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# Speed-Gradient Inverse Optimal Control for Discrete-Time Nonlinear Systems

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***Abstract*— This paper presents a speed-gradient-based inverse optimal control approach for the asymptotic stabilization of discrete-time nonlinear systems. With the solution presented, we avoid to solve the associated Hamilton-Jacobi-Bellman equation, and a meaningful cost function is minimized. The proposed stabilizing optimal controller uses the speed-gradient algorithm and is based on the proposal of what is called a discrete-time control Lyapunov function. This combined approach is referred to as the speed-gradient inverse optimal control. An example is used to illustrate the methodology. Several simulations are provided.**

## I. INTRODUCTION

In optimal nonlinear control, we deal with the problem of finding a stabilizing control law for a given system such that a criterion, which is a function of the state variables and the control inputs, is minimized. The major drawback for this is the need to solve the associated Hamilton-Jacobi-Bellman (HJB) equation [1], [2]. The HJB equation, as far as we are aware, has not been solved for general nonlinear systems. It has been only solved for the linear regulator problem, for which it is particularly well-suited [3].

In this paper, we treat the discrete-time nonlinear version of the inverse optimal control problem, which was proposed originally by Kalman [4] for linear systems using quadratic cost functions. The aim of the inverse optimal control is to avoid to find the solution of the HJB equation [5]. In the inverse approach, we distinguish two main steps. First, a stabilizing feedback control law, based on an *a priori* known control Lyapunov function (CLF), is designed. Second, it is ensured that the stabilizing control law optimizes a meaningful cost functional.

An integral characteristic of the inverse optimal control problem is that the meaningful cost function is determined *a posteriori* once the stabilizing feedback control law is established. For the inverse optimal control in the continuous-time setting, we refer the reader to the results presented in [2], [3], [4], [5], [6], [7], [8], [9]. Although the inverse optimal control has been solved for continuous-time systems, the discrete-time case has not been widely analyzed. This is surprising, if we take into account the need of discrete-time schemes for the efficient implementation of real-time control systems.

As far as we are aware, there are very few results for the discrete-time nonlinear inverse optimal control [10]. Another example is [11], here, an inverse optimal control scheme is proposed based on passivity-related concepts, where a storage function is used as the Lyapunov function and the output feedback is used as the stabilizing control law.

In this paper, we propose a CLF which depends on a time-variant parameter. A CLF implies stabilizability [2]. This parameter is adjusted by means of the speed-gradient (SG) algorithm [12] in order to establish the stabilizing control law and to minimize a cost functional. We refer this combined approach to as the *SG inverse optimal control*. The use of the SG algorithm within the control loop is other novel contribution of this paper. Although the SG has been successfully applied in the control synthesis for continuous-time systems, there are very few results of the SG algorithm application for stabilization purposes in the nonlinear discrete-time setting [13].

This paper is organized as follows. Section II gives a brief review on optimal control, Lyapunov stability, the inverse optimal control problem and the SG algorithm. Section III establishes the SG algorithm application for the proposed control law. In Section IV, the SG inverse optimal control and its solution by means of a quadratic CLF are established. Section V illustrates the applicability of the proposed method by means of an example.

## II. MATHEMATICAL PRELIMINARIES

### A. Optimal Control

This section is devoted to briefly discuss the optimal control methodology and their limitations.

Consider the affine-in-the-input discrete-time nonlinear system:

$$x(k+1) = f(x(k)) + g(x(k))\, u(k) \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^m$ is the control input, $f(x)$ and $g(x)$ are smooth maps with $f(x) \in \mathbb{R}^n$, $g(x) \in \mathbb{R}^{n\times m}$, $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. We consider that $\overline{x}$ is an isolated fixed point of $f(x) + g(x)\overline{u}$ with $\overline{u}$ constant, that is, $f(\overline{x}) + g(\overline{x})\overline{u} = \overline{x}$. Without loss of generality, we consider $\overline{x} = 0$ for some $\overline{u}$ constant, $f(0) = 0$ and $rank\{g(x)\} = m \ \forall x_k \neq 0$.

From now on, we will write system (1) as:

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$$x_{k+1} = f(x_k) + g(x_k)\, u_k \quad (2)$$

and the subscript $k$ will stand for the value of the functions and/or variables at the time step $k$.

