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Input-Output Finite-Time Stability and Stabilization of Stochastic Markovian Jump Systems

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Abstract— **In this paper, we introduce a new concept named input-output finite-time mean square stability (IO-FTMSS) for stochastic Markovian jump systems. In contrast to the available notions of stochastic stability, IO-FTMSS characterizes the input-output behavior of dynamics on a finite time horizon. Concerning a class of random input signals L_T^2, the problems of input-output finite-time mean square stability and stabilization are investigated for both linear and nonlinear stochastic systems perturbed by Markovian processes. Sufficient conditions are derived in terms of coupled linear matrix inequalities (LMIs) and Hamilton-Jacobi inequalities (HJIs), respectively. In addition, a numerical example is supplied to illustrate the proposed technique.**

I. INTRODUCTION

As an important issue of stability theory, the study of finite-time stability can be traced back at least to the 1950s; see [1]. Compared with the classical Lyapunov stability, finite-time stability is focussed on the performance of dynamics over a finite time interval. More specifically, it describes the phenomenon that the trajectory of system state is not asymptotically stable, but stays within an acceptable bound during a short period of time. In practice, some dynamics such as missile systems and certain aircraft maneuvers, etc., are only required to perform satisfactorily on a fixed time horizon. In these cases, the concept of finite-time stability finds its theoretical significance and applications. Therefore, many researchers have been attracted to this field. For instance, Amato et al. discussed the finite-time stabilization via state feedback and dynamic output feedback for the deterministic linear systems (see [2] and [3]). In [4], a finite-time disturbance attenuation problem was considered for a class of nonlinear systems. Moreover, Yang et al. [13] generalized the results of [2] to the stochastic hybrid systems with impulse perturbation. Recently, a novel notion called input-output finite-time stability is introduced for a class of linear continuous-time systems [9]. Consistently with the tradition of finite-time stability, this concept is used to quantify the input-output behavior of the dynamics within a prescribed finite time interval.

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Since a large number of applications in engineering and finance, Markovian jump systems have received a growing attention in the control community([5], [10], [20]). In [8] and [14], the robust state and output feedback H_∞ control problems have been fully developed for Itô-type stochastic Markovian jump systems. The robust H_∞ filtering was also explored for nonlinear time-delay systems with Markovian jump parameters [19]. In [6], a finite time horizon H_2/H_∞ control problem was addressed for the discrete-time stochastic Markovian jump systems. Particularly, the Lyapunov asymptotic stability has been extensively investigated for Itô-type stochastic systems and Markovian jump systems, see [7], [11], [15], [16], [17], and the references therein. It can be seen that the Lyapunov stability theory of stochastic systems has reached a certain maturity. However, to our best knowledge, the notion as well as its theoretical framework of input-output finite-time stability have not been established for stochastic Markovian jump systems up to now.

In this paper, our main objective is to extend the deterministic results of input-output finite-time stability to stochastic Itô-type systems with Markovian jumps, which, to some extent, may be viewed as a stochastic counterpart of [9]. Above all, the concept of IO-FTMSS is defined for a general class of stochastic system with Markovian jump parameters related to a set of random input signals L_T^2. Based on which, the stability analysis and control synthesis are tackled for two special kinds of nonlinear and linear stochastic Markovian jump systems, respectively. We note that in the considered nonlinear dynamics, the control and input signal can not be simultaneously involved in the diffusion term due to some technical reasons; while the linear stochastic system consists of (x, u, v)-dependent noise, which is not the trivial specification of the nonlinear case.

The outline of this paper is as follows. Section 2 presents the definition of IO-FTMSS and some preliminaries. By means of a set of coupled HJIs, the first part of Section 3 deals with the input-output finite-time stability and stabilization problems for nonlinear stochastic Markovian jump systems. Then, the second part proceeds with the discussion of the linear dynamics via some coupled LMIs. Finally, we end this paper with a brief concluding remark in Section 4.

For convenience, the following notations are adopted throughout this paper. R^n: n-dimensional Euclidean space with the usual 2-norm $\|\cdot\|$; $R^{n\times m}$: the space of all $n\times m$ real matrices; S^n: the set of all $n\times n$ symmetric matrices; $M > 0$ ($M \geq 0$) means M is a positive (semi-)definite matrix; S_+^n: the set of all $n\times n$ positive definite matrices; M': the transpose of a matrix M; I: the identity matrix of appropriate dimension; $D = \{1, 2, \cdots, N\}$; $C^2(R^k)$: the class of functions $V(x)$ twice continuously differentiable with respect to $x \in R^k$.

