2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Is stabilization of switched positive linear systems equivalent to the existence of an Hurwitz convex combination of the system matrices?

Franco Blanchini[1] and Patrizio Colaneri[2] and Maria Elena Valcher[3]

Index Terms— **Positive systems, switched systems, stabilization.**

Abstract— **In this paper exponential stabilizability of continuous-time positive switched systems is investigated. It is proved that, when dealing with two-dimensional systems, exponential stabilizability can be achieved if and only if there exists an Hurwitz convex combination of the (Metzler) system matrices. However, for systems of higher dimension this is not true.**

In general, exponential stabilizability corresponds to the existence of a (positively homogeneous, concave and co–positive) control Lyapunov function, but this function is not necessarily smooth. The existence of an Hurwitz convex combination is equivalent to the stronger condition that the system is not only exponentially stable, but it also admits a smooth control Lyapunov function. These two conditions, in turn, are equivalent to the fact that the stabilizing switching law can always be based on a linear co–positive control Lyapunov function. Finally, the characterization of exponential stabilizability is exploited to provide a description of all the "switched equilibrium points" of a positive affine switched system.

I. Introduction

Continuous-time positive switched systems are dynamic systems that switch among a family of (continuous-time) positive state-space models, each of them characterized by a Metzler matrix. The interest in this class of systems is relatively recent and strongly motivated by the possibility of employing them in system biology and pharmacokinetic [11], [12]. These systems represent quite a challenge from a theoretical point of view. Indeed, properties like reachability and controllability cannot be investigated as special cases of the analogous properties for standard switched systems, due the positivity constraint on their system matrices and on the soliciting inputs [22]. On the other hand, properties like stability and stabilizability, meanwhile inheriting the general results derived for non-positive switched systems, offer new tools and new testing criteria that find no equivalent in the general case.

Most of the literature on positive switched systems has focused on the stability problem, namely on the possibility of ensuring a good asymptotic behavior to the system trajectories, independently of the switching law [10], [14], [17]. On the other hand, there are situations when stability property can not be ensured (this is the case when modeling viral mutation and escape in patients affected by HIV [11], [12]), but we are interested in determining switching strategies that ensure the convergence to zero of the state trajectories. Stabilizability of positive switched systems is a topic only partially explored [2], [8], [11], [16], [24], that offers a wide range of interesting open problems.

In this paper we focus on the exponential stabilizability of continuous-time positive switched systems. For two-dimensional systems this property is equivalent to the existence of an Hurwitz convex combination of the (Metzler) system matrices[1]. For systems of higher dimension the existence of such an Hurwitz combination represents a stronger property. Indeed, while exponential stabilizability corresponds to the existence of a (positively homogeneous, concave and co–positive) control Lyapunov function, the existence of an Hurwitz combination is equivalent to the existence of a control Lyapunov function that, in addition to the previous properties, is also smooth. These conditions are, in turn, equivalent, to the possibility of designing a stabilizing switching law based on a linear co–positive control Lyapunov function. It is worthwhile to underline that these results are not only interesting by themselves, but they also enlighten that stabilizability characterization in the continuous-time and in the discrete-time cases are significantly different [8]. Finally, the characterization of exponential stabilizability is exploited to provide a description of all the "switched equilibrium points" of a positive affine switched system.

Before proceeding we introduce some basic notation. The set of nonnegative real numbers is $\mathbb{R}_+$. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is a *nonnegative matrix*. The matrix A is *positive* ($A > 0$) if nonnegative and nonzero, and *strictly positive* ($A \gg 0$) if all its entries

[1] Dipartimento di Mathematica e Informatica, Università di Udine, 33100 Udine, Italy, email: blanchini@uniud.it

[2] Dipartimento di Elettronica e Informazione, Politecnico di Milano, 20133 Milano, Italy, email: colaneri@elet.polimi.it

[3] Dipartimento di Ingegneria dell'Informazione, Università di Padova, 35131 Padova, Italy, email: meme@dei.unipd.it

