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An Algebraic Solution Method for the Unsteady Hamilton-Jacobi Equation

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***Abstract*— The unsteady Hamilton-Jacobi equation (HJE) plays an important role in the analysis and control of nonlinear systems and is very difficult to solve for general nonlinear systems. In this paper, the unsteady HJE for a Hamiltonian with coefficients belonging to meromorphic functions of time and rational functions of the state is considered, and its solutions with algebraic gradients are characterized in terms of commutative algebra. It is shown that there exists a solution with an algebraic gradient if and only if an H-invariant and involutive maximal ideal exists in a polynomial ring over the meromorphic functions of time and the rational functions of the state. If such an ideal is found, an algebraic gradient can be obtained by only solving a set of algebraic equations.**

I. Introduction

The unsteady Hamilton-Jacobi equation (HJE) is one of the most important equations in system control theory. For example, the optimal regulator problem [1] and the H^∞ control problem [2] lead to the HJE. However, it is difficult to solve the HJE analytically and numerically. Some numerical methods have been studied [3], but they suffer from rapid growth in the number of parameters to be determined with increasing dimension of the state space. Even the HJE for a time-invariant Hamiltonian suffers from the so-called curse of dimensionality.

On the other hand, Hamilton's canonical equations are often related to the unsteady HJE. In nonlinear optimal control problems, a stationary condition of optimality is described by Hamilton's canonical equations. A state feedback control law satisfying the stationary condition can be constructed if the costate can be obtained as a function of time and the state. For example, a gradient of a solution to the unsteady HJE satisfies the canonical equations and is a function of time and the state. Hamilton's canonical equations are also difficult to solve analytically. In particular, in the case of finite-horizon, nonlinear, time-varying optimal control problems, in solving the canonical equations we encounter the nonlinear two-point boundary-value problem, although some numerical solution methods for this problem have been developed [4].

Recently, a new representation for solution to the HJE has been proposed [5] in terms of commutative algebra [6], [7], [8], which is a different viewpoint from the standard differential geometric approach [9], [10] and the viscosity solution [11]. In this approach, a set of algebraic equations that are satisfied by the gradient of a solution to the HJE, is defined, and existence conditions for the set of algebraic equations are clarified by restricting the Hamiltonian to be a polynomial in the gradient of a solution with coefficients consisting of rational functions of the state.

On the basis of this approach, in this paper, we aim to find a set of algebraic equations that implicitly define the gradient of a solution to the unsteady HJE by restricting the Hamiltonian to be a polynomial in the gradient of the solution with coefficients consisting of meromorphic functions of time and rational functions of the state. Once a set of algebraic equations is found, the gradient of the solution at each time and the state can be numerically obtained by various techniques for solving nonlinear equations. Thus, there is no need to store a function over a domain in time and the state space, and, consequently, the curse of dimensionality is also avoided, which is similar to the case of a time-invariant Hamilton [5]. Note that in this paper, we consider the HJE with a time-varying Hamiltonian that is not a rational function of time but a meromorphic function of time. Therefore, our approach can deal with a large class of the unsteady HJE with respect to time.

The remainder of this paper is organized as follows. In Section II, the class of the unsteady HJE treated in this paper is stated, and to characterize its solution, the class of Hamilton's canonical equations treated in this paper is also stated. A solution to Hamilton's canonical equations with an algebraic function and a solution to the unsteady HJE with an algebraic gradient are defined. In Section III, an existence condition for a solution with an algebraic function is given for the canonical equations, and using this condition, an existence condition for a solution with an algebraic gradient is also given for the unsteady HJE. Each condition guarantees necessity and sufficiency, and the notions of H-invariant maximal ideals and involutive maximal ideals characterize each condition. In Section IV, a class of a nonlinear time-varying optimal regulator problem is given such that explicit solutions are obtained as algebraic functions, and an example of an explicit solution is also presented.

Notation: Throughout the paper, $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$ denotes the n-dimensional state vector of a dynamical system. For a scalar-valued function $V(t, x)$, we denote a row vector consisting of the partial derivatives of V with respect to x_i $(i = 1, 2, \ldots, n)$ as $\partial V/\partial x$, and the column vector $(\partial V/\partial x)^{\mathrm{T}}$, which is the transpose of $\partial V/\partial x$, as ∇V. Since the set of all real-valued analytic functions with a variable t is an integral domain [12], its quotient field is well-defined and is denoted by $\mathbb{R}_t$. That is, the set of all real-valued

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meromorphic functions with a variable t is denoted by $\mathbb{R}_t$. The field of rational functions with variable x over $\mathbb{R}_t$ is denoted by $K_t = \mathbb{R}_t(x)$. The polynomial ring over K_t with variables p_i $(i = 1, 2, \ldots, n)$ is denoted by $K_t[p]$ with $p = [p_1, p_2, \ldots, p_n]^{\mathrm{T}}$. Furthermore, the algebraic closure of K_t is denoted by $\bar{K}_t$. The indeterminate of a single-variable polynomial is denoted by X.