The following meaningful cost functional is associated with system (2):

$$V(x_k) = \sum_{n=k}^{\infty} \left( l(x_n) + u_n^T R(x_n)\, u_n \right) \quad (3)$$

where $V(x) : \mathbb{R}^n \to \mathbb{R}^+$; $l(x) : \mathbb{R}^n \to \mathbb{R}^+$ is a positive semidefinite[1] function and $R(x) : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is a real symmetric positive definite[2] weighting matrix. The meaningful cost functional (3) is a performance measure [14]. The entries of $R$ can be functions of the system state in order to vary the weighting on control efforts according to the state value [14]. Considering the state feedback control design problem, we assume that the full state $x_k$ is available.

Equation (3) can be rewritten as

$$\begin{aligned} V(x_k) &= l(x_k) + u_k^T R(x_k)\, u_k \\ &\quad + \sum_{n=k+1}^{\infty} l(x_n) + u_n^T R(x_n)\, u_n \\ &= l(x_k) + u_k^T R(x_k)\, u_k + V(x_{k+1}) \quad (4) \end{aligned}$$

where we require the boundary condition $V(0) = 0$ so that $V(x_k)$ becomes a Lyapunov function.

From Bellman's optimality principle [15], [16], it is known that, for the infinite horizon optimization case, the value function $V(x_k)$ becomes time invariant and satisfies the discrete-time Hamilton-Jacobi-Bellman (DT HJB) equation [16], [17], [18]

$$V(x_k) = \min_{u_k} \left\{ l(x_k) + u_k^T R(x_k)\, u_k + V(x_{k+1}) \right\} \quad (5)$$

where $V(x_{k+1})$ depends on both $x_k$ and $u_k$ by means of $x_{k+1}$ in (2). Note that the DT HJB equation is solved backward in time [17].

In order to establish the conditions that the optimal control law must satisfy, we define the discrete-time Hamiltonian $\mathcal{H}$ ([19], pages 830–832) as

$$\mathcal{H}(x_k, u_k) = l(x_k) + u_k^T R(x_k)\, u_k + V(x_{k+1}) - V(x_k). \quad (6)$$

A necessary condition that the optimal control law $u_k$ should satisfy is $\frac{\partial \mathcal{H}}{\partial u_k} = 0$ [14], which is equivalent to calculate the gradient of (5) right-hand side with respect to $u_k$, then

$$\begin{aligned} 0 &= 2R(x_k)\, u_k + \frac{\partial V(x_{k+1})}{\partial u_k} \\ &= 2R(x_k)\, u_k + g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}}. \quad (7) \end{aligned}$$

[1]A function $l(z)$ is positive semidefinite (or nonnegative definite) function if for all vectors $z$, $l(z) \geq 0$. In other words, there are vectors $z$ for which $l(z) = 0$, and for all others $z$, $l(z) > 0$ [14].

[2]A real symmetric matrix $R$ is positive definite if $z^T R z > 0$ for all $z \neq 0$ [14].

Therefore, the optimal control law is formulated as

$$u_k^* = -\frac{1}{2} R^{-1}(x_k)\, g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} \quad (8)$$

with the boundary condition $V(0) = 0$; $u_k^*$ is used when we want to emphasize that $u_k$ is optimal.

Moreover, if $\mathcal{H}$ has a quadratic form in $u_k$ and $R(x_k) > 0$, then

$$\frac{\partial^2 \mathcal{H}}{\partial u_k^2} > 0$$

holds as a sufficient condition such that optimal control law (8) (globally [14]) minimizes $\mathcal{H}$ and the performance index (3) [15].

