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Optimal Continuous Approximation of Basic Fractional Elements: Theory and Applications

Martin Čech and Miloš Schlegel

Abstract— **In the last two decades, a boom of fractional calculus applications started in many technical areas including automation and process control. The generalization of integrals and derivatives to arbitrary real order (FO – Fractional Order) simplifies solution of many problems especially in frequency domain. Unfortunately, switching into time domain is always quite difficult due to the necessity to approximate fractional elements by integer-order ones. For this purpose, often a high order zero/pole transfer function is employed. This paper extends the authors' previous work and summarizes the results of numerical optimization of zero/pole positions for two important fractional elements: fractional integro-differential operator and fractional pole. The optimization is done on a limited frequency band up to four decades. The quadratic difference between the frequency response of ideal FO element and its zero/pole approximation was taken as an optimality criterion. It is shown, that the optimization decreases markedly the criterion value compared to traditional methods. The paper main results are provided in a form of analytical functions parametrizing the zero/pole positions dependent on element order. Additionally, prospective applications of presented fractional elements are discussed from both controller synthesis and process modeling point of view.**

I. INTRODUCTION

A. Fractional calculus

Fractional Calculus is an interdisciplinary and emerging research area [10], [4]. In the last two decades, a boom of fractional calculus (FC) applications started in many technical areas including automation and process control. It was studied from both controller synthesis [14], [8] and process modeling point of view [18], [3]. The growing scientific effort resulted into number of practical applications [15], [16]. The generalization of integrals and derivatives to arbitrary real order (FO – Fractional Order) leads to more flexible transfer functions $F(s)$ with non-integer power of complex variable s. It is worth noting that such approach may reduce significantly the complexity of many problems especially when working in frequency domain. However, for simulation or even real-time implementation purposes one needs an integer-order (IO) time domain realization of FO element. It can be shown, that the 'ideal' realization always leads to infinite order filter [9], [7]. Hence, several limitations must be taken into account and we will speak further only about *approximation*. Remind that the fractional-order systems are still linear and their frequency response may be simply computed by substituting $s = j\omega$. Consequently, finding IO approximation may be regarded as a frequency domain filter design where the specifications are defined by the reference FO element.

In this paper, the idea of global continuous approximation is followed. It is based on assumption that one can usually specify the filter order or required approximation precision on a defined frequency band. Traditionally, the quality of approximation is measured by the quadratic difference between frequency response of ideal FO element and its corresponding IO equivalent. Charef's and Oustaloup's methods together with their modifications are typical representatives of such approach [3], [11], [7]. They approximate the fractional elements by classical transfer function with zeros and poles spread equidistantly in the logarithmic space. Unfortunately, the quality obtained is not sufficient namely for approximating filters with low order up to five. Often also the methods based on continued fraction expansion (CFE) are used [20]. The bad control of frequency band and approximation precision is their main disadvantage. In this paper, it is shown that the numerical optimization of zero/pole locations decreases markedly the criterion value compared to mentioned traditional methods. The optimization is done on two, three and four frequency decades for filters with order three, four and five, respectively. Next, the lower frequency boundary is normalized to the frequency $\omega_L = 1$. However, it is shown how to recompute the filter parameters for arbitrary value of ω_L. Afterwards, all zero/pole positions are approximated by analytical functions which can be simply evaluated for given order of fractional element. Note, that to obtain a discrete filter from the low order continuous approximation one can use any discretization method for specified sampling period. This 'two-step' technique is often called indirect discretization. Finally remark that the low order approximations eliminate a lot of numerical problems and are important especially for FO element implementation on compact or embedded devices with limited computational power.

The rest of the paper is organized as follows: Section I continues with brief introduction of two fractional elements. Their optimal continuous approximation is described in Section II. Possible applications of obtained filters are outlined in Section III. Conclusions and ideas for further work are summarized in Section IV.