II. Setting of the Problem

For a scalar-valued function $H(t, x, p)$, we consider the following first-order partial differential equation for a scalar-valued function $V(t, x)$:

$$\begin{bmatrix} \nabla V(t,x) \\ \partial V/\partial t \end{bmatrix} = \begin{bmatrix} p(t,x) \\ -H(t,x,p) \end{bmatrix}, \tag{1}$$

which we call the unsteady Hamilton-Jacobi equation (HJE) for the Hamiltonian H. The unsteady HJE plays an important role in the analysis and control of time-varying nonlinear systems.

Example 1 (Nonlinear optimal regulator): Consider $f, g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, the following state equation, and the performance index

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(x(t), t) + g(x(t), t)u(t), x(0) = x_0,$$
$$J = \frac{1}{2}\int_{t_0}^{\infty} \left(q(x(t), t) + u^2(t)\right) \mathrm{d}t,$$

where we assume that $f(t, 0) = 0$ for all t and that q is positive definite with respect to x and uniformly bounded with respect to t. The value function $V(t_0, x) = \inf_u J$ satisfies the unsteady HJE with the Hamiltonian

$$H(t,x,p) = p^{\mathrm{T}} f(t,x) + \frac{1}{2}\left(-p^{\mathrm{T}} g(t,x) g^{\mathrm{T}}(t,x) p + q(t,x)\right).$$

The optimal regulator is given as $u_{opt} = -g^{\mathrm{T}} p$. If the system is affine in control and the performance index is quadratic in control, then the Hamiltonian is always quadratic in p.

Remark 1: The unsteady HJE (1) is usually called the HJE; however, to distinguish (1) from

$$H(x,p) = 0, \ p = \nabla V,$$

we call (1) the unsteady HJE.

From the Poincaré lemma and its converse [13], on a contractible domain in $\mathbb{R} \times \mathbb{R}^n$, the right-hand side of the unsteady HJE (1), which is $[p(t,x) \ \ -H(t,x,p)]^{\mathrm{T}}$, is the partial derivative of V with respect to (t, x) if and only if the Jacobian matrix

$$\begin{bmatrix} \frac{\partial p}{\partial x}(t,x) & \frac{\partial p}{\partial t}(t,x) \\ -\frac{\partial H}{\partial x}(t,x,p) - \frac{\partial H}{\partial p}(t,x,p)\frac{\partial p}{\partial x}(t,x) & -\frac{\partial H}{\partial t}(t,x,p) - \frac{\partial H}{\partial p}(t,x,p)\frac{\partial p}{\partial t}(t,x) \end{bmatrix}$$

is a symmetric matrix, that is, the following equations are satisfied.

$$\frac{\partial p}{\partial t}(t,x) = \left(-\frac{\partial H}{\partial x}(t,x,p) - \frac{\partial H}{\partial p}(t,x,p)\frac{\partial p}{\partial x}(t,x)\right)^{\mathrm{T}}, \tag{2}$$

$$\frac{\partial p}{\partial x}(t,x) = \left(\frac{\partial p}{\partial x}\right)^{\mathrm{T}}(t,x). \tag{3}$$

From (3), (2) leads to the following equation:

$$\frac{\partial p}{\partial t}(t,x) + \frac{\partial p}{\partial x}(t,x)\left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}}(t,x,p) = -\left(\frac{\partial H}{\partial x}\right)^{\mathrm{T}}(t,x,p). \tag{4}$$

In summary, $p(t, x)$ satisfying (3) and (4) is identical to $\nabla V(t, x)$ for some scalar-valued function $V(t, x)$ on a contractible domain in $\mathbb{R} \times \mathbb{R}^n$.

On the other hand, Hamilton's canonical equations,

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}}(x(t), t, p(t)), \tag{5}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t}(t) = -\left(\frac{\partial H}{\partial x}\right)^{\mathrm{T}}(x(t), t, p(t)), \tag{6}$$

are often related to the unsteady HJE. In nonlinear optimal control problems such as Example 1, an input satisfying the stationary condition is obtained by solving Hamilton's canonical equations with suitable boundary conditions. Here, we consider a particular problem in which p is a function of x and t as $p_0(t, x)$ with the assumption that the canonical equations (5) and (6) have solutions at all times for all initial conditions. A state feedback control law satisfying the stationary condition can be composed if p can be obtained as a function of x and t. For example, p satisfies the canonical equations (5) and (6) and is a function of x and t if p is a gradient of a solution to the unsteady HJE. By substituting $p = p_0(t, x)$ in (5), we obtain the following:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \frac{\partial H}{\partial p}(t, x, p_0(t,x)). \tag{7}$$