Substituting (8) into (5), we obtain

$$\begin{aligned} V(x_k) &= l(x_k) + \left( -\frac{1}{2} R^{-1}(x_k)\, g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} \right)^T \\ &\quad \times R(x_k) \left( -\frac{1}{2} R^{-1}(x_k)\, g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} \right) \\ &\quad + V(x_{k+1}) \\ &= l(x_k) + V(x_{k+1}) + \frac{1}{4} \frac{\partial V^T(x_{k+1})}{\partial x_{k+1}} g(x_k) \times \\ &\quad R^{-1}(x_k) g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} \quad (9) \end{aligned}$$

which can be rewritten as

$$\begin{aligned} & l(x_k) + V(x_{k+1}) - V(x_k) + \frac{1}{4} \frac{\partial V^T(x_{k+1})}{\partial x_{k+1}} g(x_k) \times \\ & \qquad R^{-1}(x_k) g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} = 0. \quad (10) \end{aligned}$$

The problem of solving the HJB partial-differential equation (10) for $V(x_k)$ is not straightforward. This is one of the main disadvantages in discrete-time optimal control for nonlinear systems. To overcome this problem, we propose to solve the inverse optimal control problem.

### *B. Lyapunov Stability*

Due to the fact that the inverse optimal control is based on a Lyapunov function, we establish the following definitions.

**Definition 1** (**Radially Unbounded Function** [20]). *A function $V(x_k)$ satisfying the condition $V(x_k) \to \infty$ as $\|x_k\| \to \infty$ is said to be radially unbounded.*

**Definition 2** (**Control Lyapunov Function** [21]). *Let $V(x_k)$ be a radially unbounded function, with $V(x_k) > 0$, $\forall x_k \neq 0$ and $V(0) = 0$. If for any $x_k \in \mathbb{R}^n$, there exist real values $u_k$ such that*

$$\Delta V(x_k, u_k) < 0$$

*where the Lyapunov difference $\Delta V(x_k, u_k)$ is defined as $V(x_{k+1}) - V(x_k) = V(f(x_k) + g(x_k)\, u_k) - V(x_k)$. Then $V(\cdot)$ is said to be a "discrete-time control Lyapunov function" (CLF) for system (2).*

In order to establish stability, let recall the following result.

**Theorem 1** (**Global Asymptotic Stability** [22]). *The equilibrium $x_k = 0$ of (2) is globally asymptotically stable if* *there is a function $V : \mathbb{R}^n \to \mathbb{R}$ such that $(i)$ $V$ is a positive definite function, decrescent and radially unbounded, and $(ii)$ $-\Delta V(x_k, u_k)$ is a positive definite function.*

### C. Inverse Optimal Control

**Definition 3 (Inverse Optimal Control Law).** *Let define the control law*

$$u_k^* = -\frac{1}{2} R^{-1}(x_k) g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} \tag{11}$$

*to be inverse optimal (globally) stabilizing if:*

(i) *it achieves (global) asymptotic stability of $x = 0$ for system (2);*

(ii) *$V(x_k)$ is (radially unbounded) positive definite function such that inequality*

$$\overline{V} := V(x_{k+1}) - V(x_k) + u_k^{*T} R(x_k)\, u_k^* \leq 0 \tag{12}$$

*is satisfied.*

*When we select $l(x_k) := -\overline{V}$, then $V(x_k)$ is a solution for (10).*

As it is established in Definition 3, the inverse optimal control problem is based on the knowledge of $V(x_k)$. Thus, we propose a CLF, $V(x_k)$, such that (i) and (ii) are guaranteed. That is, instead of solving (10) for $V(x_k)$, we propose a control Lyapunov function $V_c(x_k)$ with the form:

$$V_c(x_k) = \frac{1}{2} x_k^T P_k\, x_k, \qquad P_k = P_k^T > 0 \tag{13}$$

for control law (11) in order to ensure stability of the fixed point of system (2). This will be achieved by defining an appropriate matrix $P_k$. Moreover, it will be established that the control law (11) with (13), which is referred to as the *inverse optimal* control law, optimizes a meaningful cost functional of the form (3).

Consequently, by considering $V(x_k) = V_c(x_k)$ as in (13), the control law (11) takes the following form:

$$u_k^* = -\frac{1}{2}\left(R(x_k) + \frac{1}{2} g^T(x_k)\, P_k\, g(x_k)\right)^{-1} \times g^T(x_k)\, P_k\, f(x_k). \tag{14}$$

It is worth to point out that $P_k$ and $R$ are positive definite and symmetric matrices; thus, the existence of the inverse in (14) is ensured.