M. Čech is with Department of Cybernetics, University of West Bohemia in Pilsen, Czech Republic mcech@kky.zcu.cz

M. Schlegel is with Department of Cybernetics, University of West Bohemia in Pilsen, Czech Republic schlegel@kky.zcu.cz

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B. Fractional integro-differential operator

The fractional integro-differential operator is described by the transfer function

$$F(s) = \frac{1}{s^m}, \quad m \in \mathbb{R} \tag{1}$$

and corresponding frequency response

$$F(\mathrm{j}\omega) = \frac{1}{(\mathrm{j}\omega)^m}, \quad m \in \mathbb{R}. \tag{2}$$

It is called fractional integrator ($m > 0$) or derivator ($m < 0$). It is easy to check that the frequency response (2) is a straight line crossing the origin of the complex plane. Thus the fractional-order integrator (1) may be called also a *constant-phase filter* (or filter with the constant phase response). Point out, that in the filter synthesis field this term often denotes filters with constant phase delay which have in our context linear phase response. Also some relation to Hilbert transformer may be observed [12]. The constant phase response requirement is not so obvious and even in newest filter design tools (f.e. Matlab signal processing toolbox [6]) there is no straightforward way to design such filter. Although one can use the `invfreqs()` Matlab function [6] to design filter with reference frequency model, the quality obtained is also not satisfactory[1]. The fractional integrator is most often used in electrochemistry as a model of capacitive elements [17]. Another applications may be found in [2]. In Section III, the utilization of FO integrator in process control applications will be explored.

C. Fractional pole

Fractional-order poles described by the transfer function

$$F(s) = \frac{1}{(\tau s + 1)^m}, \quad \tau, m \in \mathbb{R}^+ \tag{3}$$

and frequency response

$$F(\mathrm{j}\omega) = \frac{1}{(\tau \mathrm{j}\omega + 1)^m}, \quad \tau, m \in \mathbb{R}^+ \tag{4}$$

are basic building elements in process modeling as shown in [3]. Their series connection

$$F(s) = \prod_{i=0}^{n} \frac{1}{(\tau_i s + 1)^{m_i}}, \quad \tau_i,\ m_i \in \mathbb{R}^+,\ i = 1, 2, \ldots, n \tag{5}$$

allows to shape a frequency response $F(\mathrm{j}\omega)$ by only few parameters τ_i, m_i. The model (5) may be used for approximation of transcendent transfer functions describing real physical systems with distributed parameters (like heat transfer [1]). It also approximates very well processes with stable zeros. Compared to classical integer-order transfer functions, the number of parameters is markable lower while the approximation precision remains the same.

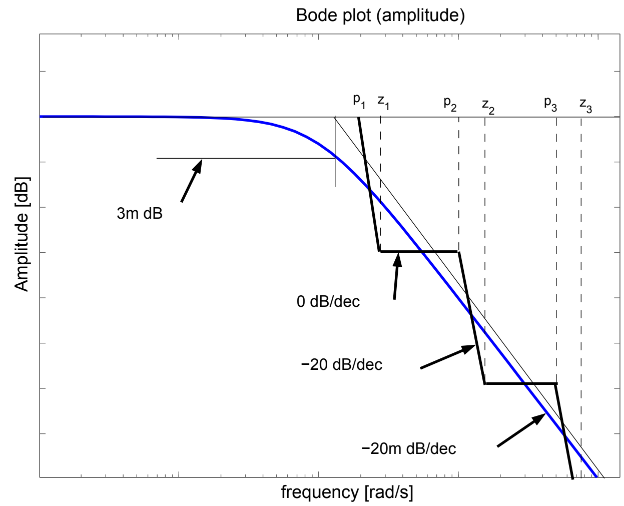


Fig. 1. Example of fractional pole approximation by alternating zeros and poles (equidistant zeros and poles positions); ideal response (blue), asymptotic zero/pole characteristic (black)

II. FILTER OPTIMIZATION

Our aim is to find a 'best' low-order IO approximation of basic fractional elements (1) and (3). Without loss of generality one can restrict to the case where $|m| < 1$. For m outside this limit the fractional element may be modeled as a series connection of classical integer part and fractional part with order inside the limit. Note that both elements generate the amplitude slope of $-20m$ $[\mathrm{dB/dec}]$.

