)\right]^TW\left[f_{1i}(\cdot)+f_{2i}(\cdot)\right], \\ G_i =\ & G_i(x_t) := \left[C_i\dot{x}(t-\tau_1)+A_ix(t)+\bar{B}_ix(t-\tau_1)\right].\end{aligned} \quad (14)$$

We look for an upper bound on F_i. Considering that for any vectors $a,b\in R^n$, the inequality $2a^Tb\leq a^Ta+b^Tb$ is satisfied, and taking into account the bounds (8), we can see that

$$\begin{aligned}2x^T(t)Pf_{1i}(\cdot) &\leq x^T(t)PPx(t)+f_{1i}(\cdot)^Tf_{1i}(\cdot) \\ &\leq x^T(t)PPx(t)+\alpha_{1i}^2x^T(t)x(t),\end{aligned}$$

$$\begin{aligned}2x^T(t)Pf_2(\cdot) &\leq x^T(t)PPx(t)+f_{2i}(\cdot)^Tf_{2i}(\cdot) \\ &\leq x^T(t)PPx(t)+\alpha_{2i}^2x^T(t-\tau_1)x(t-\tau_1),\end{aligned}$$

similarly,

$$\begin{aligned}G_i^TWf_{1i}(\cdot)+f_{1i}^T(\cdot)WG &\leq G_i^TG_i+f_{1i}^T(\cdot)WWf_{1i}(\cdot) \\ &\leq G_i^TG_i+\alpha_{1i}^2x^T(t)WWx(t),\end{aligned}$$

$$\begin{aligned}G_i^TWf_{2i}(\cdot)+f_{2i}^T(\cdot)WG_i &\leq G_i^TG_i+f_{2i}^T(\cdot)WWf_{2i}(\cdot) \\ &\leq G_i^TG_i \\ &\quad +\alpha_{2i}^2x^T(t-\tau_1)WWx(t-\tau_1),\end{aligned}$$

and

$$\begin{aligned}&\left[f_{1i}(\cdot)+f_{2i}(\cdot)\right]^TW\left[f_{1i}(\cdot)+f_{2i}(\cdot)\right]=f_{1i}(\cdot)^TWf_{1i}(\cdot) \\ &+f_{2i}^T(\cdot)Wf_{2i}(\cdot)+f_{1i}(\cdot)^TWf_{2i}(\cdot)+f_{2i}(\cdot)^TWf_{1i}(\cdot) \\ &\leq\alpha_{1i}^2x^T(t)Wx(t)+\alpha_{2i}^2x^T(t-\tau_1)Wx(t-\tau_1) \\ &+\alpha_{1i}^2x^T(t)WWx(t)+\alpha_{2i}^2x^T(t-\tau_1)x(t-\tau_1).\end{aligned}$$

Substituting the above inequalities into (14) yields

$$\begin{aligned}F_i \leq\ & 2x^T(t)PPx(t)+\alpha_{1i}^2x^T(t)x(t) \\ & +2\alpha_{2i}^2x^T(t-\tau_1)x(t-\tau_1)+2G_i^TG_i \\ & +2\alpha_{1i}^2x^T(t)WWx(t)+\alpha_{2i}^2x^T(t-\tau_1)WWx(t-\tau_1) \\ & +\alpha_{1i}^2x^T(t)Wx(t)+\alpha_{2i}^2x^T(t-\tau_1)Wx(t-\tau_1).\end{aligned}$$

Then, the derivative of $V(x_t)$ along the trajectories of any subsystem ith of (9) satisfies

$$\begin{aligned}\dot{V}(x_t) \leq\ & 2x^T(t)PG_i-x^T(t-\tau_1)Q_1x(t-\tau_1) \\ & +x^T(t)Q_1x(t)-\dot{x}^T(t-\tau_1)Q_2\dot{x}(t-\tau_1)+G_i^TWG_i \\ & -(x(t)-x(t-\tau_1))^TR_1(x(t)-x(t-\tau_1)) \\ & +2x^T(t)PPx(t)+\alpha_{1i}^2x^T(t)x(t) \\ & +2\alpha_{2i}^2x^T(t-\tau_1)x(t-\tau_1)+2G_i^TG_i \\ & +2\alpha_{1i}^2x^T(t)WWx(t)+\alpha_{2i}^2x^T(t-\tau_1)WWx(t-\tau_1) \\ & +\alpha_{1i}^2x^T(t)Wx(t)+\alpha_{2i}^2x^T(t-\tau_1)Wx(t-\tau_1).\end{aligned}$$