To compute $P_k$, which ensures stability of the fixed point of system (2) with (14), we will use the speed-gradient (SG) algorithm.

### D. Speed-Gradient Algorithm

In [12], a discrete-time application of the SG algorithm is formulated as finding a control law $u_k$ which ensures the control goal:

$$\mathcal{Q}(x_{k+1}) \leq \Delta, \qquad \text{for } k \geq k^*, \tag{15}$$

where $\mathcal{Q}$ is a control goal function, a constant $\Delta > 0$, and $k^* \in \mathbb{Z}_+$ is the time step at which the control goal is achieved. $\mathcal{Q}$ ensures stability if it is a positive definite function.

## III. Speed-Gradient Algorithm for the Inverse Optimal Control

Digressing from the SG application proposed by [12], in this paper, the control law is given by (14), and $\Delta$ in (15), is a state dependent function $\Delta(x_k)$.

Control law (14) at every time step depends on the matrix $P_k$. Let define the matrix $P_k$ at every time step $k$ as:

$$P_k = p_k\, P'$$

where $P' = P'^T > 0$ is a given constant matrix and $p_k$ is a scalar parameter to be adjusted by the SG algorithm. Then, (14) is transformed into:

$$u_k^* = -\frac{p_k}{2}\left(R(x_k) + \frac{p_k}{2} g^T(x_k)\, P'\, g(x_k)\right)^{-1} \times g^T(x_k)\, P'\, f(x_k). \tag{16}$$

The SG algorithm is now reformulated for the inverse optimal control problem.

**Definition 4 (SG Goal Function).** *Consider a time-varying parameter $p_k \in \mathcal{P} \subset \mathbb{R}^+$, with $p_k > 0$ for all $k$, and $\mathcal{P}$ is the set of admissible values for $p_k$. A nonnegative $\mathcal{C}^1$ function $\mathcal{Q} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the form*

$$\mathcal{Q}(x_k, p_k) = V_{sg}(x_{k+1}), \tag{17}$$

*where $V_{sg}(x_{k+1}) = \frac{1}{2} x_{k+1}^T\, P'\, x_{k+1}$ with $x_{k+1}$ as defined in* (2)*, is referred to as SG goal function for system* (2)*. We define $\mathcal{Q}_k(p) := \mathcal{Q}(x_k, p_k)$.*

**Definition 5 (SG Control Goal).** *Consider a constant $p^* \in \mathcal{P}$. The SG control goal for system* (2) *with* (16) *is defined as finding $p_k$ so that the SG goal function $\mathcal{Q}_k(p)$ as defined in* (17) *fulfills:*

$$\mathcal{Q}_k(p) \leq \Delta(x_k), \qquad \textit{for} \quad k \geq k^*, \tag{18}$$

*where*

$$\Delta(x_k) = V_{sg}(x_k) - \frac{1}{p_k} u_k^T\, R(x_k)\, u_k \tag{19}$$

*with $V_{sg}(x_k) = \frac{1}{2} x_k^T\, P'\, x_k$ and $u_k$ as defined in* (16)*; $k^* \in \mathbb{Z}_+$ is the time step at which the SG control goal is achieved.*

**Remark 1.** *Solution $p_k$ must guarantees that $V_{sg}(x_k) > \frac{1}{p_k} u_k^T R(x_k)\, u_k$ in order to obtain a positive definite function $\Delta(x_k)$.*

To conclude, the SG algorithm is used to compute $p_k$ in order to achieve the SG control goal defined above.

**Proposition 1.** *Consider a discrete-time nonlinear system of the form* (2) *with* (16) *as input. Let $\mathcal{Q}$ be a SG goal function as defined in* (17)*, and denoted by $\mathcal{Q}_k(p)$. Let $\bar{p}, p^* \in \mathcal{P}$ be positive constant values, $\Delta(x_k)$ be a positive definite function with $\Delta(0) = 0$ and $\epsilon^*$ be a sufficiently small positive constant. Assume that:*

- *A1. There exist $p^*$ and $\epsilon^*$ such that*

$$\mathcal{Q}_k(p^*) \leq \epsilon^* \ll \Delta(x_k) \tag{20}$$

- *A2. For all $p_k \in \mathcal{P}$ due to convexity of the SF Goal Function* (17) *for $p_k$:*

$$(p^* - p_k)^T \nabla_p \mathcal{Q}_k(p) \leq \epsilon^* - \Delta(x_k) < 0 \quad (21)$$