Assumption 1: The zero/pole positions of an optimal filter are not 'far' from those given by Charef's or Oustaloup's method, e.g. almost equidistant around logarithmic frequency axes (see Fig. 1).

Hence, the IO approximation filter form is chosen as

$$\widehat{F}(s) = 10^{K_0} \frac{\prod_{i=1}^{N} (10^{-\omega_{Z_i}} s + 1)}{\prod_{i=1}^{N} (10^{-\omega_{P_i}} s + 1)}. \tag{6}$$

Our task is to find for each m the set of parameters $\mathbf{x} = [K_0, \omega_{P_1} \ldots \omega_{P_N}, \omega_{Z_1} \ldots \omega_{Z_N}]$ which minimizes the criterion

$$J = \int_{\omega_L}^{\omega_H} |\widehat{F}(\mathrm{j}\omega) - F(\mathrm{j}\omega)|^2 \mathrm{d}\omega, \tag{7}$$

where $F(\mathrm{j}\omega)$ is defined by (2) or (4). The standard Matlab `fmincon` [5] procedure was used for this constrained numerical optimization. One can check in Fig. 1, that to follow the $-20m$ $[\mathrm{dB/dec}]$ slope, the zeros and poles must be orderly alternating.

Assumption 2: The optimal solution for $1 > m > 0$ satisfies the condition

$$\omega_{P_1} < \omega_{Z_1} < \omega_{P2} < \cdots < \omega_{P_N} < \omega_{Z_N}. \tag{8}$$

[1]The quality of known methods is not satisfactory namely in the case when one needs to hold the constant phase property.

Corollary 1: Its suitable to define the initial parameter vector $\mathbf{x_0}$ according to Oustaloup's method which satisfies the condition (8). Such starting vector speeds up the optimization and ensures that the global minimum will be found. Both fractional elements are approximated by IO filters with order three, four and five on the bandwidth two, three and four decades, respectively.

Remark 1: The filter order N and the bandwidth (ω_L, ω_H) were chosen experimentally to ensure that the maximal phase response error will be less than 1 degree in the worst case ($m = 0.5$). Naturally, the filters may be recomputed for arbitrary bandwidth and filter order.

Finally, the zero/pole positions computed for discrete values of order m are approximated by analytical functions in the form

$$f_1(m) = a_0 + a_1 m + a_2 m^2, \quad (9)$$

$$f_2(m) = a_0 e^{a_1 m + a_2 m^2 + a_3 m^3 + a_4 m^4} \quad (10)$$

allowing to evaluate the filter parameters for arbitrary value of m. Let us point out that this step is necessary for practical implementation.

A. Fractional integro-differential operator

The filter zero/pole positions are computed for $m = -0.95, -0.9, -0.85, \ldots, 0.9, 0.95$. The optimization results for all filter orders are summarized in Fig. 2. It can be observed that the zero/pole positions are symmetrical in the logarithmic frequency space around both the central frequency and the m axes. After selecting the order m one can find in the figure the positions of all filter poles (marked by '+') and zeros (marked by 'o'). However these positions for $m > 0$ may be computed simply from approximating functions summarized in Table I.

Remark 2: It is evident that the fractional derivator of given order may be obtained from the integrator of the same order simply by substituting poles by zeros and vice versa.

Next, the optimal approximating filter is compared to the others obtained by well known methods. It can be verified in Fig. 4 that the proposed filter is the only one whose phase response lies for $m = 0.5$ in 1 deg band around the nominal value 45 deg. Moreover, the phase response is composed by regular waves known from Chebyshev polynomials.

Remark 3: The value $m = 0.5$ was chosen for all comparisons as it is the worst case. For $m \neq 0.5$, the phase response wave amplitude (phase error) is always lower.

The proposed filter has a lowest value of criterion (7) as it is demonstrated in Table II. Definitely, it is useful to test the filter behavior also in time domain. The theoretical fractional integrator step response $h(t)$ may be computed as [13]

$$h(t) = \frac{t^m}{m\Gamma(m)}, \quad (11)$$

where the Γ function is the generalization of factorial [13]. The comparison of theoretical and filter step response is shown in Fig. 3.