If we give an initial condition $x_0(t_0)$ for (7), its solution $x_0(t)$ can be determined. When substituting $x_0(t)$ into x of $p(t, x)$ in (6), $(x_0(t), p_0(t, x_0(t)))$ is a solution to the canonical equations (5) and (6) if and only if

$$\frac{\partial p_0}{\partial t}(t, x_0(t)) + \frac{\partial p_0}{\partial x}(t, x_0(t))\left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}}(t, x_0(t), p_0(t, x_0(t))) = -\left(\frac{\partial H}{\partial x}\right)^{\mathrm{T}}(t, x_0(t), p_0(t, x_0(t)))$$

is satisfied. If this equation holds at any initial time $t_0 \in \mathbb{R}$ for any initial state $x_0(t_0) \in \mathbb{R}^n$, then (4) holds. Thus, $p_0(t, x)$ such that $(x_0(t), p_0(t, x_0(t)))$ is a solution to the canonical equations satisfies (4). In addition, if $p_0(t, x)$ also satisfies (3), then $p_0(t, x)$ is the gradient of a solution to the unsteady HJE on a contractible domain in $\mathbb{R} \times \mathbb{R}^n$.

In general, it is difficult to solve (3) and (4) for $p(t, x)$. In this paper, we characterize a solution of the unsteady HJE with $H \in K_t[p]$.

On the basis of the definition of algebraic functions, we define an algebraic gradient and an algebraic costate as follows.

Definition 1: An analytic function $\rho : U \to \mathbb{C}$ defined on an open set $U \subset \mathbb{R} \times \mathbb{R}^n$ is said to be an algebraic function with respect to x if there exists a nonzero irreducible polynomial $\psi \in K_t[X]$ such that

$$\psi(t, x, \rho(t, x)) = 0$$

holds for all $(t, x) \in U$. For a solution $V(t, x)$ to (1), $\nabla V(t, x)$ is said to be an algebraic gradient if all of its components $\partial V(t, x)/\partial x_i$ $(i = 1, \dots, n)$ are algebraic functions with respect to x.

Definition 2: A vector-valued function $p_0(t, x)$ such that $(x_0(t), p_0(t, x_0(t)))$ satisfies Hamilton's canonical equations is said to be an algebraic costate if all of its components p_i $(i = 1, \dots, n)$ are algebraic functions with respect to x.

Remark 2: The argument of p_0 is often omitted. If we describe the argument of a function as (t, x, p_0), then the argument of p_0 is (t, x), and if we describe the argument of a function as $(t, x_0(t), p_0)$, then the argument of p_0 is $(t, x_0(t))$ with x_0 a solution of (7).

III. Algebraic Solution

A. Existence of a Solution

In this section, we give a necessary and sufficient condition for the existence of a solution to the unsteady HJE with an algebraic gradient. First, we show an existence condition for an algebraic costate, then, on the basis of the condition, an existence condition for an algebraic gradient is shown.

The Poisson bracket for two functions Ψ and Φ is defined as

$$\{\Psi, \Phi\} := \sum_{i=1}^{n} \left(\frac{\partial \Psi}{\partial x_i} \frac{\partial \Phi}{\partial p_i} - \frac{\partial \Psi}{\partial p_i} \frac{\partial \Phi}{\partial x_i} \right).$$

If the class of functions Ψ and Φ is restricted to the polynomial ring $K_t[p]$, the Poisson bracket can be viewed as a mapping $\{\cdot, \cdot\} : K_t[p] \times K_t[p] \to K_t[p]$ because $K_t[p]$ is closed under partial differentiation. We also define the involutiveness and H-invariance of an ideal with respect to the Poisson bracket as follows.

Definition 3: An ideal I of $K_t[p]$ is involutive if

$$\{\Psi, \Phi\} \in I, \ \forall \Psi \in I, \forall \Phi \in I.$$

Definition 4: For a given $H \in K_t[p]$, an ideal I of $K_t[p]$ is H-invariant if

$$\frac{\partial \Psi}{\partial t} + \{\Psi, H\} \in I, \ \forall \Psi \in I.$$

The existence of an algebraic costate is characterized in terms of an H-invariant ideal as follows.

Theorem 1: Hamilton's canonical equations (5) and (6) have an algebraic costate if and only if there exists an H-invariant maximal ideal.

Proof: (Necessity) If $p_0(t, x)$ is an algebraic costate, then all elements of p_0 are algebraic functions with respect to x. Let $\varphi : K_t[p] \to K_t[p_0]$ be a mapping that substitutes $p = p_0(t, x)$ for an element of $K_t[p]$, which is a surjective ring homomorphism over K_t. Kerφ is a maximal ideal and generated by n elements [8], which we denote by $I = \langle F_1, F_2, \dots, F_n \rangle$. Since $F_i \in \mathrm{Ker}\varphi$, we have $F(t, x, p_0) = 0$ for $F = [F_1, F_2, \dots, F_n]^{\mathrm{T}}$. By substituting $x = x_0(t)$ for an element of K_t, we also have $F(t, x_0(t), p_0(t, x_0(t))) = 0$. By differentiating the equality with respect to t, since $(x_0(t), p_0(t, x_0(t)))$ is a solution to the canonical equations, we have