*where $\nabla_p \mathcal{Q}_k(p)$ denotes the gradient of $\mathcal{Q}_k(p)$ with respect to $p_k$.*

*Then, for any initial condition $p_0 > 0$, there exists a $k^* \in \mathbb{Z}^+$ such that the SG Control Goal* (18) *is achieved by means of the following dynamic variation of parameter $p_k$:*

$$p_{k+1} = p_k - \gamma_{d,k} \nabla_p \mathcal{Q}_k(p), \quad (22)$$

*with*

$$\gamma_{d,k} = \gamma_c \, \delta_k \left|\nabla_p \mathcal{Q}_k(p)\right|^{-2}, \qquad 0 < \gamma_c \leq 2\,\Delta(x_k)$$

*and*

$$\delta_k = \begin{cases} 1 & for \quad \mathcal{Q}_k(p) > \Delta(x_k) \\ 0 & otherwise. \end{cases} \quad (23)$$

*Finally, for $k \geq k^*$, $p_k$ becomes a constant value denoted by $\bar{p}$ and the SG algorithm is completed.*

*Proof:* We follow similar arguments as the ones given for the SG discrete-time version [12]. Let us consider the positive definite Lyapunov function $V_p(p_k) = |p_k - p^*|^2$. Then, the respective Lyapunov difference is given as

$$\begin{aligned} \Delta V_p(p_k) &= |p_{k+1} - p^*|^2 - |p_k - p^*|^2 \\ &= (p_{k+1} - p_k)^T [(p_{k+1} - p_k) + 2(p_k - p^*)] \\ &= -\gamma_{d,k}\, \nabla_p \mathcal{Q}_k(p)\, [-\gamma_{d,k}\, \nabla_p \mathcal{Q}_k(p) + \\ &\quad 2(p_k - p^*)] \\ &\leq -2\,\gamma_{d,k}\, (\Delta(x_k) - \epsilon^*) + \gamma_{d,k}^2 \left|\nabla_p \mathcal{Q}_k(p)\right|^2 \\ &\leq -2\,\gamma_c\, \delta_k\, (\Delta(x_k) - \epsilon^*) \left|\nabla_p \mathcal{Q}_k(p)\right|^{-2} + \\ &\quad \gamma_c^2\, \delta_k^2 \left|\nabla_p \mathcal{Q}_k(p)\right|^{-4} \left|\nabla_p \mathcal{Q}_k(p)\right|^2 \\ &= -\frac{\gamma_c \left[2\,\Delta(x_k) \left(1 - \left(\epsilon^*/\Delta(x_k)\right)\right) - \gamma_c\right]}{\left|\nabla_p \mathcal{Q}_k(p)\right|^2}. \end{aligned}$$

From (20), $1 - \left(\epsilon^*/\Delta(x_k)\right) \approx 1$, hence

$$\begin{aligned} \Delta V_p(p_k) &\approx -\frac{\gamma_c \left[2\,\Delta(x_k) - \gamma_c\right]}{\left|\nabla_p \mathcal{Q}_k(p)\right|^2} \\ &< 0 \end{aligned}$$

for $\mathcal{Q}_k(p) > \Delta(x_k)$, $\delta_k = 1$ and boundness of $p_k$ is guaranteed if $0 < \gamma_c \leq 2\,\Delta(x_k)$. Finally, when $k \geq k^*$, then $\delta_k = 0$, which means the algorithm concludes; hence, $\mathcal{Q}_k(p) \leq \Delta(x_k)$ and $p_k$ becomes a constant value denoted by $\bar{p}$ ($p_k = \bar{p}$). ■

Since the parameter $p_k$ is a scalar value, the gradient $\nabla_p \mathcal{Q}_k(p)$ in (22) is reduced to be the partial derivative of $\mathcal{Q}_k(p)$ with respect to $p_k$, i.e., $\frac{\partial}{\partial p_k} \mathcal{Q}_k(p)$.