Remark 4: Integrating with a time constant not equal to one (e.g. $1/(\tau_i s)^m$) needs just to multiply the filter gain by $1/(\tau_i)^m$.

TABLE I

FRACTIONAL INTEGRATOR: APPROXIMATION FUNCTIONS ($m > 0$)

		2 decades (3-th order filter)
K_0	=	$2.0299\ e^{-0.0164m+1.4952m^2-2.7888m^3+1.8795m^4}$
Z	=	$1.9373\ e^{-0.0057m+1.5584m^2+2.8336m^3+1.8859m^4}$
P	=	$1.9088\ e^{-0.2936m+0.0341m^2-0.0348m^3+0.0063m^4}$
ω_{Z1}	=	Z
ω_{Z2}	=	$1.0000 + 0.4281m$
ω_{Z3}	=	$-P + 2$
ω_{P1}	=	P
ω_{P2}	=	$1.0000 - 0.4281m$
ω_{P3}	=	$-Z + 2$
		3 decades (4-th order filter)
K_0	=	$2.0301\ e^{-0.0274m+1.5077m^2-2.8068m^3+1.8906m^4}$
Z	=	$2.9158\ e^{0.0272m+1.2249m^2+2.2259m^3+1.4754m^4}$
P	=	$2.8822\ e^{-0.1987m+0.0310m^2-0.0254m^3+0.0060m^4}$
ω_{Z1}	=	Z
ω_{Z2}	=	$1.0599 + 0.4435m - 0.0050m^2$
ω_{Z3}	=	$1.9401 + 0.4436m + 0.0049m^2$
ω_{Z4}	=	$-P + 3$
ω_{P1}	=	P
ω_{P2}	=	$1.0599 - 0.4435m - 0.0050m^2$
ω_{P3}	=	$1.9401 - 0.4436m + 0.0049m^2$
ω_{P4}	=	$-Z + 3$
		4 decades (5-th order filter)
K_0	=	$2.0304\ e^{-0.0380m+1.5210m^2-2.8258m^3+1.9021m^4}$
Z	=	$3.8982\ e^{0.0358m+1.0075m^2+1.8306m^3+1.2101m^4}$
P	=	$3.8610\ e^{-0.1491m+0.0230m^2-0.0150m^3+0.0026m^4}$
ω_{Z1}	=	Z
ω_{Z2}	=	$1.0993 + 0.4539m - 0.0062m^2$
ω_{Z3}	=	$2.0000 + 0.4508m - 0.0001m^2$
ω_{Z4}	=	$2.9006 + 0.4539m + 0.0060m^2$
ω_{Z5}	=	$-P + 4$
ω_{P1}	=	P
ω_{P2}	=	$1.0994 - 0.4539m - 0.0062m^2$
ω_{P3}	=	$2.0000 - 0.4508m - 0.0001m^2$
ω_{P4}	=	$2.9006 - 0.4539m + 0.0060m^2$
ω_{P5}	=	$-Z + 4$

TABLE II

FRACTIONAL INTEGRATOR CRITERION VALUES ($m = 0.5,\ N = 4$)

method	value of J
Oustaloup / Charef	0.0229
modified Oustaloup	0.0233
invfreqs()	0.0081
CFE	0.127
proposed filter	0.0057