II. PROBLEM STATEMENT AND PRELIMINARIES

Given a filtered probability space $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$, where there is a standard one-dimensional Brownian motion $w(t)$ on $[0, T]$ with $w(0) = 0$ and a Markovian jump process $r_t \in D$ with the generator $\Pi = (\lambda_{ij})$, and $F_t = \sigma(w(s), r_s|0 \leq s \leq t)$. Moreover, r_t is independent of $w(t)$ in this paper. Denote by $L_T^2(R^k)$ the space of all Borel measurable functions $\phi(t, \omega) \in R^k$, which is adapted to $\mathscr{F}_t$ on [0,T] and satisfies $\|\phi(\cdot, \cdot)\|_{l^2(0,T;R^k)} = (E\int_0^T \|\phi(t,\omega)\|^2 dt)^{(1/2)} < \infty$.

Consider the following stochastic system with Markovian jumps:

$$\left\{\begin{array}{l} dx(t) = \alpha(x, v, r_t)dt + \beta(x, v, r_t)dw(t), x(0) = 0, \\ y(t) = \gamma(x, r_t), \end{array}\right. \tag{1}$$

where $x(t) \in R^n$, $v(t) \in R^{n_v}$ and $y(t) \in R^{n_y}$ are the system state, exogenous input (disturbance) signal and measurement output, respectively. The triple (r_t, P, D) is a homogeneous Markovian chain and its transition probability is given by

$$P[r_{t+h} = j|r_t = i] = \left\{\begin{array}{ll} \lambda_{ij}h + o(h), & i \neq j, \\ 1 + \lambda_{ii}h + o(h), & i = j, \end{array}\right.$$

where $\lambda_{ij} \geq 0$ is the transition rate from mode i to j when $i \neq j$, $\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$ and $o(h)$ satisfies that $lim_{h\to 0} o(h)/h = 0$. It is well known that for any $v(t) \in L_T^2(R^{n_v})$, the considered system admits a unique strong solution on $[0, T]$ corresponding to any initial state $(x_0, i) \in R^n \times D$ if both α and β are Borel measurable and Lipschitz satisfying a linear growth condition, see [14].

In what follows, we will introduce the definition of IO-FTMSS associated with the system (1) over a specified time interval.

Definition 1: Given a set of input signals $\mathcal{W} \subset L_T^2(R^{n_v})$, a prescribed time $T > 0$ and a positive definite weighting matrix $Q(r_t)$, we say system (1) is input-output finite-time mean square stable, or also IO-FTMSS without confusion, with respect to $(\mathcal{W}, Q(r_t), T)$ if, for any $v(\cdot) \in \mathcal{W}$, there holds $E[y(t)^T Q(r_t) y(t)] \leq 1$, $t \in (0, T]$.

Compared with the notion of input-output finite-time stability for the deterministic system [9], mathematical expectation is introduced to evaluate the bound of the measurement output in above definition, which partially reveals the essential difference between the deterministic and the stochastic systems. In addition, we note that a concept named *finite time input-output stability*, with a very different meaning, has also been introduced in [12], where the authors are concerned with a class of non-smooth systems whose state trajectory approaches zero after a finite time.

In this paper, we are interested in the following mean square integrable signals:

$$\begin{aligned} \mathcal{W} = \{&v(\cdot) \in L_T^2(R^{n_v})| \\ &E\{\int_0^T [v(t)^T R(r_t) v(t)]dt | r_0 = i\} \leq 1, i \in D\}, \end{aligned}$$

where $R(r_t)$ denotes a positive definite matrix with appropriate dimension.

Below, we present a generalized Itô's formula associated with the diffusion processes with Markovian jumps ([14]), which will play a key role in the subsequent discussions.

Lemma 1 (Generalized Itô's formula): Assume that $a(x, i)$ and $b(x, i)$ are Borel measurable and Lipschitz satisfying a linear growth condition for all $i \in D$. Consider

$$dx(t) = a(x, r_t)dt + b(x, r_t)dw(t). \tag{2}$$

For given $\phi(x, i) \in R^n \times D,\ i \in D$, if there is a $r > 0$ such that

$$\|\phi(x, i)\| + \|\phi_x(x, i)\| + \|\phi_{xx}(x, i)\| \leq K(1 + |x|^r),$$

where $K > 0$, then we have

$$\begin{aligned} &E\{\phi(x(T), r_T) - \phi(x(s), r_s)|r_s = i\} \\ &= E\{\int_s^T \Gamma_\phi(x(t), r_t)dt | r_s = i\}, \end{aligned} \tag{3}$$

where an infinitesimal generator operator $\Gamma_\phi : [0, T] \times R^n \times D \to R$ about the system (2) is given by

$$\begin{aligned} \Gamma_\phi(x(t), i) &= a(x, i)'\phi_x(x, i) \\ &\quad + \frac{1}{2} b(x, i)'\phi_{xx}(x, i) b(x, i) + \sum_{j=1}^{N} \lambda_{ij}\phi(x, j). \end{aligned}$$