$$\begin{aligned}
&\frac{\mathrm{d}F}{\mathrm{d}t}(t, x_0(t), p_0) \\
&= \frac{\partial F}{\partial t}(t, x_0(t), p_0) + \frac{\partial F}{\partial x_0}(t, x_0(t), p_0) \frac{\mathrm{d}x_0}{\mathrm{d}t}(t) \\
&+ \frac{\partial F}{\partial p}(t, x_0(t), p_0) \left(\frac{\partial p_0}{\partial t}(t, x_0(t)) + \frac{\partial p_0}{\partial x}(t, x_0(t)) \frac{\mathrm{d}x_0}{\mathrm{d}t}(t) \right) \\
&= \frac{\partial F}{\partial t}(t, x_0(t), p_0) + \frac{\partial F}{\partial x}(t, x_0(t), p_0) \frac{\partial H}{\partial p}(t, x_0(t), p_0) \\
&- \frac{\partial F}{\partial p}(t, x_0(t), p_0) \frac{\partial H}{\partial x}(t, x_0(t), p_0) = 0,
\end{aligned} \tag{8}$$

where the ith entry of the left-hand side is simply $\partial F_i/\partial t(t, x_0(t), p_0) + \{F_i, H\}(t, x_0(t), p_0)$. Equation (8) holds at any initial time $t_0 \in \mathbb{R}$ for any initial state $x_0(t_0) \in \mathbb{R}^n$. That is, $\partial F_i/\partial t(x_0(t_0), t_0, p_0) + \{F_i, H\}(x_0(t_0), t_0, p_0) = 0$ holds for all $(x_0(t_0), t_0) \in \mathbb{R} \times \mathbb{R}^n$. Since $\mathbb{R}$ is an infinite field, for $(t, x) \in K_t \times K_t^n$, we have

$$\frac{\partial F_i}{\partial t}(t, x, p_0) + \{F_i, H\}(t, x, p_0) = 0 \ (i = 1, \dots, n),$$

which is equivalent to $\partial F_i/\partial t + \{F_i, H\} \in \mathrm{Ker}\ \phi = I$. Since any $\Psi \in I$ can be expressed as $\Psi = \sum_{i=1}^{n} s_i F_i$, $s_i \in K_t[p]$, we have

$$\begin{aligned}
&\frac{\partial \Psi}{\partial t} + \{\Psi, H\} \\
&= \sum_{i=1}^{n} s_i \left(\frac{\partial F_i}{\partial t} + \{F_i, H\} \right) + F_i \left(\frac{\partial s_i}{\partial t} + \{s_i, H\} \right).
\end{aligned}$$

Since $\partial F_i/\partial t + \{F_i, H\}$ and F_i belong to I, $\partial \Psi/\partial t + \{\Psi, H\}$ also belongs to I, which implies that I is H-invariant.

(Sufficiency) Let $I \subset K_t[p]$ be an H-invariant maximal ideal. According to Hilbert's Nullstellensatz [8], there exists $p_0(t, x) \in \mathbf{V}(I) \subset \bar{K}_t \times \bar{K}_t^n$. That is, all elements of an affine algebraic variety $\mathbf{V}(I) \subset \bar{K}_t \times \bar{K}_t^n$ are algebraic functions with respect to x. In a similar manner to the proof of necessity, we construct F from the basis of the ideal I. It suffices to show if a maximal ideal I is H-invariant, then p_0 satisfies (4). Since $F(t, x, p_0) = 0$, $p_0(t, x) \in V(I)$, by substituting $x = x_0(t)$ for an element of K_t^n, we have $F(t, x_0(t), p_0(t, x_0(t))) = 0$. By differentiating the equality with respect to t, we have

$$\begin{aligned}
&\frac{\partial F}{\partial t}(t, x_0(t), p_0) + \frac{\partial F}{\partial x}(t, x_0(t), p_0) \frac{\partial H}{\partial p}(t, x_0(t), p_0) \\
&+ \frac{\partial F}{\partial p}(t, x_0(t), p_0) \bigg(\frac{\partial p_0}{\partial t}(t, x_0(t)) \\
&\qquad + \frac{\partial p_0}{\partial x}(t, x_0(t)) \frac{\partial H}{\partial p}(t, x_0(t), p_0) \bigg) = 0,
\end{aligned} \tag{9}$$

because $x_0(t)$ satisfies (7) for all $p_0(t, x_0)$. Since (9) holds at any initial time $t_0 \in \mathbb{R}$ for any initial state $x_0(t_0) \in \mathbb{R}^n$, for $(t, x) \in K_t \times K_t^n$, we have

$$\begin{aligned}&\frac{\partial F}{\partial t}(t,x,p_0) + \frac{\partial F}{\partial x}(t,x,p_0)\frac{\partial H}{\partial p}(t,x,p_0)\\&+\frac{\partial F}{\partial p}(t,x,p_0)\bigg(\frac{\partial p_0}{\partial t}(t,x)\\&\qquad\qquad+\frac{\partial p_0}{\partial x}(t,x)\frac{\partial H}{\partial p}(t,x,p_0)\bigg) = 0. \quad (10)\end{aligned}$$