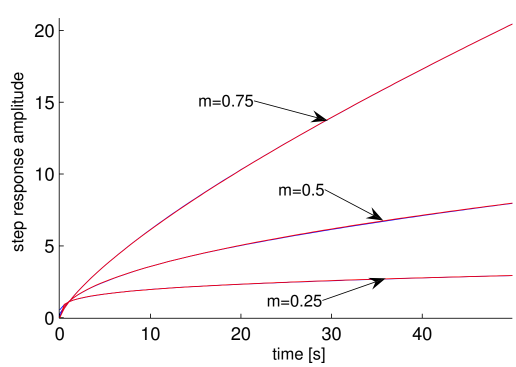
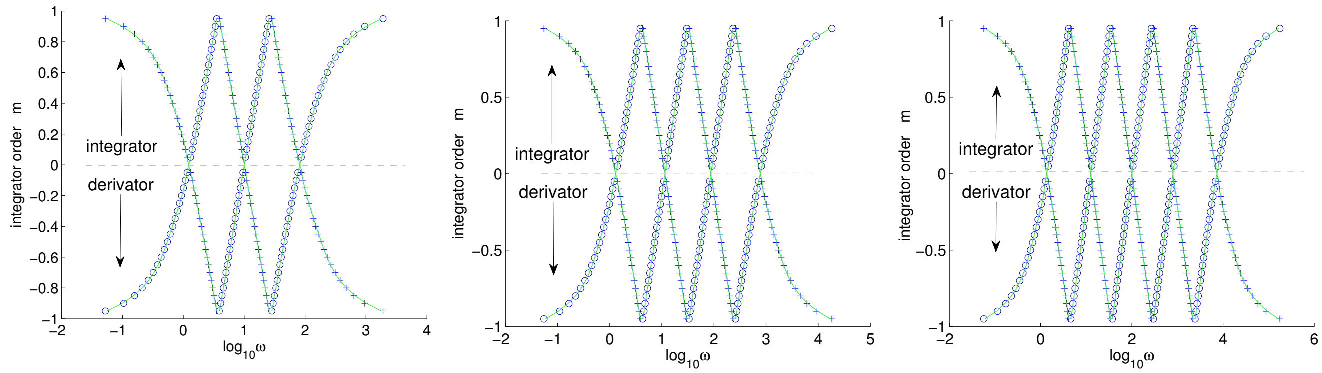


Fig. 3. Comparison of step responses of ideal fractional integrator (red) and its 4-th order IO approximation (blue).



(a) $N = 3, \omega_L = 10^0,\ \omega_H = 10^2$ (b) $N = 4, \omega_L = 10^0,\ \omega_H = 10^3$ (c) $N = 5, \omega_L = 10^0,\ \omega_H = 10^4$

Fig. 2. Fractional integrator optimization results: zeros (o) and pole (+) positions for (a) two, (b) three, (c) four decades; approximating functions – green

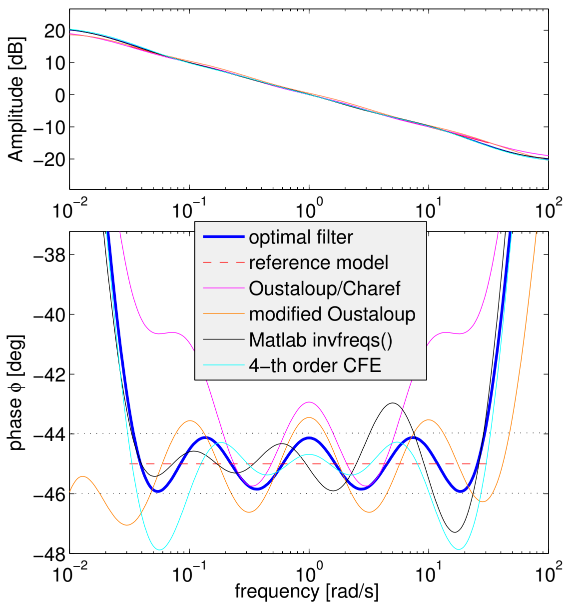


Fig. 4. Comparison of optimal 4-th order approximation at 3 decades for $m = 0.5$ (worst case) with well-known methods. Results are shifted to the central frequency $\omega_C = 1$.