Setting $\xi(t)=(x^T(t)\quad x^T(t-\tau_1)\quad \dot{x}^T(t-\tau_1))$, the above inequality is written as

$$\dot{V}(x_t)\leq\xi(t)\Phi_i(P,Q_1,Q_2,R_1)\xi^T(t) \quad (15)$$

where

$$\Phi_i=\begin{pmatrix}\Phi_{i11} & \Phi_{i12} & \Phi_{i13} \\ * & \Phi_{i22} & \Phi_{i23} \\ * & * & \Phi_{i33}\end{pmatrix}, \quad (16)$$

$$\begin{aligned}\Phi_{i11} &= PA_i+A_i^TP+Q_1+A_i^TWA_i-R_1+2PP \\ &\quad +\alpha_{1i}^2I+2\alpha_{1i}^2WW+\alpha_{1i}^2W+2A_i^TA_i \\ \Phi_{i12} &= P\bar{B}_i+A_i^TW\bar{B}_i+R_1+2A_i^T\bar{B}_i \\ \Phi_{i13} &= PC_i+A_i^TWC_i+2A_i^TC_i \\ \Phi_{i22} &= -Q_1+\bar{B}_i^TW\bar{B}_i-R_1+2\alpha_{2i}^2I+\alpha_{2i}^2WW \\ &\quad +\alpha_{2i}^2W+2\bar{B}_i^T\bar{B}_i \\ \Phi_{i23} &= \bar{B}_i^TWC_i+2\bar{B}_i^TC_i \\ \Phi_{i33} &= -Q_2+C_i^TWC_i+2C_i^TC_i \\ \bar{B}_i &= B_i+D_iK \\ W &= Q_2+\tau_1^2R_1.\end{aligned}$$

By Schur's complement it follows that $\Phi_i<0$ in (16) is equivalent to $\Psi_i<0$ in (11). Let us set $\Omega_i=\{x\in R^3 : x^T\Psi_i(P,Q_1,Q_2,R_1)x<0\}$. Then by strict completeness of the system of matrices $\Psi_i(P,Q_1,Q_2,R_1)$, it follows from (10) that $\bigcup_{i=1}^N\Omega_i=R^3\backslash\{0\}$. Define the sets $\tilde{\Omega}_1=\Omega_1,\quad \tilde{\Omega}_i=\Omega_i\backslash\bigcup_{j=1}^{i-1}\tilde{\Omega}_j,\quad i=2,3,...N$. Its clear that $\bigcup_{i=1}^N\tilde{\Omega}_i=R^3\backslash\{0\}$, $\tilde{\Omega}_i\cap\tilde{\Omega}_j=\emptyset,\quad i\neq j$. Consequently, for any $(x^T(t)\quad x^T(t-\tau_1)\quad \dot{x}^T(t-\tau_1))^T\in R^3,\ t\geq 0$, there exists $i\in\{1,2,...,N\}$ such that $(x^T(t)\quad x^T(t-\tau_1)\quad \dot{x}^T(t-\tau_1))^T\in\tilde{\Omega}_i$. For the switching rule $\sigma(x(t))=i$ whenever $x(t)\in\tilde{\Omega}_i$, from (15) we have $\dot{V}(x_t)\leq\xi(t)\Psi_i(P,Q_1,Q_2,R_1)\xi^T(t)<0$, this implies that the system is asymptotically stable. ■