[1]In [23] a detailed analysis of the stabilizability of second order non-positive switched systems is given. Necessary and sufficient geometric conditions are provided, depending on whether the origin is an unstable focus, saddle point or node for the various subsystems. Based on these conditions, stabilizing switching laws are proposed.

are positive. A positive $n \times n$ matrix A, with $n > 1$, is *reducible* if there exists a permutation matrix P such that

$$P'AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} is a $k \times k$ matrix, $1 \leq k \leq n-1$. A positive matrix which is not reducible is called *irreducible*. For an irreducible positive matrix A it is know that the spectral radius $\max\{|\lambda| : \lambda \in \sigma(a)\}$ is a simple eigenvalue of A, called the *Perron-Frobenius eigenvalue* and denoted by λ_f, and that the corresponding eigenspace is generated by a strictly positive eigenvector, called *Perron-Frobenius eigenvector*. A square matrix $A = [a_{ij}]$ is said to be *Metzler* if its off-diagonal entries are nonnegative, namely $a_{ij} \geq 0$ for every $i \neq j$.

$\mathbf{1}_n$ is the n-dimensional vector with all entries equal to 1. The suffix n will be omitted when the vector size is clear from the context. The convex polytope of the nonnegative M-tuples, $M \in \mathbb{N}$, that sum up to 1 will be denoted by:

$$\mathcal{A} \doteq \left\{ \alpha \in \mathbb{R}^M_+ : \sum_{i=1}^{M} \alpha_i = 1 \right\} = \left\{ \alpha \in \mathbb{R}^M_+ : \mathbf{1}'\alpha = 1 \right\}.$$

A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be *co–positive* if $V(x) > 0$ for every $x > 0$ and $V(0) = 0$. A *linear co–positive function* takes the form $V(x) = c'x$, for some $c \in \mathbb{R}^n, c \gg 0$.

II. Exponential stabilizability of two-dimensional positive switched systems

In this paper we consider n-dimensional continuous-time positive switched systems described by the following equation

$$\dot{x}(t) = A_{\sigma(t)}x(t), \tag{1}$$

where $\sigma(t)$ is a switching sequence, defined on $\mathbb{R}_+$ and taking values in the finite set $\{1, 2, \ldots, M\}$. For every value i taken by the switching sequence σ, $\dot{x}(t) = A_i x(t)$ is a continuous-time positive system, which amounts to saying that A_i is an $n \times n$ Metzler matrix. For these systems, we introduce the concept of exponential stabilizability. This definition requires the existence of a switching law whose value at each time t depends on the value of the state at that time, and hence a state–feedback switching law.

Definition 1: System (1) is *exponentially stabilizable* if there exist a switching control law $\sigma(t) = u(x(t))$ and real parameters $C > 0$ and $\beta > 0$ such that every trajectory (generated from any $x(0) > 0$, according to the aforementioned state-feedback switching law) satisfies

$$\|x(t)\| < Ce^{-\beta t}\|x(0)\|, \qquad \forall t \geq 0.$$

In this paper we investigate exponential stabilizability and its relationship with other important conditions, which are known to be sufficient for exponential stabilizability. To this end, we introduce the following:

Assumption 1: All the matrices $A_i, i = 1, 2, \ldots, M$, are irreducible, and hence have a strictly positive Perron-Frobenius eigenvector.

This assumption allows to simplify the calculations but it is not really restrictive. Indeed, all the results can be easily extended to arbitrary positive matrices by resorting to standard techniques (replace the matrices A_i with the irreducible matrices $A_i + \epsilon\mathbf{1}\mathbf{1}'$, with $\epsilon > 0$, and obtain all the results as limit for $\epsilon \to 0^+$). Notice, in particular, that this is possible because the Perron Frobenius eigenvalue is a continuous function of the coefficients, see [15].

It is a well known fact that the existence of an Hurwitz convex combination of the system matrices represents a sufficient condition for the exponential stabilizability [13], [20].

