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Tube MPC Scheme based on Robust Control Invariant Set with Application to Lipschitz Nonlinear Systems

Shuyou Yu, Hong Chen, Frank Allgöwer

***Abstract*— The paper presents a tube model predictive control (MPC) scheme of continuous-time nonlinear systems based on robust control invariant set. The optimization problem considered has a general cost functional rather than the quadratic one. The scheme has the same online computational burden as the standard MPC with guaranteed nominal stability. Robust stability, as well as recursive feasibility, is guaranteed if the optimization problem is feasible at the initial time instant. Furthermore, an optimization based control scheme is proposed, which inherits the robust properties of the tube MPC scheme. The related optimization problem is solved only at the initial time instant. In particular, we consider a scheme to obtain robust control invariant set for Lipschitz nonlinear systems, and show the effectiveness of the proposed schemes by a simple example.**

I. Introduction

In order to achieve robustness of the obtained closed-loop systems, a controller must stabilize the considered system for all possible realizations of the uncertainty. In model predictive control (MPC), an intuitive approach is to solve a min-max optimization problem online in the presence of disturbance and/or model mismatch [1–3]. In general, such schemes are computationally intractable since the size of the resulting optimization problem grows exponentially with the increase of the prediction horizon [2]. Constraint tightening approaches, as introduced in [4, 5], avoid computational complexity by using a nominal prediction model and tightened constraint sets. However, the constraint sets often shrink drastically because the margin, which reflects the effects of uncertainties, increases exponentially with the increase of the prediction horizon. For discrete-time linear systems subject to persistent but bounded disturbances, [6] provided a new constraint tightening scheme, namely tube MPC, which reduces online computational burden while having fixed tightened sets. The results require linearity of the considered system, and have been extended by [7] to some classes of discrete-time nonlinear systems, namely systems with matched nonlinearity and piecewise affine systems, and extended by [8] to continuous-time nonlinear systems. Recently, [9] proposed an improved tube MPC of discrete-time linear systems which removes the artificial constraint and has the same computational burden as the standard MPC with guaranteed nominal stability.

Shuyou Yu and Frank Allgöwer are with the Institute of Systems Theory and Automatic Control, University of Stuttgart, Germany. {shuyou.yu, frank.allgower}@ist.uni-stuttgart.de.

Hong Chen is with State Key Laboratory of Automobile Dynamic Simulation, and Department of Control Science and Engineering, Jilin University, PR China. chenh@jlu.edu.cn.

This paper presents an extension of the improved tube MPC scheme to general nonlinear system, and provides a scheme to construct a robust control invariant set for Lipschitz nonlinear systems. The optimization problem considered has a general cost functional rather than the quadratic one. Similar to the improved tube MPC scheme with a quadratic cost functional [9], both recursive feasibility and input-to-state stability (ISS) of the system are guaranteed if the online optimization problem has a feasible solution at the initial time instant. Furthermore, we discuss an optimization based control scheme, where the optimization problem is solved only at the initial time instant. The optimization based control scheme has the same robustness properties as the proposed tube MPC scheme.

The paper is structured as follows. In Section II we state the problem setup and preliminary results. The online optimization problem, the proposed tube MPC scheme and optimization based control scheme are discussed in Section III. Section IV discusses the construction of a robust control invariant set for Lipschitz nonlinear systems. A numerical example is given in Section V.

A. Notations and Basic Definitions

Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space. For a vector $v \in \mathbb{R}^n$, $\|v\|_\infty$ denotes the infinity norm, $\|v\|$ the 2-norm. For a matrix $M \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(M)$ ($\lambda_{\max}(M)$) is the smallest (largest) real part of eigenvalues of the matrix M, $\bar{\sigma}(M)$ the largest singular value of M. Moreover, $*$ is used to denote the symmetric part of a matrix, i.e., $\begin{bmatrix} a & b^T \\ b & c \end{bmatrix} = \begin{bmatrix} a & * \\ b & c \end{bmatrix}$. The operation $\ominus$ represents Pontryagin difference of two sets $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$. Co$\{\cdot\}$ denotes the convex hull of a set, and $\langle \cdot, \cdot \rangle$ denotes the inner production of two vectors, i.e., $\langle x, y \rangle = x^T y$.

II. Problem Setup and Preliminary Results

Consider a system described by a nonlinear ordinary differential equation (ODE):

$$\dot{x}(t) = f\left(x(t), u(t), w(t)\right), \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state of the system, $u(t) \in \mathbb{R}^{n_u}$ is the control input. The signal $w(t) \in \mathbb{R}^{n_w}$ is the exogenous disturbance or uncertainty, which is unknown but bounded, and lies in a compact set,

$$\mathcal{W} := \{w \in \mathbb{R}^{n_w} \mid \|w\| \leq w_{max}\},$$

i.e., $w(t) \in \mathcal{W}$, for all $t \geq 0$. The system is subject to constraints

$$x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U}, \qquad \forall t > 0. \tag{2}$$

Remark 2.1: In finite dimensions, the two norm and infinity norm induce equivalent metrics, so that the signal $w : \mathbb{R} \to \mathbb{R}^{n_w}$ is bounded in the infinity norm if and only if it is bounded in the two norm.