III. STABILITY ANALYSIS AND DESIGN PROBLEM

A. Nonlinear Dynamics

In this subsection, we will focus on the IO-FTMSS of the following nonlinear stochastic systems with Markovian jump parameters:

$$\left\{\begin{array}{l} dx(t) = [f(x, r_t) + g(x, r_t)v(t)]dt \\ \qquad\qquad + [h(x, r_t) + l(x, r_t)v(t)]dw(t), x(0) = 0, \\ y(t) = m(x, r_t), \end{array}\right. \tag{4}$$

where $x(t) \in R^n$, $v(t) \in R^{n_v}$ and $y(t) \in R^{n_y}$ are the same as those defined in (1). In the case $r_t = i \in D$, $f(x, i)$, $g(x, i)$, $h(x, i)$, $l(x, i)$ and $m(x, i)$ are all Borel measurable and Lipschitz satisfying a linear growth condition. For notations' convenience, we denote $f(x, i)$ by f_i etc. in the sequel.

It is worth mentioning that, in engineering terminology, the state equation of (4) can be written as (t is omitted):

$$\dot{x} = f(x, r_t) + g(x, r_t)v + [h(x, r_t) + l(x, r_t)v]W,$$

where $W(t)$ is a stationary white noise, see [18].

About the set of input signals $\mathcal{W}$, the following lemma provides a sufficient condition for the IO-FTMSS of system (4).

Lemma 2: If the following N-coupled HJIs:

$$\begin{cases} \frac{\partial V(x,i)'}{\partial x}f_i+\frac{1}{2}h_i'\frac{\partial^2 V(x,i)}{\partial x^2}h_i+\sum\limits_{j=1}^{N}\lambda_{ij}V(x,j) \\ \quad +\frac{1}{4}(g_i'\frac{\partial V(x,i)}{\partial x}+l_i'\frac{\partial^2 V(x,i)}{\partial x^2}h_i)' \\ \quad \cdot(R_i-\frac{1}{2}l_i'\frac{\partial^2 V(x,i)}{\partial x^2}l_i)^{-1}(g_i'\frac{\partial V(x,i)}{\partial x} \\ \quad +l_i'\frac{\partial^2 V(x,i)}{\partial x^2}h_i)<0, \\ R_i-\frac{1}{2}l_i'\frac{\partial^2 V(x,i)}{\partial x^2}l_i>0, \\ V(0,i)=0,\ i\in D, \end{cases} \tag{5}$$

and

$$V(x,i)\geq m_i'Q_im_i,\ i\in D \tag{6}$$

admit a set of positive solutions $V(x(t),i)\in C^2(R^n), i\in D$, then system (4) is IO-FTMSS with respect to $(\mathcal{W},Q(r_t),T)$.

Proof: First of all, by applying the generalized Itô formula to $V(x(t),r_t)$ associated with system (4), it will be calculated that

$$\begin{aligned} & E[V(x(s),r_s)|r_0=i] \\ = \ & E[V(x(s),r_s)-V(x(0),r_0)|r_0=i] \\ = \ & E\{\int_0^s\{\frac{\partial V(x,r_t)'}{\partial x}[f(x,r_t)+g(x,r_t)v(t)] \\ & +\sum_{j=1}^{N}\lambda_{r_t,j}V(x,j)+\frac{1}{2}[h(x,r_t)+l(x,r_t)v(t)]' \\ & \cdot\frac{\partial^2 V(x,i)}{\partial x^2}[h(x,r_t)+l(x,r_t)v(t)]\}dt|r_0=i\}. \end{aligned}$$

Taking into account that $V(x,i)$ verifies the HJIs (5), we may write the following inequality:

$$\begin{aligned} & E[V(x(s),r_s)|r_0=i] \\ < \ & E\{\int_0^s\{\frac{\partial V(x,r_t)'}{\partial x}g(x,r_t)v(t)+h(x,r_t)'\frac{\partial^2 V(x,i)}{\partial x^2} \\ & \cdot l(x,r_t)v(t)-\frac{1}{4}[g(x,r_t)'\frac{\partial V(x,r_t)}{\partial x}+l(x,r_t)' \\ & \cdot\frac{\partial^2 V(x,r_t)}{\partial x^2}h(x,r_t)]'[R(r_t)-\frac{1}{2}l(x,r_t)' \\ & \cdot\frac{\partial^2 V(x,r_t)}{\partial x^2}l(x,r_t)]^{-1}[g(x,r_t)'\frac{\partial V(x,r_t)}{\partial x} \\ & +l(x,r_t)'\frac{\partial^2 V(x,r_t)}{\partial x^2}h(x,r_t)]\}dt|r_0=i\}. \end{aligned}$$