Proposition 1: If there exists an Hurwitz convex combination of the system matrices A_i's, namely there exists $\alpha \in \mathcal{A}$ such that $A(\alpha) \doteq \sum_{i=1}^{M} \alpha_i A_i$ is Hurwitz, then system (1) is exponentially stabilizable.

When dealing with two-dimensional systems (i.e., systems (1) of dimension $n = 2$), this condition is also necessary.

Theorem 1: A two-dimensional system (1) is exponentially stabilizable if and only if there exist indices $i_1, i_2 \in \{1, 2, \ldots, M\}$ and nonnegative numbers $\alpha_1, \alpha_2 \in [0, 1]$, with $\alpha_1 + \alpha_2 = 1$, such that $\alpha_1 A_{i_1} + \alpha_2 A_{i_2}$ is Hurwitz.

Proof: The proof of this result is quite long and hence it is omitted. The interested reader is referred to [3]. ■

Remark 1: It is worth to comment on the meaning of this result. For two-dimensional systems (1), exponential stabilizability not only ensures that a convex Hurwitz combinations of the system matrices can be found, a condition which was known to be only sufficient for exponential stabilizability, but also that among all such combinations at least one can be found involving only two of the matrices $A_i, i \in \{1, 2, \ldots, M\}$. This implies that the system can be stabilized by resorting to switching sequences that take only two of the available M values.

The proof of Theorem 1 is based on geometric arguments that do not find an obvious extension to the general case, and indeed the result is not true for exponentially stable systems of higher dimension, as shown by the following example.

Example 1: Consider the three-dimensional positive switched system (1), switching among three subsystems characterized by the matrices:

$$A_1 = \begin{bmatrix} -0.99 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 1 & 0.01 \end{bmatrix} A_2 = \begin{bmatrix} 0.01 & 0 & 1 \\ 0 & -0.99 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.01 & 0 & 0 \\ 1 & 0.01 & 0 \\ 0 & 0 & -0.99 \end{bmatrix}.$$

It is a matter of simple computation to show that no convex Hurwitz combination of the three matrices can be found, however the matrix product $e^{A_1}e^{A_2}e^{A_3}$ is Schur, and hence the periodic switching law

$$\sigma(t) = \begin{cases} 3, & t \in [3k, 3k+1), \\ 2, & t \in [3k+1, 3k+2), \\ 1, & t \in [3k+2, 3k+3), \end{cases} \qquad k \in \mathbb{Z}_+,$$

makes the resulting system exponentially stable. Consequently, the system is exponentially stabilizable. A state-feedback stabilizing switching rule can be found as follows:

- Introduce
$$\begin{array}{rcl} c_1 & = & \left[\begin{array}{ccc} 1 & 1.1 & 1 \end{array}\right] \\ c_2 & = & \left[\begin{array}{ccc} 1 & 1 & 1.1 \end{array}\right] \\ c_3 & = & \left[\begin{array}{ccc} 1.1 & 1 & 1 \end{array}\right]. \end{array}$$
- Define $I(x) := \{i \in \{1,2,3\} : c_i' x \leq c_j' x, \forall\ j\}$.
- The stabilizing switching law is
$$\sigma(t) = u(x(t)) = \arg\min_k (\min_{i \in I(x)} c_i' A_k x).$$

See [3] for more details.

III. Exponential stabilizability and control Lyapunov functions

To investigate what is the relationship between exponential stabilizability and the existence of an Hurwitz convex combination of the matrices A_i's, we need to introduce the concept of control Lyapunov function. This definition is the same as the one used for standard switched systems, with the exception that instead of imposing that the Lyapunov function takes positive values everywhere, but in the origin, we only require it to be co–positive, namely to take positive values in the positive orthant (again, except for the origin).

Definition 2: We say that a co–positive function $V(x)$ is a *Lyapunov function* for the system (1) corresponding to a given sequence $\sigma(\cdot)$, if there exists a co–positive function $\phi(x)$ such that condition

$$V(x(t+h)) \leq V(x(t)) - \int_t^{t+h} \phi(x(\tau))d\tau, \ \forall\ h, t \geq 0 \quad (2)$$

holds. We say that $V(x)$ is a *control Lyapunov function* if it is a Lyapunov function for every sequence $\sigma(\cdot)$ generated by some state–feedback law $\sigma(t) = u(x(t))$.