TABLE III

FRACTIONAL POLE: APPROXIMATION FUNCTIONS ($m > 0$)

		2 decades (3-th order filter)
K_0	=	1
ω_{Z1}	=	$0.3678 + 0.3558m - 0.0265m^2$
ω_{Z2}	=	$1.1191 + 0.3766m - 0.0054m^2$
ω_{Z3}	=	$1.9912\ \mathrm{e}^{-0.0100m+1.5423m^2-2.8039m^3+1.8684m^4}$
ω_{P1}	=	$0.3691 - 0.3692m - 0.0022m^2$
ω_{P2}	=	$1.1175 - 0.3752m - 0.0411m^2$
ω_{P3}	=	$1.9590 - 0.5019m + 0.0365m^2$
		3 decades (4-th order filter)
K_0	=	1
ω_{Z1}	=	$0.4035 + 0.3878m - 0.0428m^2$
ω_{Z2}	=	$1.2099 + 0.4057m - 0.0321m^2$
ω_{Z3}	=	$2.0290 + 0.4071m - 0.0057m^2$
ω_{Z4}	=	$2.9535\ \mathrm{e}^{-0.0345m+1.2173m^2-2.2135m^3+1.4686m^4}$
ω_{P1}	=	$0.4056 - 0.4036m - 0.0050m^2$
ω_{P2}	=	$1.2088 - 0.3997m - 0.0598m^2$
ω_{P3}	=	$2.0276 - 0.4029m - 0.0401m^2$
ω_{P4}	=	$2.9165 - 0.5178m + 0.0350m^2$
		4 decades (5-th order filter)
K_0	=	1
ω_{Z1}	=	$0.3607 + 0.3830m - 0.0395m^2$
ω_{Z2}	=	$1.2020 + 0.4295m - 0.0437m^2$
ω_{Z3}	=	$2.0732 + 0.4315m - 0.0266m^2$
ω_{Z4}	=	$2.9468 + 0.4332m - 0.0032m^2$
ω_{Z5}	=	$3.9206\ \mathrm{e}^{-0.0381m+1.0022m^2-1.8237m^3+1.2066m^4}$
ω_{P1}	=	$0.3641 - 0.4106m + 0.0428m^2$
ω_{P2}	=	$1.2013 - 0.4314m - 0.0658m^2$
ω_{P3}	=	$2.0723 - 0.4297m - 0.0538m^2$
ω_{P4}	=	$2.9456 - 0.4310m - 0.0354m^2$
ω_{P5}	=	$3.8804 - 0.5346m + 0.0339m^2$

B. *Fractional-order pole*

Similarly to previous Section II-A, the filter zero/pole positions were computed for discrete values of $m = 0.05, 0.1, \ldots, 0.95$. The optimization results for $m > 0$ are shown in Fig. 5 and the approximating functions are summarized in Table III. Furthermore, the frequency responses of reference model (4) are compared to those obtained from approximating filter in Fig. 6. The step response of fractional pole (3) may be computed by numerical integration of its impulse response $g(t)$ given by

$$g(t) = \frac{t^{m-1}}{\Gamma(m)} \mathrm{e}^{-t}. \qquad (12)$$

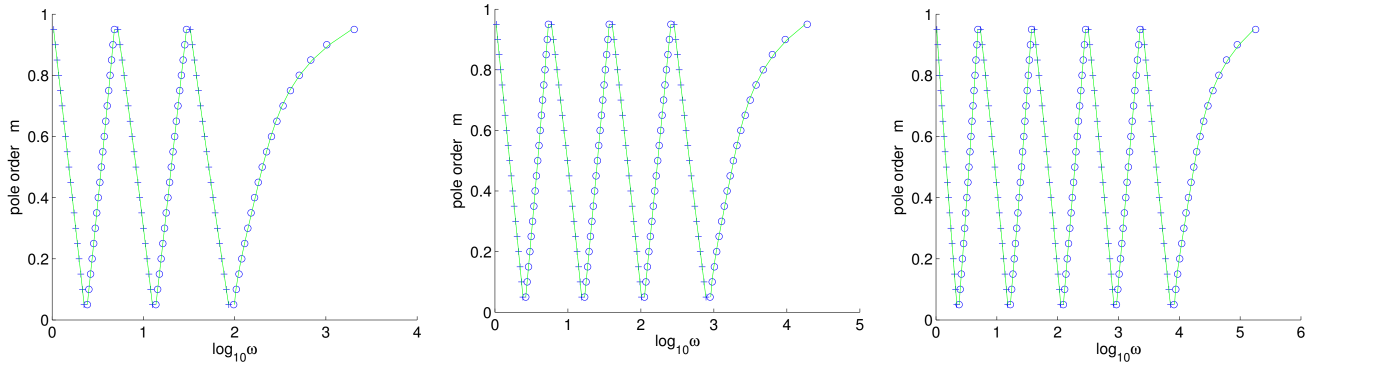
The comparison of ideal step response and 4-th order approximation step response is depicted in Fig. 7.