On the other hand, the H-invariance of I leads to the following equality:

$$\begin{aligned}&\frac{\partial F}{\partial t}(t,x,p_0) + \frac{\partial F}{\partial x}(t,x,p_0)\frac{\partial H}{\partial p}(t,x,p_0)\\&\qquad -\frac{\partial F}{\partial p}(t,x,p_0)\frac{\partial H}{\partial x}(t,x,p_0) = 0. \quad (11)\end{aligned}$$

Since the right-hand sides of (10) and (11) are equivalent, we have

$$\begin{aligned}&\frac{\partial F}{\partial p}(t,x,p_0)\left(\frac{\partial p_0}{\partial t}(t,x) + \frac{\partial p_0}{\partial x}(t,x)\frac{\partial H}{\partial p}(t,x,p_0)\right)\\&= -\frac{\partial F}{\partial p}(t,x,p_0)\frac{\partial H}{\partial x}(t,x,p_0).\end{aligned}$$

From Lemma 2 in [5], $\partial F/\partial p(t,x,p_0)$ is nonsingular at all elements in $V(I)$, which implies

$$\frac{\partial p_0}{\partial t}(t,x) + \frac{\partial p_0}{\partial x}(t,x)\frac{\partial H}{\partial p}(t,x,p_0) = -\frac{\partial H}{\partial x}(t,x,p_0).$$

This is simply (4). Thus, for $p_0(t,x) \in V(I)$ and $x_0(t)$ satisfying (7), $(x_0(t), p_0(t, x_0(t)))$ satisfies Hamilton's canonical equations (5) and (6). ■

The existence of an algebraic gradient is characterized in terms of an H-invariant and involutive ideal as follows.

Theorem 2: The unsteady HJE (1) has a solution V with an algebraic gradient on a contractible domain in $\mathbb{R} \times \mathbb{R}^n$ if and only if there exists an H-invariant and involutive maximal ideal.

Proof: Since we assume that a solution to the unsteady HJE is defined on a contractible domain in $\mathbb{R} \times \mathbb{R}^n$, from the Poincaré lemma and its converse [13], $p(t, x)$ satisfying (3) and (4) is the gradient of a solution to the unsteady HJE. Therefore, a costate of Hamilton's canonical equations that satisfies (4) is the gradient of a solution to the unsteady HJE if and only if the costate satisfies (3) and, consequently, an algebraic costate is an algebraic gradient if and only if it satisfies (3). From Theorem 1, the existence of an algebraic costate and the existence of an H-invariant maximal ideal are equivalent. Thus, it suffices to show that the algebraic costate characterized by the H-invariant maximal ideal satisfies (3) if and only if the H-invariant maximal ideal is involutive. This can be proved in a similar manner to the proof of Theorem 1 in [5]. ■

Remark 3: The maximality of an ideal implies that a map $p_{0_i} : U \to \mathbb{C}$ is defined as an implicit function by a set of algebraic equations $F(t, x, p) = 0$. The H-invariance of an ideal implies that $(x_0(t), p_0(t, x_0(t)))$ is a solution of Hamilton's canonical equations, and especially that $\mathrm{d}p_0/\mathrm{d}t(t, x_0(t)) = \{p_0, H\}(t, x_0(t), p)$. Moreover, the involutiveness of an ideal implies that $p_0(t, x_0(t))$ satisfies (3).

Remark 4: In analytical dynamics, functions F_i $(i = 1, \ldots, n)$ are said to be first integrals [14] if $\partial F_i/\partial t + \{F_i, H\} = 0$ holds identically over $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. It is readily shown that if the functions F_i $(i = 1, \ldots, n)$ are first integrals, then the ideal $I = \langle F_1, \ldots, F_n\rangle$ is H-invariant. However, the converse is not true in general, and the H-invariance of an ideal is a weaker condition than the first integrability of functions. According to [5], the involutiveness of an ideal is a weaker condition than the involutiveness of functions in analytical dynamics. Therefore, even if there exists a solution with an algebraic gradient, Liouville's theorem [14] is not necessarily applicable to guarantee the existence of a complete solution with n arbitrary constants. Similarly, the existence of a solution with an algebraic gradient does not necessarily imply algebraic complete integrability [15], an algebraic expression of Liouville' s theorem.