Note that the previous definition is quite generic as it does not require $V(x)$ to be differentiable. If this is the case, then (2) becomes equivalent to the standard condition: $\dot{V}(x) \leq -\phi(x)$.

The existence of a co–positive control Lyapunov function for (1) ensures the exponential stabilizability of the system. This is true in particular when the function is convex. When dealing with linear (i.e., not necessarily positive) switched systems, exponential stabilizability does not imply the existence of a convex control Lyapunov function [5]. On the other hand, it has been shown [11] that exponential stabilizability of discrete–time positive switched systems implies the existence of a concave co–positive control Lyapunov function. This result extends to continuous–time positive switched systems described as in (1).

Proposition 2: System (1) is exponentially stabilizable if and only if it admits a positively homogeneous, concave and co–positive control Lyapunov function.

Proof: Sufficiency is known, so we only need to prove the necessity. Given any positive initial state x_0, consider the cost function

$$J(x_0, \sigma(\cdot)) = \int_0^{\infty} \mathbf{1}' x(t) dt,$$

where $x(t)$ is the state trajectory generated from $x(0) = x_0$ corresponding to σ, and let $V(x_0)$ be its optimal value w.r.t all possible switching sequences, i.e.

$$V(x_0) = \inf_{\sigma(\cdot)} J(x_0, \sigma(\cdot)).$$

We want to prove that this is the control Lyapunov function we are searching for. Clearly, $V(0) = 0$ and $V(x_0) > 0$ for $x_0 > 0$, and hence V is a co–positive function.

By the exponential stabilizability assumption, $V(x_0) < +\infty$ for all $x_0 \geq 0$. Let ϵ be an arbitrarily small positive number, and let $\tilde{\sigma}(\cdot)$ be a switching sequence which corresponds to the ϵ–optimal cost, by this meaning that the system trajectory $\tilde{x}(t)$ generated from x_0, by switching according to $\tilde{\sigma}$, corresponds to a value of the cost function equal to $J(x_0, \tilde{\sigma}(\cdot)) = V(x_0) + \epsilon$. Express x_0 as the convex combination of two nonnegative vectors x_1 and x_2, i.e. $x_0 = \gamma x_1 + (1-\gamma)x_2$ for some $\gamma \in [0,1]$. If we denote by $x_1(t)$ and $x_2(t)$ the trajectories starting from the initial conditions x_1 and x_2, respectively, and achieved through the same switching sequence $\tilde{\sigma}(\cdot)$, clearly $\tilde{x}(t) = \gamma x_1(t) + (1-\gamma)x_2(t)$. We have

$$\begin{array}{rcl} V(x_0) + \epsilon & = & J(x_0, \tilde{\sigma}(\cdot)) = \displaystyle\int_0^{\infty} \mathbf{1}' \tilde{x}(t) dt = \\ & & \gamma \displaystyle\int_0^{\infty} \mathbf{1}' x_1(t) dt + (1-\gamma) \displaystyle\int_0^{\infty} \mathbf{1}' x_2(t) dt \\ & \geq & \gamma V(x_1) + (1-\gamma) V(x_2). \end{array}$$

Since, ϵ can be arbitrarily small, it follows that

$$V(x_0) \geq \gamma V(x_1) + (1-\gamma) V(x_2),$$

which ensures the function concavity.

The fact that the function $V(x_0)$ is positively homogeneous of order one is a trivial consequence of the way V is defined.