By use of the technique of completing square, the above inequality yields that

$$\begin{aligned} & E[V(x(s),r_s)|r_0=i] \\ < \ & E\{\int_0^s\{v(t)'R(r_t)v(t)-[v(t)-(R(r_t)-\frac{1}{2}l(x,r_t)' \\ & \cdot\frac{\partial^2 V(x,r_t)}{\partial x^2}l(x,r_t))^{-1}(g(x,r_t)'\frac{\partial V(x,r_t)}{\partial x} \\ & +l(x,r_t)'\frac{\partial^2 V(x,r_t)}{\partial x^2}h(x,r_t))]'[R(r_t)-\frac{1}{2}l(x,r_t)' \\ & \cdot\frac{\partial^2 V(x,r_t)}{\partial x^2}l(x,r_t)][v(t)-(R(r_t)-\frac{1}{2}l(x,r_t)' \\ & \cdot\frac{\partial^2 V(x,r_t)}{\partial x^2}l(x,r_t))^{-1}(g(x,r_t)'\frac{\partial V(x,r_t)}{\partial x} \\ & +l(x,r_t)'\frac{\partial^2 V(x,r_t)}{\partial x^2}h(x,r_t))]\}dt|r_0=i\}. \end{aligned}$$

Recalling that $R(r_t)-\frac{1}{2}l(x,r_t)'\frac{\partial^2 V(x,r_t)}{\partial x^2}l(x,r_t)>0$ for any $t\in(0,T]$, we can easily deduce that

$$\begin{aligned} E[V(x(s),r_s)|r_0=i] &< E[\int_0^s v(t)'R(r_t)v(t)dt|r_0=i] \\ &\leq E[\int_0^T v(t)'R(r_t)v(t)dt|r_0=i] \end{aligned}$$

for any $(s,i)\in(0,T]\times D$. Further, due to the input signal $v(t)\in\mathcal{W}$, it follows that $E[V(x(s),r_s)|r_0=i]<1$ for any $(s,i)\in(0,T]\times D$. Combining with the condition (6), we will arrive at the desired result:

$$\begin{aligned} & E[y(s)^TQ(r_s)y(s)] \\ = \ & \sum_{i=0}^{N}P(r_0=i)E[y(s)^TQ(r_s)y(s)|r_0=i] \\ = \ & \sum_{i=0}^{N}P(r_0=i)E[m(x,r_s)^TQ(r_s)m(x,r_s)|r_0=i] \\ < \ & \sum_{i=0}^{N}P(r_0=i)E[V(x(s),r_s)|r_0=i] \\ < \ & 1, \qquad \forall s\in(0,T], \end{aligned}$$

which completes the proof.

Based on the preceding lemma, we begin to discuss the control synthesis problem of the following nonlinear stochastic Markovian jump systems:

$$\begin{cases} dx(t)=[f(x,r_t)+g(x,r_t)v(t)+k(x,r_t)u(t)]dt \\ \qquad\quad +[h(x,r_t)+l(x,r_t)v(t)]dw(t), x(0)=0, \\ y(t)=m(x,r_t). \end{cases} \tag{7}$$

that is, how to seek a state feedback controller that guarantees the closed-loop of system (7) to be IO-FTMSS with respect to $(\mathcal{W},Q(r_t),T)$.

The following theorem is the main result of this subsection, which supplies a sufficient condition for the aforementioned control synthesis problem.

Theorem 1: For system (7), if the following N-coupled HJIs:

$$\begin{cases} H_1(V):=\frac{\partial V(x,i)'}{\partial x}f_i+\frac{1}{2}h_i'\frac{\partial^2 V(x,i)}{\partial x^2}h_i \\ \quad +\sum\limits_{j=1}^{N}\lambda_{ij}V(x,j)-\frac{1}{4}\frac{\partial V(x,i)'}{\partial x}k_ik_i'\frac{\partial V(x,i)}{\partial x} \\ \quad +\frac{1}{4}(g_i'\frac{\partial V(x,i)}{\partial x}+l_i'\frac{\partial^2 V(x,i)}{\partial x^2}h_i)' \\ \quad \cdot(R_i-\frac{1}{2}l_i'\frac{\partial^2 V(x,i)}{\partial x^2}l_i)^{-1}(g_i'\frac{\partial V(x,i)}{\partial x} \\ \quad +l_i'\frac{\partial^2 V(x,i)}{\partial x^2}h_i)<0, \\ R_i-\frac{1}{2}l_i'\frac{\partial^2 V(x,i)}{\partial x^2}l_i>0, \\ V(0,i)=0,\ i\in D, \end{cases} \tag{8}$$

and

$$V(x,i)\geq m_i'Q_im_i,\ i\in D \tag{9}$$

admit a set of positive solutions $V(x(t),i)\in C^2(R^n)$, $i\in D$, then $u(t)=-\frac{1}{2}k_i'\frac{\partial V(x,i)}{\partial x}$ is a state feedback controller guaranteeing the closed-loop of system (7) to be IO-FTMSS with respect to $(\mathcal{W},Q(r_t),T)$.