Remark 5: In a finite-horizon linear-quadratic regulator problem for time-invariant systems, the numerator and denominator of each element of a solution to the Riccati differential equation (RDE) are exponential functions with a variable t [1], which implies $s_{ij}(t) \in K_t$, where $s_{ij}(t)$ is the (i, j) element of $S(t)$, which is a solution to the RDE. In the RDE, a gradient p of a solution to the unsteady HJE is expressed as $p = S(t)x$. In this case, the basis of an H-invariant and involutive maximal ideal is given by $F_i = p_i - \sum_{j=1}^{n} s_{ij}x_j$. In a linear-quadratic regulator problem for time-varying systems, it is only guaranteed that $\dot{s}_{ij}(t)$ and $\ddot{s}_{ij}(t)$ exist and are continuous [1]. However, it is not always guaranteed that $s_{ij}(t)$ are meromorphic functions and, consequently, belong to $\bar{K}_t$.

It can be determined whether a maximal ideal $I = \langle F_1, \ldots, F_n\rangle$ is H-invariant or not by computing the Poisson bracket for two functions H and F_i according to the proof of Theorem 1, and a similar argument holds for involutiveness. If an H-invariant maximal ideal $I = \langle F_1, \ldots, F_n\rangle$ can be found for the unsteady HJE, then an algebraic gradient $p_0(t, x)$ is obtained by only solving $F(t, x, p) = 0$ with respect to p, where $p_0(t, x)$ is in $\mathbf{V}(I) \subset \bar{K}_t \times \bar{K}_t^n$. Therefore, the gradient of a solution can be determined by solving a set of algebraic equations pointwise without storing the solution over a domain in time and the state space. However, the H-invariance condition leads to a set of partial differential equations for unknown meromorphic functions in time t and rational functions in the state x, and the involutiveness condition leads to a set of partial differential equations for unknown rational functions in the state x. These equations are still difficult to solve in general. However, an example of a class of involutive maximal ideals is given as follows.

Proposition 1: [5] Let $\Phi_i \in \mathbb{R}(x_i)[X_i] \setminus \mathbb{R}[X_i]$ be a monic irreducible polynomial, and assume $a_i \in K$ satisfies $\partial a_i/\partial x_j = \partial a_j/\partial x_i$. Then, for $F_i(p) = \Phi_i(p_i - a_i)$, $\langle F_1, \ldots, F_n\rangle$ is an involutive maximal ideal.

It is straightforward to extend Proposition 1 as follows.

Proposition 2: Let $\Phi_i \in \mathbb{R}_t(x_i)[X_i] \setminus \mathbb{R}_t[X_i]$ be a monic irreducible polynomial, and assume $a_i \in K_t$ satisfies $\partial a_i/\partial x_j = \partial a_j/\partial x_i$. Then, for $F_i(p) = \Phi_i(p_i - a_i)$, $\langle F_1, \ldots, F_n \rangle$ is an involutive maximal ideal.

B. Stabilizing Solution

If an H-invariant and involutive maximal ideal can be found for the unsteady HJE, then an algebraic gradient is obtained by only solving a set of algebraic equations. However, the solution to the set of equations is not unique. For the analysis and design of control systems, the gradient of the stabilizing solution is the most important branch. We characterize the gradient of a stabilizing solution in the following.

Definition 5: A solution $V(t,x)$ to the unsteady HJE is called a stabilizing solution if it is defined on an open set containing the origin, if $V(t,0) = 0$ and $\nabla V(t,0) = 0$ hold for all t, and if the origin is an uniformly asymptotically stable equilibrium of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} (t, x, \nabla V). \tag{12}$$

We define the following matrices:

$$A(t) := \frac{\partial^2 H}{\partial p \partial x}(t,0,0),\ B(t) := \frac{\partial^2 H}{\partial p^2}(t,0,0),$$
$$C(t) := \frac{\partial^2 H}{\partial x^2}(t,0,0),\ X(t) := \frac{\partial^2 V}{\partial x^2}(t,0,0).$$

Then, the linearized system of (12) at the origin is expressed as

$$\mathrm{d}\xi/\mathrm{d}t = (A(t) + B(t)X(t))\xi. \tag{13}$$

Moreover, it is readily shown by differentiating the unsteady HJE that $X(t)$ is a solution to the following RDE:

$$\begin{aligned} -\mathrm{d}X/\mathrm{d}t(t) = {} & A^{\mathrm{T}}(t)X(t) + X(t)A(t) \\ & + X(t)B(t)X(t) + C(t). \end{aligned}$$

For the RDE, its solution $X(t)$ is called a stabilizing solution if the origin is an uniformly asymptotically stable equilibrium of system (13).

The following proposition relates a stabilizing solution to the unsteady HJE with a stabilizing solution to the RDE.

Proposition 3: [2] Let V be a solution to the unsteady HJE with $V(t,0) = 0$ and $\nabla V(t,0) = 0$ for all t. If $X(t)(= \partial^2 V/\partial x^2(t,0))$ is a stabilizing solution to the RDE, then $V(t,x)$ is a stabilizing solution to the unsteady HJE.

This proposition was originally given for the unsteady HJE and the RDE in the H_∞ problem: however, it also holds in the present problem setting.