Function V is non–differentiable. However, being concave, it is locally Lipschitz, so we can resort to the generalized Dini derivative

$$D^+V(x(t)) \doteq \lim\sup_{h\to 0^+} \frac{V(x(t+h)) - V(x(t))}{h},$$

to claim that the optimal trajectory satisfies

$$D^+V(x(t)) = -\mathbf{1}'x(t)$$

almost everywhere [19]. The corresponding state feedback switching law is then implicitly defined by the following optimization problem

$$\sigma(t) = u(x(t)) = \arg\min_{\sigma} D^+V(x(t))$$

■

IV. Main result

As we have seen in section II, the existence of an Hurwitz convex combination of the system matrices is only a sufficient condition for the exponential stabilizability of a positive switched system (1). This latter, in turn, is equivalent to the existence of a positively homogeneous and co–positive control Lyapunov function. Such a Lyapunov function, however, is not necessarily smooth, and indeed the one we resorted to within the proof of Proposition 2 is not.

On the other hand, when such an Hurwitz combination can be found, it is well-known that stabilizing switching laws $\sigma(t) = u(x(t))$ can be designed, based on linear co–positive (and hence smooth) control Lyapunov functions. So, the intuitive idea arises that, in order to ensure that the convex combination $A(\alpha)$ is Hurwitz for some $\alpha \in \mathcal{A}$, system (1) should not only be exponentially stable, but also endowed with a smooth control Lyapunov function. This intuition proves to be correct and indeed these two conditions prove to be equivalent, as well as equivalent to the existence of a linear co–positive control Lyapunov function.

To derive our main theorem, we need the following technical result regarding the existence of an Hurwitz convex combination $A(\alpha), \alpha \in \mathcal{A}$.

Theorem 2: Assume that system (1) is exponentially stabilizable and it admits a positively homogeneous, co–positive and smooth control Lyapunov function $V(x)$ satisfying:

$$\exists\ \beta > 0 : \quad \min_i \nabla V(x)A_i x \le -\beta V(x), \qquad \forall x \ge 0. \quad (3)$$

Then there exists $\alpha \in \mathcal{A}$ such that $A(\alpha) = \sum_{i=1}^{M} \alpha_i A_i$ is Hurwitz.

Proof: Assume that system (1) is exponentially stabilizable and there exists a positively homogeneous, co–positive and smooth control Lyapunov function $V(x)$ satisfying (3), or, equivalently such that

$$\exists\ \beta > 0 : \min_{\alpha\in\mathcal{A}} \nabla V(x)A(\alpha)x \le -\beta V(x), \quad \forall x \ge 0. \quad (4)$$

Clearly, condition (4) holds, in particular, for all the nonnegative vectors x with unitary norm, i.e. in

$$\mathcal{S} \doteq \{x \ge 0 :\ \mathbf{1}'x = 1\}.$$

By the continuity of the involved functions and the compactness of $\mathcal{S}$, it is not difficult to show that there exists a small perturbation $\epsilon > 0$ such that ($\beta - \epsilon > 0$ and), for every $x \in \mathcal{S}$, the set

$$\Omega(x) \doteq \{\alpha \in \mathcal{A} :\ \nabla V(x)A(\alpha)x \le -(\beta-\epsilon)V(x)\}, \quad (5)$$

has non-empty relative interior.

We first associate with each $x \in \mathcal{S}$ the vector $\alpha \in \Omega(x)$ of smallest Euclidean norm:

$$\Phi_\alpha(x) \doteq \arg\min_{\alpha\in\Omega(x)} \|\alpha\|_2.$$

On the other hand, we associate with each $\alpha \in \mathcal{A}$, the (right) eigenvector of $A(\alpha)$ of unitary norm, and hence belonging to $\mathcal{S}$, corresponding to the Perron-Frobenius eigenvalue $\lambda_{f,\alpha}$. Notice that this eigenvalue is unique thanks to the assumption that matrices in $\mathcal{A}$ are irreducible. Formally, the corresponding map is

$$\Phi_v(\alpha) = v,$$

where v is the only solution in $\mathcal{S}$ of the equation $A(\alpha)v = \lambda_{f,\alpha}v$. Finally, we define the composite map $\Phi : \mathcal{S} \to \mathcal{S}$ given by

$$v = \Phi(x) \doteq [\Phi_v \odot \Phi_\alpha](x)$$

The following considerations are in order:

1) The set-valued map $\Omega(x)$ is convex-valued and continuous with non empty relative interior.
2) The function $\Phi_\alpha(x)$ is the minimal selection map and it is continuous in view of Michael's selection theorem [1], [9].
3) The function $\Phi_v(\alpha)$ is continuous in view of the continuity of the Perron-Frobenius eigenvector with respect to parameter variations of positive matrices, see [15]. Actually, the result holds for positive matrices, so to prove continuity in our case, we just note that the eigenvectors of a Metzler matrix $A(\alpha)$ coincide with those of $A(\alpha) + \tau I$, for any $\tau > 0$.

By the above considerations, we can conclude that $\Phi(x)$ is a continuous map from the compact set $\mathcal{S}$ into itself. Consequently, by the well known Brouwer Kakutani fixed point result (see for instance [1], pag. 84), the equation $\Phi(x) = x$ has a solution, say $\bar{x} \in \mathcal{S}$. Set $\bar{\alpha} \doteq \Phi_\alpha(\bar{x})$. By construction we have

$$\nabla V(\bar{x})A(\bar{\alpha})\bar{x} \le -(\beta-\epsilon)V(\bar{x}).$$

On the other hand, $A(\bar{\alpha})\bar{x} = \lambda_{f,\bar{\alpha}}\bar{x}$ implies

$$\nabla V(\bar{x})A(\bar{\alpha})\bar{x} = \lambda_{f,\bar{\alpha}}\nabla V(\bar{x})\bar{x},$$

and hence

$$\lambda_{f,\bar{\alpha}}\nabla V(\bar{x})\bar{x} \le -(\beta-\epsilon)V(\bar{x}).$$

The right hand-side of the previous inequality is negative, and hence so is the left hand-side. Being $V(x)$ positively homogeneous, it readily follows that $\nabla V(\bar{x})\bar{x} = \mu V(\bar{x})$, for a suitable positive μ. Hence, $\nabla V(\bar{x})\bar{x} > 0$ implies $\lambda_{f,\bar{\alpha}} < 0$, and this ensures that $A(\bar{\alpha})$ is Hurwitz. ■

We are now in a position to provide a complete characterization of the existence of an Hurwitz convex combination of the system matrices $A_i, i = 1, 2, \ldots, M$.

Theorem 3: The following statements are equivalent:
(i) System (1) admits a positively homogeneous, smooth for $x \neq 0$, co–positive control Lyapunov function such that

$$\min_i \nabla V(x) A_i x \leq -\phi(x), \qquad \forall x \geq 0, \tag{6}$$

for some co–positive continuous function $\phi(x)$.
(ii) System (1) is exponentially stabilizable and admits a positively homogeneous, smooth for $x \neq 0$ and co–positive control Lyapunov function which satisfies (4).
(iii) There exists $\alpha \in \mathcal{A}$ such that $A(\alpha)$ is Hurwitz.
(iv) System (1) is exponentially stabilizable and admits a linear co–positive control Lyapunov function $V_L(x) = c'x$, with $c \gg 0$.

Proof: We prove the statements in circular order.

$(i) \rightarrow (ii)$. Assume that (6) is true for some homogeneous smooth co–positive $V(x)$ and some co–positive continuous function $\phi(x)$. Set

$$\beta \doteq \min_{x \geq 0 : V(x)=1} \phi(x) > 0.$$

Since V is homogeneous, by scaling we immediately have for every $x \geq 0$

$$\min_i \nabla V(x) A_i x \leq -\beta V(x).$$

In turn this condition implies $\dot{V}(x) \leq -\beta V(x)$, if one chooses as switching law $\sigma = u(x) = \arg\min_i \nabla V(x) A_i x$, and hence exponential stabilizability (see for instance [4]).

$(ii) \rightarrow (iii)$ It follows from Theorem 2.