Proof: Similar to the argument of Lemma 2, applying the generalized Itô formula to $V(x(t), r_t)$ related with system (7), we have that

$$\begin{aligned}
& E[V(x(s),r_s)|r_0=i] \\
= \ & E[V(x(s),r_s)-V(x(0),r_0)|r_0=i] \\
= \ & E\{\int_0^s \{H_1(V)+v(t)'R(r_t)v(t) \\
& -u(t)'u(t)-[v(t)-(R(r_t)-\frac{1}{2}l(x,r_t)' \\
& \cdot\frac{\partial^2 V(x,r_t)}{\partial x^2}l(x,r_t))^{-1}(g(x,r_t)'\frac{\partial V(x,r_t)}{\partial x} \\
& +l(x,r_t)'\frac{\partial^2 V(x,r_t)}{\partial x^2}h(x,r_t))]'[R(r_t)-\frac{1}{2}l(x,r_t)' \\
& \cdot\frac{\partial^2 V(x,r_t)}{\partial x^2}l(x,r_t)][v(t)-(R(r_t)-\frac{1}{2}l(x,r_t)' \\
& \cdot\frac{\partial^2 V(x,r_t)}{\partial x^2}l(x,r_t))^{-1}(g(x,r_t)'\frac{\partial V(x,r_t)}{\partial x} \\
& +l(x,r_t)'\frac{\partial^2 V(x,r_t)}{\partial x^2}h(x,r_t))]+[u(t) \\
& +\frac{1}{2}k(x,r_t)'\frac{\partial V(x,r_t)}{\partial x}]'[u(t)+\frac{1}{2}k(x,r_t)' \\
& \cdot\frac{\partial V(x,r_t)}{\partial x}]\}dt|r_0=i\}. \qquad (10)
\end{aligned}$$

Having in mind that $V(x(t), r_t)$ satisfies the coupled HJIs (8), the above equality leads to that:

$$\begin{aligned}
& E[V(x(s),r_s)|r_0=i] \\
& < E\{\int_0^s \{v(t)'R(r_t)v(t)+[u(t)+\frac{1}{2}k(x,r_t)'\frac{\partial V(x,r_t)}{\partial x}]' \\
& \cdot[u(t)+\frac{1}{2}k(x,r_t)'\frac{\partial V(x,r_t)}{\partial x}]\}dt|r_0=i\}.
\end{aligned}$$

Thus, if we impose the state feedback control $u(t) = -\frac{1}{2}k_i'\frac{\partial V(x,i)}{\partial x}$ on system (7), the above inequality gives that

$$E[V(x(s),r_s)|r_0=i] < E[\textstyle\int_0^s v(t)'R(r_t)v(t)dt|r_0=i].$$

Then, by following the line of Lemma 2, it is easy to show the remainder of the proof and hence the detail is omitted.

Remark 1: It should be pointed out that there has not been a general approach to solve the HJIs like (8) to date. However, in the special case that the diffusion term of system (7) consists of only state-dependent noise, a Takagi-Sugeno fuzzy approach can be applied to approximate the nonlinear dynamics (7) via a set of linear subsystems [18], which avoids the difficulty of solving (8).

B. Linear Dynamics

In what follows, we are concentrated on the analysis and synthesis problems of IO-FTMSS for a very broad class of linear stochastic systems with Markovian jump parameters:

$$\begin{cases} dx(t)=[A(r_t)x(t)+B(r_t)u(t)+G(r_t)v(t)]dt \\ \qquad +[\bar{A}(r_t)x(t)+\bar{B}(r_t)u(t)+\bar{G}(r_t)v(t)]dw(t), \\ x(0)=0, \\ y(t)=C(r_t)x(t). \end{cases} \qquad (11)$$

Before dealing with the control synthesis problem, we firstly present a sufficient condition guaranteeing system (12) to be IO-FTMSS, which is actually a reduced form of Lemma 2.

Lemma 3: For the stochastic unforced linear system with Markovian jump parameters:

$$\begin{cases} dx(t)=[A(r_t)x(t)+G(r_t)v(t)]dt \\ \qquad +[\bar{A}(r_t)x(t)+\bar{G}(r_t)v(t)]dw(t), \\ x(0)=0, \\ z(t)=C(r_t)x(t), \end{cases} \qquad (12)$$

if the following coupled LMIs:

$$\begin{bmatrix} \Theta(i) & P(i)G(i) & \Psi(i) \\ G(i)'P(i) & -R(i) & \Xi(i) \\ \Psi(i)' & \Xi(i)' & \Omega(i) \end{bmatrix} < 0, \ i\in D, \qquad (13)$$

$$P(i)\geq C(i)'Q(i)C(i), \qquad i\in D \qquad (14)$$

admit a set of positive solutions $P(i)$, $i \in D$, then system (12) is IO-FTMSS with respect to $(\mathcal{W}, Q(r_t), T)$, where in (13),

$$\begin{aligned}
\Theta(i) &= A(i)'P(i)+P(i)A(i)+\lambda_{ii}P(i) \\
\Psi(i) &= [\bar{A}(i)'P(i) \ \sqrt{\lambda_{i1}}P(i) \ \cdots \ \sqrt{\lambda_{i,i-1}}P(i) \\
& \qquad \sqrt{\lambda_{i,i+1}}P(i)\cdots \ \sqrt{\lambda_{iN}}P(i)], \\
\Xi(i) &= [\bar{G}(i)'P(i) \ \ 0 \ \ \cdots \ \ 0], \\
\Omega(i) &= diag(-P(i), \ -P(1), \ \cdots, -P(i-1), \\
& \qquad -P(i+1), \ \cdots, \ -P(N)).
\end{aligned}$$