Some fundamental assumptions are stated in the following, which are similar to the general assumptions of MPC with guaranteed nominal stability [10], but take the disturbance input into account.

Assumption 1: $f(x, u, w) : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \to \mathbb{R}^{n_x}$ is twice continuously differentiable in x, u and w. Furthermore, $f(0, 0, 0) = 0$, thus $0 \in \mathbb{R}^{n_x}$ is an equilibrium of the system (1).

Assumption 2: $\mathcal{U} \subset \mathbb{R}^{n_u}$ is compact, $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ is connected and the point $(0, 0, 0)$ is contained in the interior of $\mathcal{X} \times \mathcal{U} \times \mathcal{W}$.

Assume that $x(t)$ can be measured in real time, and define a nominal system

$$\dot{\bar{x}}(t) := f\left(\bar{x}(t), \bar{u}(t), 0\right), \tag{3}$$

i.e., $w(t) \equiv 0$, $\bar{x}(t) \in \mathcal{X}$ and $\bar{u}(t) \in \mathcal{U}$.

Denote $v(t) := x(t) - \bar{x}(t)$ as the error (deviation) between the actual system (1) and the nominal system (3). The dynamics of the *error system* is given as

$$\dot{v}(t) = f(x(t), u(t), w(t)) - f(\bar{x}(t), \bar{u}(t), 0). \tag{4}$$

We will design a control signal which consists of a nominal input and a state feedback control, i.e.,

$$u(t) := \bar{u}(t) + \kappa(x(t), \bar{x}(t)),$$

with $\kappa(x(t), \bar{x}(t)) : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^{n_u}$.

The main objective of this paper is to find an effective control scheme for constrained continuous-time nonlinear systems with respect to bounded disturbances, in particular, for Lipschitz nonlinear systems.

Before proceeding, we introduce the definition of robust control invariant set, and a way to obtain robust control invariant set.

Definition 1: A set $\Omega \subset \mathbb{R}^n$ is a robust control invariant set for the error system (4) if and only if there exists an ancillary feedback control law $\kappa(\cdot, \cdot)$ such that for all $v(t_0) \in \Omega$ and for all $w \in \mathcal{W}$, $v(t) \in \Omega$ for all $t \geq t_0$.

Furthermore, if the control law $\kappa(\cdot, \cdot)$ is chosen, Ω is a *robust invariant set* of the closed-loop error system.

The following lemma provides us a way to construct robust control invariant set for the error system (4).

Lemma 1: [8] Let $S : \mathbb{R}^{n_x} \to [0, \infty)$ be a continuously differentiable function and $\alpha_1(\|v\|) \leq S(v) \leq \alpha_2(\|v\|)$. Suppose $u : \mathbb{R} \to \mathbb{R}^{n_u}$ is chosen, and there exist $\lambda > 0$ and $\mu > 0$ such that

$$\frac{d}{dt} S(v) + \lambda S(v) - \mu w^T w \leq 0, \quad \forall w \in \mathcal{W}, \tag{5}$$

where α_1, α_2 are class $\mathcal{K}_\infty$ functions and $v \in \mathcal{X}$. Then, the system trajectory starting from $v(t_0) \in \Omega \subseteq \mathcal{X}$ will remain in the set Ω, where

$$\Omega := \left\{ v \in \mathbb{R}^{n_x} \mid S(v) \leq \frac{\mu w_{max}^2}{\lambda} \right\}. \tag{6}$$

Assumption 3: Suppose that there exists a robust control invariant set Ω for the error system (4) with the control law $\kappa(\cdot, \cdot)$, such that Ω lies in the interior of $\mathcal{X}$ and $\kappa(x, \bar{x})$ lies in the interior of $\mathcal{U}$ for all $x - \bar{x} \in \Omega$.

III. Tube MPC with a General Cost Functional

Tube MPC, proposed originally by [6] for discrete-time linear systems, uses the repeated online solution to an optimization problem subject to the *nominal* dynamics (3) and the tightened constraints in which the initial state of the nominal model is a decision variable. Here, we consider continuous-time nonlinear systems. For this, define the nominal cost functional

$$J(\phi(t_k), \varphi(\cdot)) := \int_{t_k}^{t_k + T_p} l\left(\phi(\tau), \varphi(\tau)\right) d\tau + E\left(\phi(t_k + T_p)\right),$$

where the stage cost $l : \mathbb{R}^{n_x \times n_u} \to \mathbb{R}$ is uniformly continuous of ϕ and φ, satisfies $l(0, 0) = 0$. Furthermore, $l\left(\phi(\tau), \varphi(\tau)\right) > 0$ for all $(\phi(\tau), \varphi(\tau)) \neq (0, 0)$. The terms $\phi(\cdot)$ and $\varphi(\cdot)$ are piecewise continuous trajectories of t from time instant t_k to $t_k + T_p$. The terminal penalty function $E(\cdot)$ is positive semidefinite, and the prediction horizon $T_p \geq 0$.