**Remark 2.** *Parameter $\gamma_c$ in* (22) *is selected such that solution $p_k$ ensures the requirement $V_{sg}(x_k) > \frac{1}{p_k} u_k^T R(x_k)\, u_k$ in Remark 1. Then, we have a positive definite function $\Delta(x_k)$.*

**Remark 3.** *With $\mathcal{Q}(x_k, p_k)$ as defined in* (17)*, the dynamic variation of parameter $p_k$ in* (22) *results in*

$$p_{k+1} = p_k + 8\,\gamma_{d,k}\, \frac{f^T(x_k)\, P'\, g(x_k)\, R(x_k)^2\, g^T(x_k)\, f(x_k)}{\left(2\,R(x_k) + p_k\, g^T(x_k)\, P'\, g(x_k)\right)^3}$$

*which is positive for all time step $k$ if $p_0 > 0$. Therefore positiveness for $p_k$ is ensured and requirement $P_k = P_k^T > 0$ for* (13) *is guaranteed.*

When SG Control Goal (18) is achieved, then $p_k = \bar{p}$ for $k \geq k^*$. Thus, matrix $P_k$ in (14) is considered constant and $P_k = P$ where $P$ is computed as $P = \bar{p}\, P'$, with $P'$ a *design* positive definite matrix. Under these constraints, we obtain:

$$\begin{aligned} \alpha(x_k) &:= u_k^* \\ &= -\frac{1}{2}\left(R(x_k) + \frac{1}{2}\, g^T(x_k)\, P\, g(x_k)\right)^{-1} \times \\ &\qquad g^T(x_k)\, P\, f(x_k). \end{aligned} \quad (24)$$

## IV. SG Inverse Optimal Control

Once the control law (24) has been established, we demonstrate that it ensures stability and optimality for (2) without solving the HJB equation (10). Thus, the main contribution of this paper is stated as the following theorem.

**Theorem 2.** *Consider that system* (2) *with* (16) *has achieved the SG control goal* (18) *by means of* (22)*. Let $V(x_k) = \frac{1}{2} x_k^T\, P\, x_k$ be a Lyapunov function candidate with $P = P^T > 0$. Then, control law* (24) *is an inverse optimal control law, in accordance with Definition 3, which makes the fixed point $x_k = 0$ of system* (2) *be globally asymptotically stable. Moreover, with $V(x_k) = \frac{1}{2} x_k^T\, P\, x_k$ as CLF and $P = \bar{p}\, P'$, control law* (24) *is inverse optimal in the sense that it minimizes the meaningful functional given by*

$$\mathcal{J} = \sum_{k=0}^{\infty} \left(l(x_k) + u_k^T\, R(x_k)\, u_k\right) \quad (25)$$

*where*

$$l(x_k) := -\overline{V} \quad (26)$$

*with $\overline{V}$ defined as*

$$\overline{V} = V(x_{k+1}) - V(x_k) + \alpha^T(x_k)\, R(x_k)\, \alpha(x_k).$$

*Proof:* Considering that system (2), (16) and (22) has achieved the SG Control Goal (18) for $k \geq k^*$, then (18) can be rewritten as:

$$\begin{aligned} &V_{sg}(x_{k+1}) - V_{sg}(x_k) + \frac{1}{\bar{p}}\, \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\ &= \frac{1}{2}\, x_{k+1}^T\, P'\, x_{k+1} - \frac{1}{2}\, x_k^T\, P'\, x_k + \frac{1}{\bar{p}}\, \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\ &\leq 0. \end{aligned} \quad (27)$$

Multiplying (27) by the positive constant $\bar{p}$, we obtain

$$\begin{aligned} \overline{V} &:= \frac{\bar{p}}{2}\, x_{k+1}^T\, P'\, x_{k+1} - \frac{\bar{p}}{2}\, x_k^T\, P'\, x_k + \\ &\quad \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\, x_{k+1}^T\, P\, x_{k+1} - \frac{1}{2}\, x_k^T\, P\, x_k + \\
&\quad \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\
&= V(x_{k+1}) - V(x_k) + \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\
&\leq 0 \qquad (28)
\end{aligned}
$$

and condition (12) is fulfilled. From (28), obviously $V(x_{k+1}) - V(x_k) < 0$ for all $x_k \neq 0$ with $V(x_k)$ a positive definite and radially unbounded function, then global asymptotic stability is achieved in accordance with Theorem 1.