$(iii) \rightarrow (iv)$ Given the convex combination $A(\bar{\alpha})$, let c' be a left strictly positive dominant eigenvector associated with it. Then

$$c' A(\bar{\alpha}) = -\beta c'$$

for some positive β. We have

$$\min_\sigma c' A_\sigma x \leq c' A(\bar{\alpha}) x = -\beta c' x, \tag{7}$$

and therefore $V_L = c'x$ is a control Lyapunov function, in the sense of Definition 2, for $\phi(x) = \beta c' x$.

$(iv) \rightarrow (i)$ Obvious. ■

V. Switched equilibria of affine systems

The results of section IV are now used to cope with the case of continuous-time positive affine switched systems, i.e. systems described by

$$\dot{x}(t) = A_{\sigma(t)} x(t) + b_{\sigma(t)}, \tag{8}$$

where $\sigma(\cdot)$ is a switching sequence, defined on $\mathbb{R}_+$ and taking values in the finite set $\{1, 2, \ldots, M\}$, and for every value i taken by σ, A_i is an $n \times n$ Metzler matrix and b_i a nonnegative vector. We assume that system (8) is exponentially stabilizable, which amounts to saying that when all the b_i's are set to zero in (8) the state trajectories can be driven to zero (by resorting to some state feedback switching law). For the sake of simplicity, we assume that all input vectors b_i are strictly positive.

We say that a state $\bar{x} > 0$ is a *switched equilibrium point* of (8) if the origin is included in the convex hull of $A_i \bar{x} + b_i$ (see [1] pag. 101 for details on discontinuous differential equations). Notice that, in general, $\bar{x}$ is not an equilibrium point of any of the affine subsystems. However, if $\bar{x} > 0$ is a switched equilibrium point, then [6] there exists $\alpha \in \mathcal{A}$ such that

$$0 = A(\alpha)\bar{x} + b(\alpha), \tag{9}$$

where $b(\alpha) \doteq \sum_{i=1}^M \alpha_i b_i$, and α can be seen as the "equivalent derivative" of our discontinuous system [21].

By exploiting the proof of the main theorem, we want to provide a characterization of all the switched equilibria that can be reached under some state-feedback stabilizing switching law $\sigma(t) = u(x(t))$.

Theorem 4: Suppose that system (8) satisfies any of the equivalent conditions of Theorem 3 and assume $b_i \gg 0$ for every $i \in \{1, 2 \ldots, M\}$. Then the set of all switched equilibria of system (8) that can be achieved by resorting to some state feedback switching law $\sigma(t) = u(x(t))$ is given by

$$\mathcal{E} = \{\bar{x} : \bar{x} = -A(\alpha)^{-1} b(\alpha), \exists\, \alpha \in \mathcal{A}_H\},$$

where $\mathcal{A}_H \doteq \{\alpha \in \mathcal{A} : A(\bar{\alpha}) \text{ is Hurwitz}\}$.

Proof: We preliminary notice that, by Theorem 3, the set $\mathcal{A}_H$ is not empty, and hence $\mathcal{E} \neq \emptyset$. We have already seen that if $\bar{x}$ is an equilibrium point, then (9) holds. But then $A(\bar{\alpha})\bar{x} = -b(\alpha) \ll 0$ which implies that $A(\bar{\alpha})$ is Hurwitz. Note that if $b(\alpha)$ is strictly positive, the $\bar{x}$ must be strictly positive since no equilibrium exists on the boundary of the orthant. This ensures that $\bar{x} \in \mathcal{E}$.