An existence condition for a stabilizing solution to the RDE is known to be as follows.

Proposition 4: [16] Suppose that $B(t) = -\hat{B}^{\mathrm{T}}(t)\hat{B}(t)$ for some matrix $\hat{B}(t)$ and that $C(t) = -\hat{C}^{\mathrm{T}}(t)\hat{C}(t)$ for some matrix $\hat{C}(t)$. An RDE has a stabilizing solution that is also positive semidefinite and uniformly bounded if $(A(t), \hat{B}(t))$ is stabilizable and $(A(t), \hat{C}(t))$ is detectable.

If the assumptions in Proposition 4 hold, Proposition 3 implies that the stabilizing solution to the nonlinear optimal regulator problem can be obtained by taking a branch of the algebraic gradient such that $p(t,0) = 0$ for all t and $\partial p(t,0)/\partial x$ is positive semidefinite and uniformly bounded.

IV. Examples

In this section, we obtain an explicit solution to the unsteady HJE as algebraic functions based on Proposition 2. Let be $n = 2$ and $H \in K_t[p]$ be given by

$$H = f_1 p_1 + f_2 p_2 - (g_1 p_1 + g_2 p_2)^2/2 + (x_1^2 + x_2^2)/2.$$

If the state equation is defined in the neighborhood of the origin, the denominators of the entries of f and g do not vanish at the origin. Thus, f and g belong to $R_t := \mathbb{R}_t[x]_{\langle x \rangle}$. Furthermore, if the origin is an equilibrium, $f(t,0) = 0$ holds, and therefore, the component of $f = [f_1, f_2]^{\mathrm{T}}$ belongs to $M := \langle x \rangle R_t$. R_{t_i}, M_i and $R_{t_i}^{\times}$ denote intersections of $\mathbb{R}_t(x_i)$ $(i = 1,2)$, with $\mathbb{R}_t$, M and the set of units (invertible elements) of R_t, respectively.

On the basis of Proposition 2, we consider the involutive ideal generated by the following polynomials:

$$\left\{ \begin{array}{l} F_1 = p_1 + b_{11}(x_1, t), \\ F_2 = p_2^2 + b_{21}(x_2)p_2 + b_{22}(x_2). \end{array} \right. \tag{14}$$

Proposition 5: Assume that $g_1 \in R_t, g_2 \in R_{t_2}^{\times}$ and $\sqrt{b_{21}^2 - 4b_{22}} \notin K_t$. If for some $\hat{b}_{11} \in \mathbb{R}_t$ and $b_{21} \in M_2$,

$$b_{11} = x_1 \hat{b}_{11},\ b_{22} = -x_2^2/g_2^2 \neq 0, \tag{15}$$

$$f_1 = \frac{x_1}{2}\left\{\frac{1}{\hat{b}_{11}}\left(1 - \partial\hat{b}_{11}/\partial t\right) - \hat{b}_{11} g_1^2\right\}, \tag{16}$$

$$f_2 = -b_{11} g_1 g_2 - g_2^2 b_{21}/2 \tag{17}$$

are satisfied, then $f_1(t,0) = 0$ and $f_2(t,0) = 0$ for all t, and the unsteady HJE (1) has a solution with an algebraic gradient p defined as a zero of (14).

Proof: We show that $I = \langle F_1, F_2 \rangle$ is an H-invariant and involutive maximal ideal if the assumptions of Proposition 5 hold. First, $f_1 \in x_1 R_t \subset M$ and $f_2 \in \langle b_{11} \rangle + \langle b_{21} \rangle \subset \langle x_1 \hat{b}_{11} \rangle + M_2 \subset M$ imply that $f_1(0) = 0$ and $f_2(0) = 0$. From the definitions of b_{11} and b_{22}, b_{11} and b_{22} belong to $\mathbb{R}_t(x_1) \backslash \mathbb{R}_t$ and $\mathbb{R}_t(x_2) \backslash \mathbb{R}_t$, respectively. $\sqrt{b_{21}^2 - 4b_{22}} \notin K_t$ implies that F_2 is irreducible. Thus, $I = \langle F_1, F_2 \rangle$ is an involutive maximal ideal according to Proposition 2.