When function $-l(x_k)$ is set to be the (28) right-hand side, then:

$$
\begin{aligned}
l(x_k) &:= -\overline{V} \qquad (29) \\
&= -(V(x_{k+1}) - V(x_k)) - \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\
&> 0, \qquad \forall x_k \neq 0.
\end{aligned}
$$

Consequently, $V(x_k) = \frac{1}{2}\, x_k^T\, P\, x_k$ as CLF is a solution of the HJB equation (10) for $k \geq k^*$.

In order to obtain the optimal value function for the meaningful cost functional (25), we substitute $l(x_k)$ given in (29) into (25), and we obtain:

$$
\begin{aligned}
\mathcal{J} &= \sum_{k=0}^{\infty} \big( l(x_k) + u_k^T\, R(x_k)\, u_k \big) \\
&= \sum_{k=0}^{\infty} \big( -\overline{V} + u_k^T\, R(x_k)\, u_k \big) \\
&= -\sum_{k=0}^{\infty} \Big[ V(x_{k+1}) - V(x_k) \Big] + \sum_{k=0}^{\infty} \Big[ u_k^T\, R(x_k)\, u_k \\
&\quad - \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \Big]. \qquad (30)
\end{aligned}
$$

After evaluating (30) for $k = 0$, then it can be written as

$$
\begin{aligned}
\mathcal{J} &= -\sum_{k=1}^{\infty} \Big[ V(x_{k+1}) - V(x_k) \Big] - V(x_1) + V(x_0) \\
&\quad + \sum_{k=0}^{\infty} \Big[ u_k^T\, R(x_k)\, u_k - \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \Big] \\
&= -\sum_{k=2}^{\infty} \Big[ V(x_{k+1}) - V(x_k) \Big] - V(x_2) + V(x_1) \\
&\quad - V(x_1) + V(x_0) + \sum_{k=0}^{\infty} \Big[ u_k^T\, R(x_k)\, u_k \\
&\quad - \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \Big]. \qquad (31)
\end{aligned}
$$

For notation convenience in (31), the upper limit $\infty$ will be considered as $N \to \infty$, and therefore

$$
\begin{aligned}
\mathcal{J} &= -V(x_N) + V(x_{N-1}) - V(x_{N-1}) + V(x_0) \\
&\quad + \sum_{k=0}^{N} \Big[ u_k^T\, R(x_k)\, u_k - \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \Big] \\
&= -V(x_N) + V(x_0) + \sum_{k=0}^{N} \Big[ u_k^T\, R(x_k)\, u_k \\
&\quad - \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \Big].
\end{aligned}
$$

Letting $N \to \infty$ and noting that $V(x_N) \to 0$ for all $x_0$, then

$$
\mathcal{J}(x_k) = V(x_0) + \sum_{k=0}^{\infty} \left[ u_k^T R(x_k) u_k - \alpha^T(x_k) R(x_k) \alpha(x_k) \right]. \qquad (32)
$$

Thus, the minimum value of (32) is reached with $u_k = \alpha(x_k)$, with $\alpha(x_k)$ as in (24). Hence, the control law (24) minimizes the cost functional (25). The optimal value function of (25) is $\mathcal{J}^*(x_0, \alpha(x_k)) = V(x_0)$ for all $x_0$. ■

We can establish the main conceptual differences between optimal control and inverse optimal control as:

- For optimal control, the cost functions $l(x_k) \geq 0$ and $R(x_k) > 0$ are given *a priori*. Then, they are used to calculate $u(x_k)$ and $V(x_k)$ by means of DT HJB equation solution.
- For inverse optimal control, the control Lyapunov function $V(x_k)$ and the cost function $R(x_k)$ are given *a priori*. Then, these functions are used to compute $u(x_k)$, and $l(x_k)$ with function $\overline{V}$ as defined in (12).

The optimal control will in general be of the form (11) and the minimum value of the performance index will be function of the initial state $x_0$, that is, $V(x_0)$.