For the measured actual state $x(t_k)$, the optimization problem which is solved *online* is formulated as follows:

Problem 1:

$$\underset{\bar{u}(\cdot;\bar{x}(t_k),t_k)}{\text{minimize}} \qquad J\left(\bar{x}(t_k), \bar{u}(\cdot; \bar{x}(t_k), t_k)\right) \tag{7a}$$

subject to

$$\dot{\bar{x}}(\tau) = f\left(\bar{x}(\tau), \bar{u}(\tau), 0\right), \tag{7b}$$

$$\bar{x}\left(\tau; \bar{x}(t_k), t_k\right) \in \mathcal{X}_0, \quad \tau \in [t_k, t_k + T_p], \tag{7c}$$

$$\bar{u}\left(\tau; \bar{x}(t_k), t_k\right) \in \mathcal{U}_0, \quad \tau \in [t_k, t_k + T_p], \tag{7d}$$

$$\bar{x}\left(t_k + T_p; \bar{x}(t_k), t_k\right) \in \mathcal{X}_f, \tag{7e}$$

where $\mathcal{X}_0 := \mathcal{X} \ominus \Omega$ and $\mathcal{X}_f \subset \mathcal{X} \ominus \Omega$. Define $G := \{\kappa(x, \bar{x}) \in \mathbb{R}^{n_u} \mid x - \bar{x} \in \Omega, x \in \mathcal{X} \textit{ and } \bar{x} \in \mathcal{X}_0\}$, $\mathcal{U}_0 := U \ominus G$. The set $\mathcal{X}_f$ is a terminal set. Both Ω and $\mathcal{X}_f$ will be suitably derived. We use $\bar{u}\left(\cdot; \bar{x}(t_k), t_k\right)$ to emphasis that the control input is determined with the state $\bar{x}(t_k)$ at time instant t_k, and $\bar{x}\left(\cdot; \bar{x}(t_k), t_k\right)$ is the trajectory of the nominal system (3) starting from the state $\bar{x}(t_k)$ at time t_k and driven by the input function $\bar{u}\left(\cdot; \bar{x}(t_k), t_k\right)$. The term $\bar{u}^*(\cdot; \bar{x}^*(t_k), t_k)$ denotes the optimal solution to the optimization problem at the time instant t_k, and the term $\bar{x}^*\left(\cdot; \bar{x}^*(t_k), t_k\right)$ is the trajectory of the nominal system. Problem 1 is solved in discrete time with a sample time of δ, and the nominal control during the sample interval δ is

$$\bar{u}(\tau) := \bar{u}^*\left(\tau; \bar{x}^*(t_k), t_k\right), \quad \tau \in [t_k, t_k + \delta),$$

and the overall applied control input for the actual system (1) during the sampling interval δ consequently is

$$u(\tau) := \bar{u}(\tau) + \kappa\left(x(\tau), \bar{x}^*\left(\tau; \bar{x}^*(t_k), t_k\right)\right), \tau \in [t_k, t_k + \delta).$$

The nominal controller calculated online generates a nominal state trajectory, and the ancillary control law $\kappa(\cdot,\cdot)$ obtained offline keeps the trajectories of the error system in the robust control invariant set Ω centered along the nominal trajectory.

The following definition implies that if the dissipation and the constraint satisfaction conditions are satisfied in a compact set, then the compact set can be chosen as the terminal set of Problem 1.

Definition 2: [10] Set $\mathcal{X}_f := \{\bar{x} \in \mathbb{R}^{n_x} \mid E(\bar{x}) \leq \alpha\}$ with $\alpha > 0$, and function $E(\bar{x})$ are a terminal set and a terminal penalty function, respectively, if there exists an admissible control law $\pi(\bar{x})$ such that,

B0. $\mathcal{X}_f \subseteq \mathcal{X}_0$,
B1. $\pi(\bar{x}) \in \mathcal{U}_0$, for all $\bar{x} \in \mathcal{X}_f$,
B2. $E(\bar{x})$ satisfies inequalities,

$$\alpha_3(\|\bar{x}\|) \leq E(\bar{x}) \leq \alpha_4(\|\bar{x}\|) \tag{8a}$$

$$\frac{\partial E(\bar{x})}{\partial \bar{x}} f\left(\bar{x}, \pi(\bar{x})\right) + l\left(\bar{x}, \pi(\bar{x})\right) \leq 0, \forall \bar{x} \in \mathcal{X}_f, \tag{8b}$$

where $\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ are class $\mathcal{K}_\infty$ functions.
The set $\mathcal{X}_f$ is invariant for the nominal system under control $\bar{u} = \pi(\bar{x})$ since (8) holds.

Assumption 4: For the nominal system, there exist a locally asymptotically stabilizing controller $\bar{u} = \pi(\bar{x})$, a terminal set $\mathcal{X}_f \subseteq \mathcal{X}_0$, and a continuously differentiable positive definite function $E(\bar{x})$ such that conditions B0-B2 are satisfied for all $\bar{x} \in \mathcal{X}_f$.