B. Exponential stability of the closed loop system

The closed loop system (9) is said to be $\alpha-$stable or "exponentially stable" with the rate α if there exists a scalar $\kappa\geq 1$ such that for any continuously differentiable initial condition φ, the solution $x(t,t_0,\varphi)$ satisfies:

$$|x(t,t_0,\varphi)|\leq\kappa|\varphi|e^{-\alpha(t-t_0)}.$$

Using the change of variable: $z(t) := e^{\alpha t}x(t)$, we can rewrite the system (9) as

$$\begin{aligned}\dot{z}(t) - C_\sigma e^{\alpha\tau_1}\dot{z}(t-\tau_1) = \bar{A}_\sigma z(t) + e^{\alpha\tau_1}\bar{B}_\sigma z(t-\tau_1)\\ -\alpha e^{\alpha\tau_1}C_\sigma z(t-\tau_1) + f_{1\sigma}(t,z(t)) + f_{2\sigma}(t,z(t-\tau_1))\\ x(t_0+\theta) = \varphi(\theta), \quad \forall\theta\in[-\tau,0] \qquad (17)\end{aligned}$$

where $\bar{A}_\sigma = A_\sigma + \alpha I$ and $\bar{B}_\sigma = B_\sigma + D_\sigma K$. Notice that the condition (8) on the perturbations imply

$$\begin{aligned}\|f_{1\sigma}(t,z(t))\| &\leq \alpha_{1\sigma}\|z(t)\| \quad \forall t\geq 0,\\ \|f_{2\sigma}(t,z(t-\tau_1))\| &\leq \alpha_{2\sigma}\|z(t-\tau_1)\| \quad \sigma\in\{1,...,N\}.\end{aligned}$$

Our proposal is to find conditions for which the solution $z=0$ of the transformed system (17) is stable. Clearly, these conditions will assure the exponential stability of the original system (9). Applying Theorem 3 yields the following result.

Theorem 4 (Exponential stability): Given a gain matrix K, the switched neutral system (9) with the nonlinear perturbations bounded as in (8) is exponentially stable if there are symmetric positive definite matrices P, Q_1, Q_2, R_1 such that the set of matrices $\Psi_i, i=1,N$ is strictly complete, where

$$\Psi_i = \begin{pmatrix} \Psi_{i11} & \sqrt{2}P & \sqrt{2}\alpha_{1i}W & \Psi_{i14} & 0 & \Psi_{i16}\\ * & -I & 0 & 0 & 0 & 0\\ * & * & -I & 0 & 0 & 0\\ * & * & * & \Psi_{i44} & \alpha_{2i}W & \Psi_{i46}\\ * & * & * & * & -I & 0\\ * & * & * & * & * & \Psi_{i66}\end{pmatrix}. \qquad (18)$$

Here

$$\begin{aligned}\Psi_{i11} &= P\bar{A}_i + \bar{A}_i^TP + Q_1 + \bar{A}_i^TW\bar{A}_i - R_1\\ &\quad +\alpha_{1i}^2I + \alpha_{1i}^2W + 2\bar{A}_i^T\bar{A}_i\\ \Psi_{i14} &= e^{\alpha\tau_1}P\left(\bar{B}_i-\alpha C_i\right) + e^{\alpha\tau_1}\bar{A}_i^TW\left(\bar{B}_i-\alpha C_i\right)\\ &\quad +R_1 + 2e^{\alpha\tau_1}\bar{A}_i^T\left(\bar{B}_i-\alpha C_i\right)\\ \Psi_{i16} &= e^{\alpha\tau_1}PC_i + e^{\alpha\tau_1}\bar{A}_i^TWC_i + 2e^{\alpha\tau_1}\bar{A}_i^TC_i\\ \Psi_{i44} &= -Q_1 + e^{2\alpha\tau_1}\left(\bar{B}_i-\alpha C_i\right)^TW\left(\bar{B}_i-\alpha C_i\right) - R_1\\ &\quad +2\alpha_{2i}^2I + \alpha_{2i}^2W + 2e^{2\alpha\tau_1}\left(\bar{B}_i-\alpha C_i\right)^T\left(\bar{B}_i-\alpha C_i\right)\\ \Psi_{i46} &= e^{2\alpha\tau_1}\left(\bar{B}_i-\alpha C_i\right)^TWC_i + 2e^{2\alpha\tau_1}\left(\bar{B}_i-\alpha C_i\right)^TC_i\\ \Psi_{i66} &= -Q_2 + e^{2\alpha\tau_1}C_i^TWC_i + 2e^{2\alpha\tau_1}C_i^TC_i\\ \bar{A}_i &= A_i + \alpha I\\ \bar{B}_i &= B_i + D_iK\\ W &= Q_2 + \tau_1^2R_1\end{aligned}$$