It suffices to show that the involutive maximal ideal I is H-invariant. From the proof of Theorem 1, we only have to verify that $\partial F_i/\partial t + \{F_i, H\} \in I$ $(i = 1,2)$, which are satisfied if

$$\frac{\partial}{\partial x_1}\left(f_2 + b_{11} g_1 g_2 + \frac{b_{21} g_2^2}{2}\right) = 0$$

$$2\frac{\partial b_{11}}{\partial t} + \frac{\partial}{\partial x_1}\left(2 b_{11} f_1 + (b_{11} g_1)^2 - x_1^2 - b_{22} g_2^2\right) = 0$$

$$\begin{aligned} \frac{\partial}{\partial x_2}\biggl(& 2 b_{11} f_1 + (b_{11} g_1)^2 + b_{21}\left(f_2 + b_{11} g_1 g_2 + \frac{b_{21} g_2^2}{2}\right) \\ & - b_{22} g_2^2 - x_2^2 \biggr) = 0 \end{aligned}$$

$$\frac{b_{21}}{2}\frac{\partial(2b_{11}f_1+(g_1b_{11})^2)}{\partial x_2}+2b_{22}\frac{\partial(f_2+b_{11}g_1g_2+b_{21}g_2^2/2)}{\partial x_2}$$
$$+\left(f_2+b_{11}g_1g_2+\frac{b_{21}g_2^2}{2}\right)\frac{\partial b_{22}}{\partial x_2}-\frac{b_{21}}{2}\frac{\partial(b_{22}g_2^2+x_2^2)}{\partial x_2}=0.$$

They hold if (15), (16) and (17) hold. ∎

On the basis of Proposition 5, an example of an explicit solution is given.

Example 2: Suppose a state equation and cost function are given as follows:

$$\dot{x}=\begin{bmatrix} -\mathrm{e}^{-t}x_1/2 \\ x_1/(1+\mathrm{e}^{-t})-x_2^3/2 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}u,$$
$$J=\frac{1}{2}\int_{t_0}^{\infty}(x_1^2+x_2^2+u^2)\mathrm{d}t.$$

According to Proposition 5, the cost function has a value function with an algebraic gradient p defined as a zero of the following polynomials:

$$\begin{cases} p_1-x_1/(1+\mathrm{e}^{-t})=0, \\ p_2^2+x_2^3p_2-x_2^2=0. \end{cases}$$

Because of Propositions 3 and 4, a branch of the algebraic gradient p such that $\partial p(t,0)/\partial x$ is positive semidefinite is chosen as

$$p(x)=\begin{bmatrix} x_1/(1+\mathrm{e}^{-t}) & \left(-x_2^3+x_2\sqrt{4+x_2^4}\right)/2 \end{bmatrix}^{\mathrm{T}}.$$

To show that the assumption of Proposition 4 holds, we only need to verify the stabilizability of the linearized system at the origin since the linearized system is obviously detectable. In particular, we focus on showing that the linearized system of the closed-loop system using the feedback law

$$u=-g^{\mathrm{T}}p=-\frac{x_1}{1+\mathrm{e}^{-t}}-\frac{-x_2^2+x_2\sqrt{4+x_2^2}}{2}$$

is uniformly asymptotically stable at the origin. The linearized closed-loop system is computed as

$$\dot{\xi}=(A(t)+B(t)X(t))\xi,$$
$$A(t)+B(t)X(t)=\begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2}-\frac{1}{1+\mathrm{e}^{-t}} & -1 \\ 0 & -1 \end{bmatrix},$$

and $A(t)+B(t)X(t)$ is continuous. By transforming the RDE to the Lyapunov equation for time-varying linear systems, we have

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t)+X(t)(A(t)+B(t)X(t))$$
$$+(A(t)+B(t)X(t))^{\mathrm{T}}X(t)=-(C(t)+X(t)B(t)X(t)),$$

where

$$X(t)=\frac{\partial p}{\partial x}(t,0)=\begin{bmatrix} \frac{1}{1+\mathrm{e}^{-t}} & 0 \\ 0 & 1 \end{bmatrix},$$

which is continuously differentiable, positive definite and uniformly bounded. Then $C(t)+X(t)B(t)X(t)$ is continuous, positive definite and uniformly bounded, where

$$B(t)=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ C(t)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From the Lyapunov stability theorem for time-varying linear systems [17], the linearized system is uniformly asymptotically stable at the origin. Thus, p is the gradient of a stabilizing solution. The value function is expressed as

$$V(t,x)=\int_0^x\frac{\partial V}{\partial \xi}(\xi)\mathrm{d}\xi=\int_0^1 p^{\mathrm{T}}(sx)x\mathrm{d}s.$$

The line integral yields the explicit value function

$$V(t,x)=\frac{1}{8}\left(\frac{4x_1^2}{1+\mathrm{e}^{-t}}-x_2^4+x_2^2\sqrt{4+x_2^4}+4\sinh^{-1}\frac{x_2^2}{2}\right).$$

V. Conclusion

In this paper, the polynomial-type HJE for a Hamiltonian with coefficients belonging to meromorphic functions of time and rational functions of the state was considered. A necessary and sufficient condition for the existence of a solution with an algebraic gradient was characterized in terms of an H-invariant and involutive maximal ideal. If an H-invariant and involutive maximal ideal is found, an algebraic gradient can be obtained by only solving a set of algebraic equations. Although the algebraic gradient is not unique, a method of choosing its branches using a stabilizing solution is introduced. Finally, a class of nonlinear optimal regulator problems has been given such that the gradients of explicit solutions are obtained as algebraic functions, and an example of an explicit solution was also presented.

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