A. Tube MPC Algorithm

Associated with Problem 1, consider the algorithm:

Algorithm 1: Step 0. At time t_0, set $\bar{x}(t_0) = x(t_0)$ in which $x(t_0)$ is the current state.
Step 1. At time t_k and current state $(\bar{x}(t_k), x(t_k))$, solve Problem 1 to obtain the nominal control action $\bar{u}(t_k)$ and the actual control action $u(t_k) = \bar{u}(t_k) + \kappa(x(t_k), \bar{x}(t_k))$.
Step 2. Apply the control $u(t_k)$ to the actual system being controlled, during the sampling interval $[t_k, t_{k+1}]$, where $t_{k+1} = t_k + \delta$.
Step 3. Measure the state $x(t_{k+1})$ at the next time instant t_{k+1} of the system being controlled and compute the successor state $\bar{x}(t_{k+1})$ of the nominal system under the nominal control $\bar{u}(t_k)$.
Step 4. Set $(\bar{x}(t_k), x(t_k)) = (\bar{x}(t_{k+1}), x(t_{k+1}))$, $t_k = t_{k+1}$, and go to Step 1.
Notice that a similar algorithm was proposed in [9], which considers a tube MPC scheme of discrete-time linear systems.

Remark 3.1: Since only the nominal model is used for prediction and the nominal control action is calculated online in Problem 1, the scheme has the same online computational burden as the standard MPC with guaranteed nominal stability.
The control action $\bar{u}(t)$ depends on the initial state $x(t_0)$. Therefore, the control action applied to the system (1) can not be calculated offline *a priori*. On the other hand, Problem 1 is solved online, which depends on the state of the nominal system rather than the state of actual system. Thus, Algorithm 1 is not a general MPC scheme. The following theorem states the properties of the proposed algorithm. Before it, we will introduce a useful definition.

Definition 3: A system is asymptotically ultimately bounded if the system converges asymptotically to a bounded set [11].

Theorem 1: Suppose that Problem 1 is feasible at time $t_0 = 0$. Then, for a small sample time $\delta > 0$,

(1). it is feasible for all $t > 0$,
(2). according to Algorithm 1, the trajectory of the system (1) under MPC control law is asymptotically ultimately bounded,
(3). the closed-loop system is input-to-state stable.

Proof: (1). Since only the "measured" state of the nominal system and the nominal system dynamics are used to solve Problem 1 at the next time instant, the online optimization is not related to the disturbances at all. Thus, recursive feasibility is guaranteed provided that Problem 1 has a feasible solution at the initial time instant [10].

(2). and (3). Because of the asymptotic stability of the nominal system [10], there exists a class $\mathcal{KL}$ function $\beta(\bar{x}, t)$ [11] such that

$$\|\bar{x}(t)\| \leq \beta(\bar{x}(t_0), t), \quad \forall t \geq t_0.$$

Due to $S\left(v(t)\right) \leq \frac{\mu w_{\max}^2}{\lambda}$ for all $t \geq t_0$ and $v(t) \in \Omega$, there exists a class $\mathcal{K}$ function such that

$$\|v(t)\| \leq \gamma \left(\sup_{t_0 \leq \tau \leq t} \|w(\tau)\| \right), \quad \forall t \geq t_0.$$

Since $x(t) = \bar{x}(t) + v(t)$ and $\bar{x}(t_0) = x(t_0)$,

$$\|x(t)\| \leq \beta(x(t_0), t) + \gamma \left(\sup_{t_0 \leq \tau \leq t} \|w(\tau)\| \right), \quad \forall t \geq t_0.$$

Therefore, the solution of system (1) *under the MPC control law according to Algorithm 1 is asymptotically ultimately bounded and the closed-loop system is input-to-state stable [11].* □

Algorithm 1 can be implemented in a parallel/offline way if the initial state $x(t_0)$ can be known *a priori*. That is, calculate $\bar{u}(t_i)$, $i \in [0, \infty)$, and store it for future use.

Algorithm 2: **(Parallel/Offline)**
Step 0. At time t_0, set $\bar{x}(t_0) = x(t_0)$ in which $x(t_0)$ is the current state.
Step 1. At time t_k, solve Problem 1 to obtain the nominal control action $\bar{u}(t_k)$, and store it.
Step 2. Compute the state $\bar{x}(t_{k+1})$ of the nominal system under the nominal control $\bar{u}(t_k)$, where $t_{k+1} = t_k + \delta$.
Step 3. Set $\bar{x}(t_k) = \bar{x}(t_{k+1})$, $t_k = t_{k+1}$, and go to Step 1.
(Online)
Step a. Apply the control $u(t_k) := \bar{u}(t_k) + \kappa(x(t_k) - \bar{x}(t_k))$ to the system being controlled, during the sampling interval $[t_k, t_{k+1})$, where $t_{k+1} = t_k + \delta$.
Step b. Measure the state $x(t_{k+1})$ at the next time instant t_{k+1} of the system being controlled.
Step c. Set $x(t_k) = x(t_{k+1})$, $t_k = t_{k+1}$, and go to Step a.