V. NUMERICAL RESULT

The behavior of the drilling system at the bottom end is described by the neutral-type equation (5) in which the nonlinear part of the function that describes the torque on the bit: $W_{ob}R_b\left(\mu_{cb} + (\mu_{sb}-\mu_{cb})e^{-\frac{\gamma_b}{v_f}x_2}\right)sgn\,(x_2)$, is considered as a perturbation of the system. This nonlinear function is a switching function depending on the angular velocity at the bottom end of the drillstring, x_2.

If we approximate the switching rule (6) for the functions $f_{1\sigma}$, $f_{2\sigma}$ of the system (5) by the following one,

$$\begin{cases}\text{for } 0\leq x_2(t) < 0.1:\\ f_{11}(t,x_2(t)) = f_{21}(t,x_2(t-\tau_1)) = 0\\ \\ \text{for } x_2(t) > 0.1\\ f_{12}(t,x_2(t)) = -c_1 - c_2e^{-\frac{\gamma_b}{v_f}x_2(t)}\\ f_{22}(t,x_2(t-\tau_1)) = c_1\Upsilon + c_2\Upsilon e^{-\frac{\gamma_b}{v_f}x_2(t-\tau_1)}\end{cases} \qquad (19)$$

with $c_1 = \frac{W_{ob}R_b}{I_B}\mu_{cb}$, and $c_2 = \frac{W_{ob}R_b}{I_B}(\mu_{sb}-\mu_{cb})$, then, the conditions (8) on $f_{12}(t,x_2(t))$ and $f_{22}(t,x_2(t-\tau_1))$ are satisfied for some relatively small constants α_1, α_2. The approximate switching law (19) means that for small values of the angular velocity at the bottom end ($x_2 < 0.1 rad/seg$) the nonlinear part of the torque on the bit has no effect (this actually happens when $x_2 = 0$).

According to (19) we have that for $0\leq x_2(t) < 0.1$

$$\begin{aligned}\|f_{11}(t,x_2(t))\| &= 0\leq\alpha_1\|x_2(t)\| \qquad (20)\\ \|f_{21}(t,x(t-\tau_1))\| &= 0\leq\alpha_2\|x_2(t-\tau_1)\|\end{aligned}$$

and for $x_2(t)\geq 0.1$

$$\begin{aligned}\|f_{12}(t,x_2(t))\| &= \left\|-c_1 - c_2e^{-\frac{\gamma_b}{v_f}x_2(t)}\right\| \qquad (21)\\ &\leq \alpha_1\|x_2(t)\|\\ \|f_{22}(t,x(t-\tau_1))\| &= \left\|c_1\Upsilon + c_2\Upsilon e^{-\frac{\gamma_b}{v_f}x_2(t-\tau_1)}\right\|\\ &\leq \alpha_2\|x_2(t-\tau_1)\|.\end{aligned}$$

where $\Upsilon = \frac{c_a-\sqrt{IGJ}}{c_a+\sqrt{IGJ}}$, $c_1 = \frac{W_{ob}R_b}{I_B}\mu_{cb}$, and $c_2 = \frac{W_{ob}R_b}{I_B}(\mu_{sb}-\mu_{cb})$.

The model parameters used in the sequel are:

$$\begin{array}{ccc} G = 79.3x10^9N/m^2, & I = 0.095Kg\cdot m, & L = 1172m,\\ J = 1.19x10^{-5}m^4 & Rb = 0.155575, & v_f = 1,\\ W_{ob} = 97347N, & I_B = 89Kgm^2 & c_a = 2000Nms,\\ \mu_{cb} = 0.5, & \mu_{sb} = 0.8, & \gamma_b = 0.9\\ c_b = 0.03Nms/rad. & & \end{array} \qquad (22)$$

and the simulations are performed using the variable step Matlab-Simulink solver ode45 (Dormand Prince Method).