Proof: In Lemma 2, taking $V(x, r_t) = x(t)'P(r_t)x(t)$ and replacing f_i with $A(i)$, etc., we will derive the following coupled algebraic Riccati inequalities:

$$\begin{cases} A(i)'P(i)+P(i)A(i)+\bar{A}(i)'P(i)\bar{A}(i)+[G(i)'P(i) \\ \quad +\bar{G}(i)'P(i)\bar{A}(i)]'[R(i)-\bar{G}(i)'P(i)\bar{G}(i)]^{-1} \\ \quad \cdot[G(i)'P(i)+\bar{G}(i)'P(i)\bar{A}(i)]+\sum_{j=1}^{N}\lambda_{ij}P(j)<0, \\ R(i)-\bar{G}(i)'P(i)\bar{G}(i)>0, \\ P(i)\geq C(i)'Q(i)C(i), i\in D. \end{cases} \qquad (15)$$

Via the well-known Schur's complement, the desired result is immediately yielded.

By means of Lemma 3, we are able to show the main result of this subsection in the following theorem, which provides a sufficient condition for the control synthesis problem of IO-FTMSS associated with system (11).

Theorem 2: For system (11), if the following system of LMIs:

$$\begin{bmatrix} M_{11} & G(i) & M_{13} \\ G(i)' & -R(i) & M_{23} \\ M_{13}' & M_{23}' & M_{33} \end{bmatrix} < 0, \quad i\in D, \qquad (16)$$

$$\begin{bmatrix} X(i) & X(i)C(i)' \\ C(i)X(i) & \Delta(i) \end{bmatrix} \geq 0, \quad i\in D \qquad (17)$$

admit a set of solutions $(X(i), Y(i)) \in S_+^n \times R^{n_u\times n}$, $i \in D$, then there exists a state feedback control $u(t) = K(r_t)x(t)$ with $K(r_t) = Y(r_t)X(r_t)^{-1}$, such that the closed-loop of system (11) is IO-FTMSS with respect to $(\mathcal{W}, Q(r_t), T)$, where in (16) and (17),

$$
\begin{aligned}
M_{11} &= A(i)X(i) + X(i)A(i)' + B(i)Y(i) \\
&\quad + Y(i)'B(i)' + \lambda_{ii}X(i), \\
M_{13} &= [X(i)\bar{A}(i)' + Y(i)'\bar{B}(i)' \ \sqrt{\lambda_{i1}}X(i) \ \cdots \\
&\quad \sqrt{\lambda_{i,i-1}}X(i) \ \sqrt{\lambda_{i,i+1}}X(i) \cdots \sqrt{\lambda_{iN}}X(i)], \\
M_{23} &= [\bar{G}(i)' \ \ 0 \ \ \cdots \ \ 0], \\
M_{33} &= diag(-X(i), \ -X(1), \ \cdots, -X(i-1), \\
&\qquad -X(i+1) \ \cdots, \ -X(N)), \\
\Delta(i) &= Q(i)^{-1}.
\end{aligned}
$$

Proof: Substituting $u(t) = K(r_t)x(t)$ into system (11), we will obtain the following closed-loop system:

$$
\begin{cases}
dx(t) = [(A(r_t) + B(r_t)K(r_t))x(t) + G(r_t)v(t)]dt \\
\qquad + [(\bar{A}(r_t) + \bar{B}(r_t)K(r_t))x(t) + \bar{G}(r_t)v(t)]dw(t), \\
x(0) = 0, \\
y(t) = C(r_t)x(t).
\end{cases} \tag{18}
$$

By employing the similar argument of Lemma 3 associated with system (18), it can be shown that if the following N-coupled algebraic Riccati inequalities:

$$
\begin{cases}
[A(i) + B(i)K(i)]'P(i) + P(i)[A(i) + B(i)K(i)] \\
\quad + [\bar{A}(i) + \bar{B}(i)K(i)]'P(i)[\bar{A}(i) + \bar{B}(i)K(i)] \\
\quad + [G(i)'P(i) + \bar{G}(i)'P(i)(\bar{A}(i) + \bar{B}(i)K(i))]' \\
\quad \cdot [R(i) - \bar{G}(i)'P(i)\bar{G}(i)]^{-1}[G(i)'P(i) \\
\quad + \bar{G}(i)'P(i)(\bar{A}(i) + \bar{B}(i)K(i))] + \sum_{j=1}^{N} \lambda_{ij}P(j) < 0, \\
R(i) - \bar{G}(i)'P(i)\bar{G}(i) > 0, \\
P(i) \geq C(i)'Q(i)C(i), i \in D
\end{cases} \tag{19}
$$

admit a set of solutions $(P(i), K(i)) \in S_+^n \times R^{n_u \times n}$, $i \in D$, then the state feedback control $u(t) = K(r_t)x(t)$ can guarantee the closed-loop of system (11) to be IO-FTMSS with respect to $(\mathcal{W}, Q(r_t), T)$.