Remark 5: The time constant τ is equal to the lower frequency limit, thus $\omega_L = \tau = 1$. Shifting to other frequency or time constant can be done simply by adding a proper value to the logarithmic zero/pole positions obtained from Table III.

III. APPLICATIONS

It this section, several promising applications of developed filters are briefly described[2].

[2]The detailed description exceeds the page limit of this paper



(a) $N = 3$, $\omega_L = 10^0$, $\omega_H = 10^2$ (b) $N = 4$, $\omega_L = 10^0$, $\omega_H = 10^3$ (c) $N = 5$, $\omega_L = 10^0$, $\omega_H = 10^4$

Fig. 5. Fractional pole optimization results: zeros (o) and poles (+) positions for (a) two, (b) three, (c) four decades; approximating functions – green

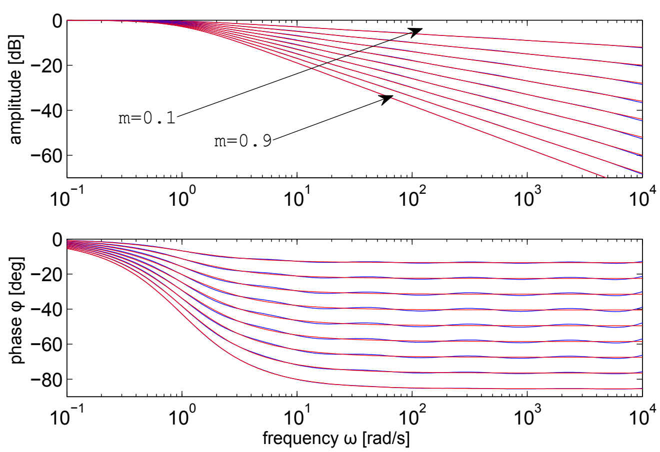


Fig. 6. Fractional-order pole: blue – approximation filter frequency responses for $N = 5$ and 4 decades frequency band $\omega_L = 10^0$, $\omega_H = 10^4$; red – ideal response

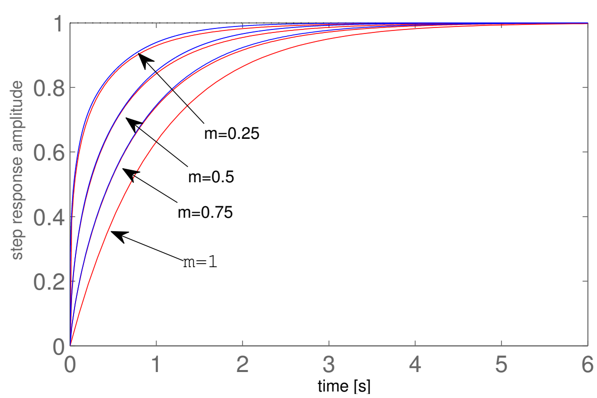


Fig. 7. Comparison of step responses of ideal fractional pole (red) and its 4-th order IO approximation (blue).