Remark 3.2: In the offline part of Algorithm 2, the symbol t_i, $i \in [0, \infty)$, is only a virtual time instant.

B. Optimization based Control Algorithm

As we emphasized in Section III, only the nominal model is used for prediction, and the nominal control action is calculated online in the proposed tube MPC scheme. Furthermore, the actual system state lies in a robust invariant set centered along the nominal trajectory since the error system is robust invariant in the set Ω. In order to further reduce the computational burden, Problem 1 can be solved only at the initial time instant.

Algorithm 3: Step 0. At time t_0, set $\bar{x}(t_0) = x(t_0)$ in which $x(t_0)$ is the current state, and solve Problem 1 to obtain the nominal control action $\bar{u}^*(\tau; \bar{x}_0, t_0)$, $\tau \in [t_0, t_0 + T_p)$.

Step 1. At time t_k and current state $(\bar{x}(t_k), x(t_k))$,

- If $t_k \in [t_0, t_0 + T_p)$, set $\bar{u}(t_k) = \bar{u}^*(t_k; \bar{x}_0, t_0)$,
- If $t_k \in [t_0 + T_p, \infty)$, set $\bar{u}(t_k) = \pi(\bar{x}(t_k))$.

Apply the control $u(t_k) = \bar{u}(t_k) + \kappa(x(t_k), \bar{x}(t_k))$ to the system being controlled, during the sampling interval $[t_k, t_{k+1}]$, where $t_{k+1} = t_k + \delta$.

Step 2. Measure the state $x(t_{k+1})$ at the next time instant t_{k+1} of the system being controlled and compute the state $\bar{x}(t_{k+1})$ of the nominal system under the nominal control $\bar{u}(t_k)$.

Step 3. Set $(\bar{x}(t_k), x(t_k)) = (\bar{x}(t_{k+1}), x(t_{k+1}))$, $t_k = t_{k+1}$, and go to Step 1.

The following result follows from Theorem 1.

Corollary 1: Suppose that Problem 1 is feasible at time $t_0 = 0$. Then, for a small sample time $\delta > 0$,

(1). according to Algorithm 3, the trajectory of the system (1) under the control law is asymptotically ultimately bounded,

(2). the closed-loop system is input-to-state stable.

Sketch of the proof: The input $\bar{u}^*(t_k; x(t_0), t_0)$ is a feasible solution to Problem 1 with the initial state $\bar{x}^*(t_k; x(t_0), t_0)$ at the time instant t_k, where $t_k \in [t_0, t_0 + T_p]$. Thus, the control input

$$\bar{u}(\tau) := \begin{cases} \bar{u}^*(\tau; x(t_0), t_0) & \tau \in [t_0, t_0 + T_p), \\ \pi\left(x(\tau; x(t_0), t_0)\right) & \tau \in [t_0 + T_p, \infty), \end{cases}$$

drives the nominal system (3) *asymptotically stable, and the state of the actual system is in the robust invariant set* Ω *around the nominal trajectory.* □

As it has been shown, robust control invariant set plays an important role in the tube MPC scheme and in the optimization based control scheme. In the next section, we provide sufficient (and conservative) conditions for the calculation of a quadratic Lyapunov function $S(v) = v^T P v$ and an ancillary linear feedback control law Kv for Lipschitz nonlinear systems based on Lemma 1. Note that both the robust control invariant set and the ancillary feedback control law are calculated *offline*.

IV. Robust Control Invariant Set for Lipschitz Nonlinear Systems

Consider the following continuous-time nonlinear system

$$\dot{x}(t) = Ax(t) + g(x(t)) + Bu(t) + B_w w(t), \tag{9}$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, $w(t) \in \mathbb{R}^{n_w}$, and $g(x) : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ represents a nonlinear function that is continuously differentiable in x.

The nonlinear function $g(x)$ is called a Lipschitz function in the set $\mathcal{X}$ with respect to x if there exists a constant $L > 0$ such that

$$\|g(x_1) - g(x_2)\| \le L\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathcal{X}, \tag{10}$$

where the smallest constant L satisfying (10) is known as the Lipschitz constant. The associated nominal system is

$$\dot{\bar{x}}(t) = A\bar{x}(t) + g(\bar{x}(t)) + B\bar{u}(t). \tag{11}$$

Chosen $u(t) := \bar{u}(t) + K(x(t) - \bar{x}(t))$, $K \in \mathbb{R}^{n_u \times n_x}$, the dynamics of the error system are

$$\dot{v}(t) = (A+BK)v(t) + B_w w(t) + [g(x(t)) - g(\bar{x}(t))]. \tag{12}$$

Lemma 2: Suppose that there exist positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, non-square matrix $Y \in^{n_u \times n_x}$, and scalars $\lambda_0 > \lambda > 0$ and $\mu > 0$ such that

$$\begin{bmatrix} (AX+BY)^T + AX + BY + \lambda_0 X & B_w \\ * & -\mu I \end{bmatrix} \le 0. \tag{13}$$

and

$$L \le \frac{(\lambda_0 - \lambda)\alpha_{min}(P)}{2\|P\|}. \tag{14}$$

Then, the error system (12) is robust invariant in the set $\Omega := \left\{x \in \mathbb{R}^{n_x} \mid x^T P x \le \frac{\mu w_{max}^2}{\lambda}\right\}$, where $u(t) - \bar{u}(t) := Kv(t)$, $S(v) := v^T P v$, $P := X^{-1}$ and $K := YX^{-1}$.