To solve the above nonlinear inequalities, we make use of Schur's complement again, and will write that

$$
\begin{bmatrix}
\bar{M}_{11} & P(i)G(i) & \bar{M}_{13} \\
G(i)P(i)' & -R(i) & \bar{M}_{23} \\
\bar{M}_{13}' & \bar{M}_{23}' & \bar{M}_{33}
\end{bmatrix} < 0, \tag{20}
$$

$$
\begin{bmatrix}
P(i) & C(i)' \\
C(i) & Q(i)^{-1}
\end{bmatrix} \geq 0, \tag{21}
$$

where in (20),

$$
\begin{aligned}
\bar{M}_{11} &= [A(i) + B(i)K(i)]'P(i) + P(i)[A(i) \\
&\quad + B(i)K(i)] + \lambda_{ii}P(i), \\
\bar{M}_{13} &= [\bar{A}(i)'P(i) + K(i)'\bar{B}(i)'P(i) \ \sqrt{\lambda_{i1}}P(i) \ \cdots \\
&\quad \sqrt{\lambda_{i,i-1}}P(i) \ \sqrt{\lambda_{i,i+1}}P(i) \cdots \sqrt{\lambda_{iN}}P(i)], \\
\bar{M}_{23} &= [\bar{G}(i)'P(i) \ \ 0 \ \ \cdots \ \ 0], \\
\bar{M}_{33} &= diag(-P(i), \ -P(1), \ \cdots, -P(i-1), \\
&\qquad -P(i+1), \cdots, \ -P(N)).
\end{aligned}
$$

To proceed, we denote that $X(i) = P(i)^{-1}$ and $\Delta(i) = Q(i)^{-1}$. Then, pre- and post-multiplying (20) and (21) by $diag(X(i), I, M_{33})$ and $diag(X(i), I)$, respectively, we can get the following inequalities:

$$
\begin{bmatrix}
\hat{M}_{11} & G(i) & \hat{M}_{13} \\
G(i)' & -R(i) & M_{23} \\
\hat{M}_{13}' & M_{23}' & M_{33}
\end{bmatrix} < 0, \tag{22}
$$

$$
\begin{bmatrix}
X(i) & X(i)C(i)' \\
C(i)X(i) & \Delta(i)
\end{bmatrix} \geq 0, \tag{23}
$$

where in (22),

$$
\begin{aligned}
\hat{M}_{11} &= A(i)X(i) + X(i)A(i)' + B(i)K(i)X(i) \\
&\quad + X(i)K(i)'B(i)' + \lambda_{ii}X(i), \\
\hat{M}_{13} &= [X(i)\bar{A}(i)' + X(i)K(i)'\bar{B}(i)' \ \sqrt{\lambda_{i1}}X(i) \ \cdots \\
&\quad \sqrt{\lambda_{i,i-1}}X(i) \ \sqrt{\lambda_{i,i+1}}X(i) \cdots \sqrt{\lambda_{iN}}X(i)].
\end{aligned}
$$

Without loss of generality, we may set $Y(i) = K(i)X(i)$ in (22) and then the desired result is directly obtained. The proof is ended.

Remark 2: As opposed to the nonlinear case, the control design for linear stochastic Markovian jump systems is based on a set of coupled LMIs, which may be effectively solved by means of the well-known software (LMI control toolbox).

Example 1: The coefficients of system (11) corresponding to two modes are given as follows:

Table 1

Coefficients	$i = 1$	$i = 2$
$A(i)$	$\begin{bmatrix} -0.6 & 1 \\ 1 & -0.3 \end{bmatrix}$	$\begin{bmatrix} -0.6 & 2.7 \\ 1.7 & -0.9 \end{bmatrix}$
$B(i)$	$\begin{bmatrix} 1 & 0.3 \\ 0.9 & -0.4 \end{bmatrix}$	$\begin{bmatrix} 0.1 & 1 \\ 1 & -0.3 \end{bmatrix}$
$G(i)$	$\begin{bmatrix} 0.9 & 0.8 \\ 0.5 & 1.2 \end{bmatrix}$	$\begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 1 \end{bmatrix}$
$\bar{A}(i)$	$\begin{bmatrix} -0.9 & 0.4 \\ 0.1 & -0.7 \end{bmatrix}$	$\begin{bmatrix} -0.4 & 1.2 \\ 1 & 0.6 \end{bmatrix}$
$\bar{B}(i)$	$\begin{bmatrix} 0.7 & 0 \\ -1 & 0.9 \end{bmatrix}$	$\begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 1 \end{bmatrix}$
$\bar{G}(i)$	$\begin{bmatrix} -0.8 & 0 \\ 1 & -0.9 \end{bmatrix}$	$\begin{bmatrix} -0.2 & 0 \\ 0 & -0.1 \end{bmatrix}$
$C(i)$	$\begin{bmatrix} 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \end{bmatrix}$