A. Improvement of the relay identification experiment

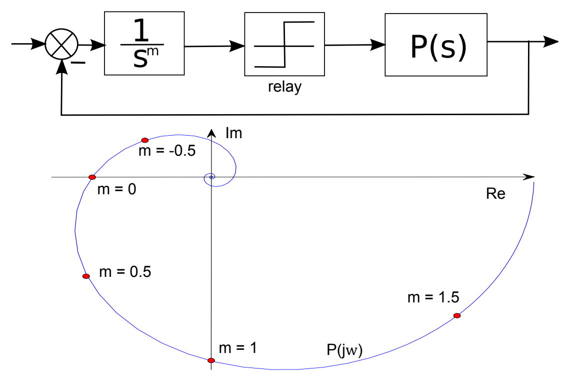


Fig. 8. Fractional integrator in the relay autotuner allows identifying points with arbitrary phase shift defined by m

Traditionally, in PID relay autotuners a sample with phase shift 180 degrees is identified. Such phase shift is usually not optimal. When one does not have additional information about the process gain, the phase shift should be decreased to 135 degrees. On the contrary, when the gain is known and the noises are low one can make faster identification at higher frequencies with the phase shift more than 200 degrees. In the past, an adaptive filter was used to get the point with arbitrary phase shift. Unfortunately, the adaptation makes the identification experiment very time consuming. Joining fractional integrator (constant-phase filter) to the relay (Fig. 8) provides the same feature without adaptation. Therefore the experiment is speed up significantly. The filter phase shift is independent on frequency in the large band (3–4 decades). The user can specify the required phase shift by one parameter m.

B. Fractional PID controller

The three-parameter PID controller is the most popular industrial controller thanks to its simplicity. Only two more parameters λ, μ arise after considering the integrator and derivator of arbitrary real order. Such a generalized controller is called fractional PID controller (FPID) and has a form (with filtered derivative part)

$$C(s) = K\left(1 + \frac{1}{T_i s^{\lambda}} + \frac{T_d s^{\mu}}{\frac{T_d}{N}s + 1}\right). \qquad (13)$$

Moreover, these two parameters λ and μ still have a clear physical interpretation. Such controller is applicable in the case when the frequency domain specifications are well defined and cannot be fulfilled by the classical PID controller[3]. The presented filters are suitable for implementation of fractional integrator and derivator inside the FPID controller. However, the astatic property must be ensured for zero steady state error. Therefore it is better to implement the fractional integrator part as $s^{(1-\lambda)}/s$.

C. Process modeling and identification

The fractional pole is essential element in process identification based on model set approach described in [18], [19]. It is shown there that a class of all processes in the form (5) with unlimited order ($n \to \infty$) is wide enough to cover almost all real monotone processes (temperature, pressure, concentration, flow, etc.) even with time delay. Hence such class is a sort of *a priori* assumption about the process. As the real processes are often very slow one can get only their few characteristic numbers from simple identification experiment (rectangle pulse, relay). Amplitude, phase and frequency of one frequency response sample is a good example of those numbers. Our aim is always to find all *a priori* admissible processes consistent with measured experimental data. Naturally, the so called *model set* obtained contains infinite number of processes. Fortunately, they create after mapping into frequency domain a compact area which boundary is created by processes – called *extremal* – with quite low count of fractional poles up to three. When $n \to \infty$, also the dead time appears in the extremal process transfer function. These bounding processes are significant for robust controller design where one needs to fulfill common design specifications (gain and phase margins, sensitivity functions limits, proper bandwidth) for all processes from the model set. Consequently, the presenting filters are usable for final time domain simulation of the designed closed loop.

IV. CONCLUSIONS AND FUTURE WORKS

In this paper the new low order IO filters approximating two basic fractional elements were presented. They parameters (zero/pole positions) were optimized on frequency band up to four decades using Matlab `fmincon()` procedure. It was shown, that this optimization gives better results compared to traditional well established methods. As a final step, the analytical functions allowing direct evaluation of zero pole positions for any order of the element were computed using least squares method. Additionally, several practical applications of fractional elements were sketched. The authors believe that the presented results may help to increase the number of industrial applications of fractional systems.

In the future a huge effort will be put to practical implementation of presented filters on embedded real-time devices. It is the prerequisite for making the mentioned application ideas alive in practice.

[3] Remind that the design specifications are contradictory in principle. Hence the closed loop tuning is always a trade-off.

ACKNOWLEDGMENTS

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