Proof: First, consider the system

$$\dot{s}(t) = (A+BK)s(t) + B_w w(t).$$

Define $\tilde{S}(s(t)) := s(t)^T P s(t)$, and denote $H(s(t)) := \dot{\tilde{S}}(t) + \lambda_0 \tilde{S}(t) - \mu w(t)^T w(t)$. Then,

$$\begin{aligned} H(s(t)) =& s(t)^T[(A+BK)^T P + P(A+BK)]s(t) \\ &+ w(t)^T B_w^T P s(t) + s(t)^T P B_w w(t) \\ &+ \lambda_0 s(t)^T P s(t) - \mu w(t)^T w(t). \end{aligned}$$

Multiplying (18) *from left and right sides with* $\operatorname{diag}\{P, I\}$ *and substituting $P = X^{-1}$, $K = YX^{-1}$, we obtain that*

$$\begin{bmatrix} (A+BK)^T P + P(A+BK) + \lambda_0 P & PB_w \\ B_w^T P & -\mu \end{bmatrix} \le 0. \tag{15}$$

Multiplying (15) *from both sides with* $\begin{bmatrix} s(t) & w(t) \end{bmatrix}$ *and* $\begin{bmatrix} s^T(t) & w^T(t) \end{bmatrix}^T$*, respectively, and* (15) *is sufficient for $H(s(t)) \le 0$. Because of Lemma 1, there exists an Ω_0 such that the system $\dot{s}(t) = (A+BK)s(t) + B_w w(t)$ is robust invariant, where $\Omega_0 := \{s \in \mathbb{R}^{n_x} \mid s^T P s \le \frac{\mu w_{max}^2}{\lambda_0}\}$.*

Denote $M(v(t)) = \dot{S}(t) + \lambda S(t) - \mu w(t)^T w(t)$, $\lambda < \lambda_0$. For the error system (12)*,*

$$\begin{aligned} M(v(t)) =& \dot{S}(t) + \lambda S(t) - \mu w(t)^T w(t) \\ =& v(t)^T[(A+BK)^T P + P(A+BK)]v(t) \\ &+ 2w(t)^T B_w^T P v(t) + \lambda v(t)^T P v(t) \\ &- \mu w(t)^T w(t) + 2[g(x(t)) - g(\bar{x}(t))]^T P v(t), \\ =& H(v(t)) + (\lambda - \lambda_0) v(t)^T P v(t) \\ &+ 2[g(x(t)) - g(\bar{x}(t))]^T P v(t). \end{aligned}$$

Since $H(v(t)) \leq 0$ and $\alpha_{\min}(P)\|v(t)\| \leq v(t)^T Pv(t) \leq \alpha_{\max}(P)\|v(t)\|$,

$$\begin{aligned} M(v(t)) \leq & (\lambda-\lambda_0)v(t)^T Pv(t) \\ & + 2[g(x(t)) - g(\bar{x}(t))]^T Pv(t) \\ \leq & (\lambda-\lambda_0)\alpha_{min}(P)\|v(t)\|^2 \\ & + 2[g(x(t)) - g(\bar{x}(t))]^T Pv(t), \end{aligned}$$

Due to (10) *and* (14)*, we have*

$$\begin{aligned} M(v(t)) \leq & (\lambda-\lambda_0)\alpha_{min}(P)\|v(t)\|^2 + 2L\|P\|\|v(t)\|^2 \\ = & (2L\|P\| + (\lambda-\lambda_0)\alpha_{min}(P))\|v(t)\|^2 \leq 0. \end{aligned}$$

Because of Lemma 1, this is sufficient to the error system (12) *being robust invariant in the set Ω.* □

Remark 4.1: Consider linear systems with norm-bounded uncertainty,

$$\dot{x}(t) = (A + \triangle A)x(t) + Bu(t) + B_w w(t). \tag{16}$$

Suppose that $\bar{\sigma}(\triangle A) \leq L$, where

$$\bar{\sigma}(\triangle A) := \sup_{x(t_1),x(t_2)} \frac{\|\triangle Ax(t_1) - \triangle Ax(t_2)\|}{\|x(t_1) - x(t_2)\|} \tag{17}$$

is the largest singular value of $\triangle A$ and $x(t_1) \neq x(t_2)$ [12]. Compared (17) with (10), we know that Lemma 2 also holds for the system (16).