Using the above parameters, the matrices A, B, C and D of the oilwell drilling model (5) take the following values:

$$A = \begin{pmatrix} 0 & 1\\ 0 & -3.3645\end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0\\ 0 & -2.4878\end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 0\\ 0 & 0.7396\end{pmatrix}, \quad D = \begin{pmatrix} 0\\ 5.8523\end{pmatrix},$$

the time delays are $\tau_2 = 0.3719$ and $\tau_1 = 2\tau_2$, and the constants $c_1 = 85.0829$, $c_2 = 51.0498$, $\Upsilon = 0.7396$. The conditions (20)-(21) are satisfied for all $\alpha_1 > 1317.1$, $\alpha_2 > 974.3$.

Simulation results for the system (5)-(6) with $u(t) = 15rad/s$ presented in Figure 1 show the stick-slip phenomenon of the drilling system. The vibrations of the drillstring lead to fatigue and diminish the accuracy of the drilling process. Thus, control actions are necessary in order to induce the suppression of this undesirable behavior. We propose a stabilizing control law that ensures the exponential convergence of the trajectory $x_2(t)$ of the drilling system (velocity at the bottom end) and consequently the suppression of the stick-slip phenomenon.

For stability issues the velocity at the bottom end must track the angular velocity at the upper part.

In [10] the wave equation describing the torsional behavior of a flexible rod with a mass interpreted as a linear delay system is analyzed. A formally stable neutral model of the form $\dot{x}(t) - C\dot{x}(t-\tau_1) = Ax(t) + Bx(t-\tau_1) + Du(t)$, where A, B, C, and D are given constant matrices is obtained, and the stabilizing control law: $u(t) = \lambda\dot{x}(t-\tau_1) + v(t)$ is studied. Here λ is a constant matrix of the form $\lambda = \begin{pmatrix} 0 & -\lambda_0 \end{pmatrix}$, with $\lambda_0 \in (0,2)$, and $v(t)$ is designed on the basis of $x(t-\tau_1)$.

In order to achieve the velocity tracking we could propose the control law $u(t) = \lambda\dot{x}(t-\tau_2) + Kx(t-\tau_1)$, where $K = \begin{pmatrix} 0 & -\lambda_1 \end{pmatrix}$. Then, the closed loop drilling system is:

$$\dot{x}(t) - (C + D\lambda)\,\dot{x}(t-\tau_1) = Ax(t) + (B + DK)\,x(t-\tau_1) + Dr(t-\tau_2) + f_{1\sigma}(t, x_2(t)) + f_{2\sigma}(t, x_2(t-\tau_1)). \quad (23)$$

We can apply the result of Theorem 4 to analyze the exponential stability of the closed loop 'switched' system (23)-(19). Notice that α_1 and α_2 satisfying (21) also satisfy (20). In this case if the matrix (18) is negative definite, then system (23)-(19) is exponentially stable.

After computing the LMI $\Psi < 0$ (Ψ given in Theorem 4) for $\lambda_0 = 0.05$, $\lambda_1 = 0.36$, $\alpha_1 = 1320$, $\alpha_2 = 975$ and $\alpha = 0.6$, we can conclude that the closed loop system (23) with the switching law (19) where the functions $f_{1\sigma}(t, x_2(t))$, $f_{2\sigma}(t, x_2(t-\tau_1))$ satisfy the conditions (20)-(21) is exponentially stable for the parameters values given in (22).

The simulation results of Figure 2 show the expected exponential convergence of the variable $x_2(t)$ of the system (5)-(6) in closed loop with the control law

$$u(t) = \lambda\dot{x}(t-\tau_2) + Kx(t-\tau_2) + r(t) \quad (24)$$

where $r(t)$ is the angular velocity reference.

VI. CONCLUSION

The exponential stability of switched neutral systems with bounded nonlinear perturbations is investigated in this paper. The proposal of an energy functional and the property of the strict completeness of matrices, allowed us to investigate the stability through the solution of linear matrix inequalities. This approach lead us to determine the exponential stability of the response of the drilling system in closed loop with a proposed control law that reduces substantially the stick-slip phenomenon.