## V. EXAMPLE

The proposed methodology is illustrated in an example. We design an inverse optimal control law for a discrete-time second order nonlinear system (unstable for $u_k = 0$) of the form (2) with:

$$
f(x_k) = \begin{bmatrix} x_{1,k}\, x_{2,k} - 0.8\, x_{2,k} \\ x_{1,k}^2 + 1.8\, x_{2,k} \end{bmatrix} \qquad (33)
$$

and

$$
g(x_k) = \begin{bmatrix} 0 \\ -2 + cos(x_{2,k}) \end{bmatrix}. \qquad (34)
$$

According to (24), the inverse optimal control law is formulated as

$$
\begin{aligned}
u_k^* &= -\frac{1}{2} \left( R(x_k) + \frac{1}{2}\, g^T(x_k)\, P_k\, g(x_k) \right)^{-1} \times \\
&\qquad g^T(x_k)\, P_k\, f\,(x_k)
\end{aligned}
$$

where the positive definite matrix $P_k = p_k\, P'$ is calculated by the SG algorithm with $P'$ as the identity matrix, that is

$$
\begin{aligned}
P_k &= p_k\, P' \\
&= p_k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

and $R(x_k)$ is a constant matrix selected as $R(x_k) = 0.5$.

The state penalty term $l(x_k)$ in (25) is calculated according to (26).

Fig. 1 shows the solution $x_k$ of this system with initial conditions $x_0 = [2 \quad -2]^T$; this figure also includes the applied inverse optimal control law, which achieves asymptotic stability.

Fig. 2 displays both the SG algorithm solution $p_k$ with initial condition $p_0 = 2.5$ and final value $\bar{p} = 3.4613$.

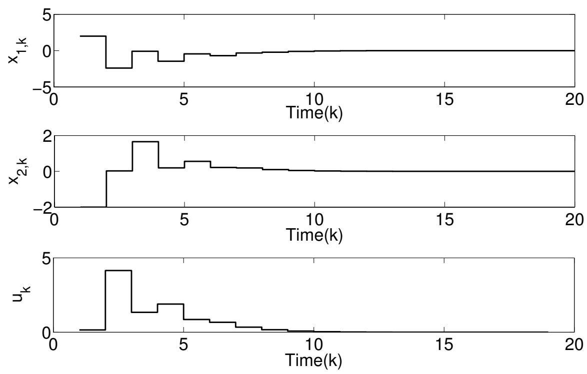


Fig. 1. Stabilization of a nonlinear system.

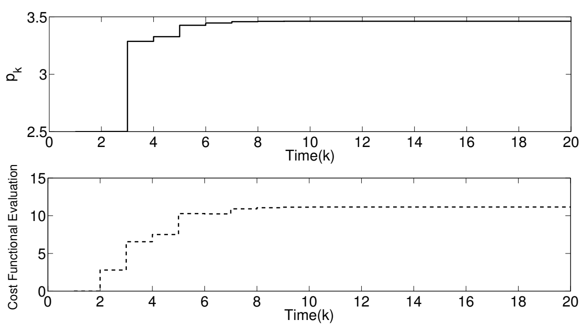


Fig. 2. $p_k$ and $\mathcal{J}$ time evolution.

Evaluation of the cost functional $\mathcal{J}$ is also shown in this figure.

Notice that the open-loop system (33), had an unstable fixed point for $u_k = 0$. In this example, according to Theorem 2, the optimal value function is calculated as $\mathcal{J}^*(x_0, \alpha(x_k)) = V(x_0) = \frac{1}{2} x_0^T P x_0 = 13.8452$, which is reached as shown in Fig. 2.

## VI. Conclusions

This paper has established the inverse optimal control problem for a class of discrete-time nonlinear systems. To avoid the solution of the Hamilton-Jacobi-Bellman equation, we propose a discrete-time control Lyapunov function (CLF) in a quadratic form adjusted by means of the speed-gradient algorithm. Based on this CLF, the inverse optimal control strategy is synthesized. Stability and the corresponding conditions for the inverse optimal control solution are established. Simulation results illustrate that the proposed controller ensures stabilization of a nonlinear system and minimizes a meaningful cost functional.

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