The admissible Lipschitz constant L is always small since $\alpha_{\min}(P) \leq \|P\|$, see (14). In order to reduce the conservativeness, we can resort to the concept of one-sided Lipschitz continuity.

Definition 4: A nonlinear function $\phi(x) : \mathbb{R}^{n_x} \to \mathbb{R}^n$ is said to be one-sided Lipschitz continuous in a set $\mathcal{D}$ if there exists a $\rho \in \mathbb{R}$ such that for all $x_1, x_2 \in \mathcal{D}$,

$$\langle \phi(x_1) - \phi(x_2), x_1 - x_2 \rangle \leq \rho\|x_1 - x_2\|^2,$$

where ρ is called a one-sided Lipschitz constant.

Any Lipschitz function is a one-sided Lipschitz function, since

$$\begin{aligned} |\langle \phi(x_1) - \phi(x_2), x_1 - x_2 \rangle| \leq & \|\phi(x_1) - \phi(x_2)\|\|x_1 - x_2\| \\ \leq & |\rho|\|x_1 - x_2\|^2. \end{aligned}$$

However, the converse is not true in general.

Corollary 2: Suppose that there exist a positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, a non-square matrix $Y \in^{n_u \times n_x}$, and scalars $\lambda_0 > \lambda > 0$ and $\mu > 0$ such that

$$\begin{bmatrix} (AX + BY)^T + AX + BY + \lambda_0 X & B_w \\ * & -\mu I \end{bmatrix} \leq 0, \tag{18}$$

and $Pf(x(t))$ is one-sided Lipschitz continuous, i.e., $\langle Pg(x(t)) - Pg(\bar{x}(t)), x(t) - \bar{x}(t) \rangle \leq \rho\|x(t) - \bar{x}(t)\|^2$ with $P := X^{-1}$.

If $\rho \leq \frac{(\lambda_0 - \lambda)\alpha_{min}(P)}{2}$, where $\alpha_{\min}(P)$ is the smallest eigenvalue of the positive definite matrix P. Then, the error system (12) is robustly invariant in the set $\Omega := \left\{ v \in \mathbb{R}^{n_x} \mid v^T Pv \leq \frac{\mu w_{max}^2}{\lambda} \right\}$, where $u(t) - \bar{u}(t) := Kv(t)$, $S(v) := v^T Pv$ and $K := YX^{-1}$.

Sketch of the proof:

$$\begin{aligned} M(v(t)) \leq & (\lambda-\lambda_0)v(t)^T Pv(t) + 2[g(x(t)) - g(\bar{x}(t))]^T Pv(t) \\ \leq & (\lambda-\lambda_0)\alpha_{min}(P)\|v(t)\|^2 + 2\rho\|v(t)\|^2, \\ = & (2\rho + (\lambda-\lambda_0)\alpha_{min}(P))\|v(t)\|^2. \end{aligned}$$

Since $\rho \leq \frac{(\lambda_0 - \lambda)\alpha_{min}(P)}{2}$, $M(v(t)) \leq 0$. Because of Lemma 1, the error system (12) *is robust invariant in the set Ω.* □
In the next section, we exemplify the derived results considering a numerical example.

V. Illustrative Example

Consider the system described by

$$\dot{x}(t) = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} x(t) + g(x(t)) + \begin{bmatrix} 0.5 \\ -2 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t),$$

with $g(x) = \begin{bmatrix} 0 & -0.25x_2^3 \end{bmatrix}^T$. The origin of this system is open-loop unstable and its linearized system is stabilizable. Assume that x_1 and x_2 can be measured instantaneously, and the control constraint is

$$-2 \leq u(t) \leq 2, \quad \forall t \geq 0.$$

The disturbance is bounded by $w(t) \in \mathbb{W} := \{w \in \mathbb{R} \mid \|w\| \leq 0.1\}$. Choose the stage penalty function as $l(x,u) = x^T Qx + u^T Ru$, where the penalty matrices $Q = \mathrm{diag}(0.5, 0.5)$ and $R = 1$.

According to the Mean-value theorem, $g(x)$ is a region Lipschitz function with a Lipschitz constant $L = 0.75x_{2,\max}^2$ provided that $x_2 \in [-x_{2,\max}, x_{2,\max}]$. Since the admissible Lipschitz constant is very small if Lemma 2 is adopted to obtain a robust control invariant set, we resort to the one-sided Lipschitz constant.
The following remark will be used in the example.