Assume that $R_1 = diag(1.5, 1.9)$, $R_2 = diag(1.8, 1.2)$, $Q(1) = 2$ and $Q(2) = 1$. Consider the exogenous input signal $v(t) = [e^{-t}sint \ \ 0]' \in L_T^2(R^2)$ with $T = 2s$ and a Markovian jump process with the state space $D = \{1, 2\}$ and the transition rate:

$$
(\lambda_{ij})_{2\times 2} = \begin{bmatrix} -0.9 & 0.9 \\ 0.4 & -0.4 \end{bmatrix}.
$$

Thus, by use of Theorem 2, we can obtain the design of state feedback controller via LMI control toolbox; see Table 2.

Table 2

Solution	$i=1$	$i=2$
$X(i)$	$\begin{bmatrix} 4.01 & -4.10 \\ -4.10 & 4.67 \end{bmatrix}$	$\begin{bmatrix} 2.41 & -1.42 \\ -1.42 & 0.94 \end{bmatrix}$
$Y(i)$	$\begin{bmatrix} -1.88 & -1.92 \\ 1.85 & -2.17 \end{bmatrix}$	$\begin{bmatrix} -0.55 & -2.09 \\ -0.10 & -0.64 \end{bmatrix}$
$K(i)$	$\begin{bmatrix} -8.85 & -8.18 \\ -0.13 & -0.58 \end{bmatrix}$	$\begin{bmatrix} -13.80 & -23.02 \\ -4.02 & -6.75 \end{bmatrix}$

Figs. 1-2 demonstrate the evolutions of $E[y(t)'Q(r_t)y(t)]$ associated with the unforced system and the closed-loop system with the state feedback control $u(t) = K(r_t)x(t)$, respectively. The simulation results explicitly exhibit the efficiency of Theorem 2.

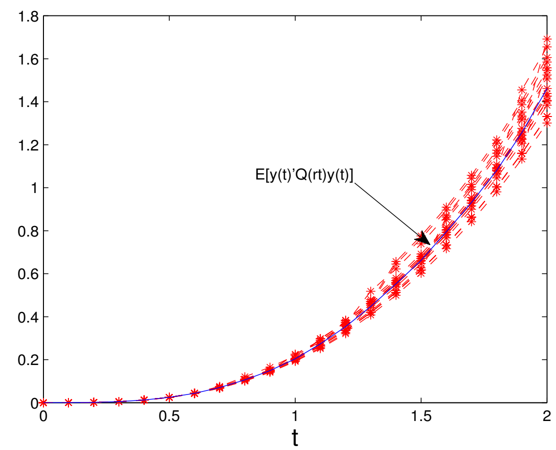

Fig. 1. $y(t)'Q(r_t)y(t)$ of the unforced system

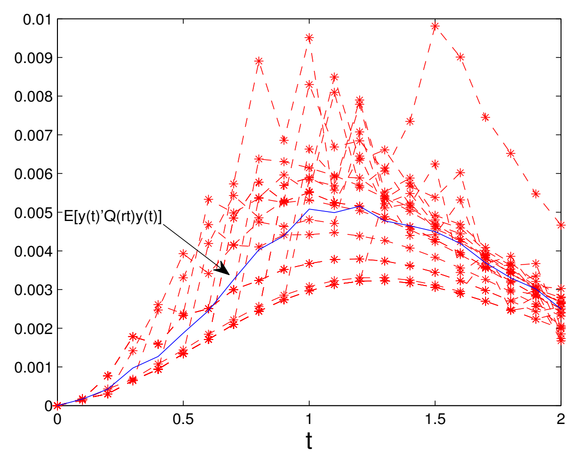

Fig. 2. $y(t)'Q(r_t)y(t)$ of the closed-loop system

IV. CONCLUSIONS

In this paper, the concept of input-output finite-time mean square stability has been introduced. For which, the stability analysis and control synthesis problems have been tackled for nonlinear and linear stochastic systems with Markovian jump parameters, respectively. Based on the solutions of some coupled HJIs or LMIs, the state feedback controller can be designed for the considered dynamics.

There are still some meaningful topics that remain open. First, how to deal with the coupled HJIs still deserves further studies. Besides, in the case that the information of system state is not available directly, the problem of output feedback control becomes important and necessary.

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