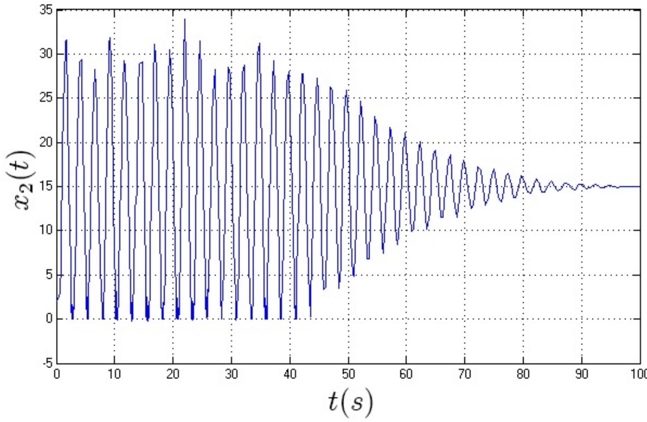

Fig. 1. Simulation of trajectory $x_2(t)$ of the drilling system (5)-(6) for $u(t) = 15rad/s$.

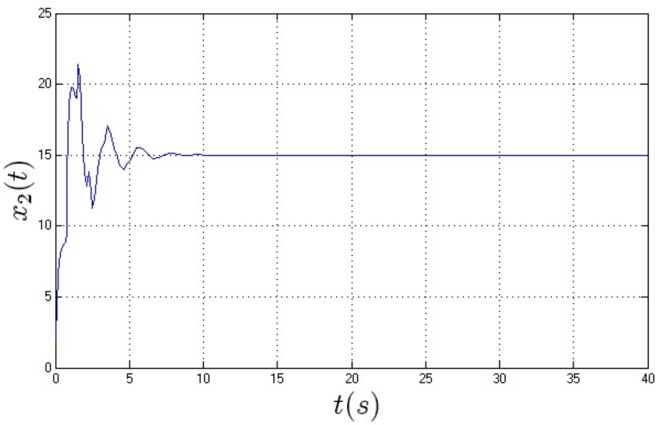

Fig. 2. Simulation of trajectory $x_2(t)$ of the drilling system (5)-(6) in closed loop with the control law (24) for $r(t) = 15rad/s$.

REFERENCES

[1] S. Kim, S. A. Campbell and X. Liu, Stability of a class of linear switching systems with time delay, IEEE Trans. Circuit Syst. 1 53 (2) (2006) 384–393.

[2] D.Y. Liu, X.Z. Liu, S.M. Zhong, Delay-dependent robust stability and control synthesis for uncertain switched neutral systems with mixed delays, Applied Mathematics and Computation, 2008;202(2):828-839.

[3] E. Navarro, R. Suárez, Practical approach to modelling and controlling stick-slip oscillations in oilwell drillstrings, Proceedings of the 2004 IEEE International Conference on Control Applications, pp. 1454–1460.

[4] M. B. Saldivar, S. Mondié, J. J. Loiseau, Reducing stick-slip oscillations in oilwell drillstrings, 6th International Conference on Electrical Engineering, Computing Science and Automatic Control, CCE, pp. 1-6 Toluca, México, 2009.

[5] X.M. Sun, J. Fu, H.F. Sun and J. Zhao, Stability of linear switched neutral delay systems, Proceedings of The Chinese Society for Electrical Engineering, 2005;25(23):42-46.

[6] F. Uhlig, A recurring theorem about pairs of quadratic forms and extensions, Linear Algebra with application, 25(1979):219-237.

[7] C. H. Wang, L. X. Zhang, H. J. Gao and L. G. Wu, Delay-dependent stability and stabilization of a class of linear switched time-varying delay systems, Proceeding of the Fourth ICMLC, Guangzhou, 18–21 August 2005, pp. 917–922.

[8] L. Xiong, S. Zhong and Mao Ye, New stability analysis for switched neutral systems, 2009 Second International Symposium on Computational Intelligence and Design.

[9] Y. Zhang, X. Liu, H. Zhu, Stability analysis and control synthesis for a class of switched neutral systems. Applied Mathematics and Computation, 2007;190: 1258-1266.

[10] M. Fliess, H. Mounier, P. Rouchon and J. Rudolph, Controllability and motion planning for linear delay systems with an application to a flexible rod, Proceedings of the 34th Conference on Decision & Control, TA16 10:40 New Orleans, LA - December 1995.