Remark 5.1: [13] If a scalar function $h(x) : \mathbb{R}^n \to \mathbb{R}$ is differentiable with respect to x, then, for any $x, \bar{x} \in \mathbb{R}^n$ there exists $\xi \in \mathrm{Co}(x, \bar{x})$ such that

$$h(x) - h(\bar{x}) = \left(\frac{\partial h}{\partial x_1}(\xi), \frac{\partial h}{\partial x_2}(\xi), \cdots, \frac{\partial h}{\partial x_n}(\xi) \right)(x - \bar{x}).$$

According to the remark, for any $P_0 = \mathrm{diag}(\alpha_1, \alpha_2)$ with $\alpha_1 > 0$ and $\alpha_2 > 0$, there exists a non-zero $\xi \in (\min(x_2, \bar{x}_2), \max(x_2, \bar{x}_2))$ such that

$$\begin{aligned} & \langle P_0 (g(x) - g(\bar{x})), x - \bar{x} \rangle \\ = & \alpha_2(-0.25x_2^3 + 0.25\bar{x}_2^3)(x_2 - \bar{x}_2) \\ = & -\alpha_2 \cdot 0.75\xi^2(x_2 - \bar{x}_2)^2 < 0 \end{aligned}$$

that is, $P_0 g(x)$ is a one-sided Lipschitz nonlinearity with the one-sided Lipschitz constant $\rho = 0$. In this case, the robust control invariant set for the linear system $\dot{x}(t) = (A + BK)x(t) + B_w w(t)$ is also a robust control invariant set for the system (12). The ancillary control law $Kx = \begin{bmatrix} -1.3693 & 5.1273 \end{bmatrix} x$ guarantees that the set Ω is robustly invariant for the error system (12), where $\Omega = \{x \in \mathbb{R}^{n_2} \mid x^T Px \leq 1\}$ with $P = \mathrm{diag}(39.0251, 486.0402)$. Both the terminal control law and the terminal penalty matrix are yielded by the solution of a convex optimization problem, see [14], $\pi(x) = \begin{bmatrix} -1.1456 & 1.3925 \end{bmatrix} x$ and

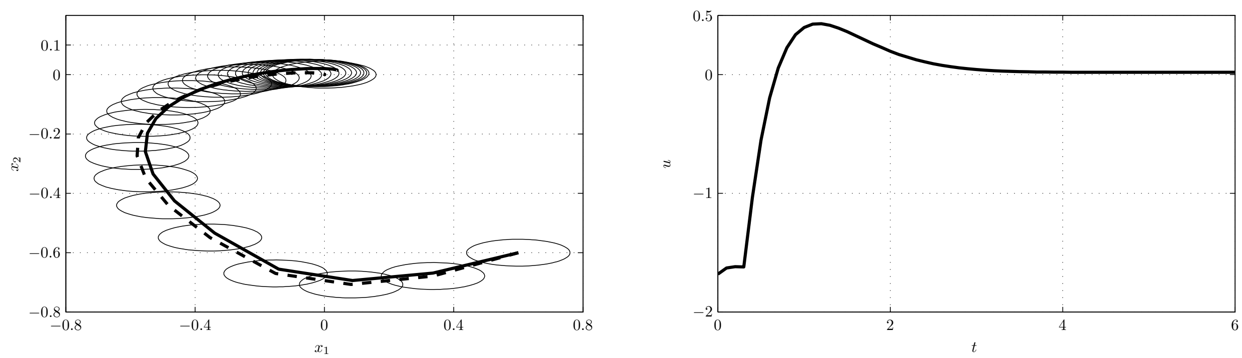

Fig. 1. Time profiles for the closed-loop system from $[0.6\ \ -0.6]^T$, solid line: trajectory of real system, dashed line: trajectory of nominal system.

$E(x) = x^T \begin{bmatrix} 7.9997 & -12.2019 \\ -12.2019 & 27.0777 \end{bmatrix} x$. The terminal set of the optimization problem is $\mathcal{X}_f = \{x \in \mathbb{R}^{n_2} \mid E(x) \leq 10\}$. The open-loop optimization problem described by Problem 1 is solved in discrete time with a sample time of $\delta = 0.1$ time units and a prediction horizon of $T_p = 1.5$ time units. Here, only Algorithm 1 is adopted. Figure 1 shows the state trajectory of the considered system starting from state $[0.6\ \ -0.6]^T$ with the disturbances $w(t) \equiv 0.1$, for all $t \geq 0$. The dashed line shows the trajectory of the nominal system, and the solid line shows the trajectory of the actual system. As it can be seen, the trajectory of the actual system under persistent but bounded disturbances remains in the "robust control invariant sets" centered along the nominal trajectory. Furthermore, the system state remains in the robust control invariant set around the origin while the time approaches to infinity.

VI. Conclusions

In this paper, we proposed a tube MPC scheme for continuous-time nonlinear systems subject to bounded disturbances based on robust control invariant sets. An ancillary control law is determined off-line which keeps the error system, which is the deviation of the actual system from the nominal system, robust invariant in a set. An optimization problem, which has the same computational burden as the standard MPC with guaranteed nominal stability, is solved online, and its solution defines the nominal trajectory. The actual trajectory of the system under the proposed tube MPC control law is in the sets centered along the nominal trajectory. Furthermore, it had been shown that both feasibility and input-to-state stability of the closed-loop system are guaranteed if the considered optimization problem is initially feasible. In order to reduce further the online computational burden, an optimization based scheme was discussed where the considered optimization problem is solved only at the initial time instant. In particular, we considered a way to obtain the robust control invariant set for Lipschitz nonlinear systems and illustrated the proposed schemes by a simple example.

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