Conversely, we want to prove that all points in $\mathcal{E}$ are equilibria achievable by means of some state-feedback stabilizing switching law. Let $A(\alpha), \alpha \in \mathcal{A}_H$, be an Hurwitz matrix and let $P = P'$ be a positive definite matrix such that $A'(\bar{\alpha})P + PA(\bar{\alpha})$ is negative definite. Let $\bar{x}$ be the element of $\mathcal{E}$ corresponding to $A(\alpha)$, and consider the control Lyapunov function $V(x - \bar{x}) = (x - \bar{x})' P (x - \bar{x})$ and the control strategy

$$u(x) = \arg\min_\sigma \dot{V}(x - \bar{x}),$$

where

$$\begin{aligned} \dot{V}(x - \bar{x}) &= (A_\sigma x + b_\sigma)' P (x - \bar{x}) + (x - \bar{x})' P (A_\sigma x + b_\sigma) \\ &= 2(x - \bar{x})' P (A_\sigma x + b_\sigma). \end{aligned} \tag{10}$$

Bearing in mind that $A(\bar{\alpha})\bar{x} = -b(\bar{\alpha})$, we have for $x \neq 0$

$$\begin{aligned}\frac{\dot{V}(x-\bar{x})}{2} &= \underbrace{(x-\bar{x})' P A(\bar{\alpha})(x-\bar{x})}_{<0} + \\ &+ (x-\bar{x})' P[(A_\sigma x + b_\sigma) - (A(\bar{\alpha})x + b(\bar{\alpha}))]\end{aligned}$$

By construction, the vector $A(\bar{\alpha})x + b(\bar{\alpha})$ belongs to the convex hull of the vectors $A_i x + b_i$, and hence

$$\min_\sigma (x-\bar{x})' P[(A_\sigma x + b_\sigma) - (A(\bar{\alpha})x + b(\bar{\alpha}))] \leq 0.$$

This ensures that the minimum of (10) is negative. ■

Concluding remarks

In this paper exponential stabilizability of continuous-time positive switched systems has been investigated. For two-dimensional systems, exponential stabilizability proves to be equivalent to the existence of an Hurwitz convex combination of the (Metzler) system matrices. Even more, Hurwitz convex combinations involving only two of the M system matrices can be found. For higher order systems these results are not true. Exponential stabilizability is equivalent to the existence of a (positively homogeneous and co–positive) control Lyapunov function, however only when a smooth control Lyapunov function can be found an Hurwitz combination exists. These two conditions turn out to be equivalent to the existence of a special class of co–positive and smooth control Lyapunov function, namely linear ones.

Finally, based on these results, a description of all the “switched equilibrium points” of a positive affine switched system has been provided.

Acknowledgments

This paper has been partially supported by IEIIT-CNR.

References

[1] J. P. Aubin and A. Cellina. *Differential inclusions*, volume 264 of “Grundlehren der Mathematischen Wissenschaften” [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1984.

[2] A. Benzaouia and F. Tadeo. Stabilization of positive switching linear discrete-time systems , *Intern. J. Innovative Computing, Information and Control*, vol. 6 (6), pp.2427-2437, 2010

[3] F. Blanchini, P. Colaneri and M.E. Valcher. Co-positive Lyapunov functions for the stabilization of switched positive linear systems, *submitted for publication*, 2011.

[4] F. Blanchini and S. Miani, *Set–Theoretic Methods in Control*, Birkhäuser, Boston, MA, 2008.

[5] F. Blanchini and C. Savorgnan. Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions. *Automatica*, Vol. 44, no. 4, pp. 1166-1170, April 2008.

[6] P. Bolzern and W. Spinelli. Quadratic stabilization of a switched affine system about a nonequilibrium point, *Proceeding of the 2004 American Control Conference*, pp. 3890-3895, Boston, MA, 2004

[7] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan. *Linear matrix inequalities in system and control theory*, SIAM Studies in Applied Mathematics, 1994.

[8] E. Fornasini and M.E. Valcher. Stabilizability of discrete-time positive switched systems. *Proceedings of the 49th IEEE Conf. on Decision and Control*, pp. 432-437, Atlanta, Georgia, 2010.

[9] R. A. Freeman and P. V. Kokotović, *Robust nonlinear control design. State-space and Lyapunov techniques.* Systems and Control: Foundations and Applications. Birkhäuser Inc., Boston, MA, xii+256 pp. edition, 1996, ISBN: 0-8176-3930-6.

[10] L. Gurvits, R. Shorten, and O. Mason. On the stability of switched positive linear systems. *IEEE Transactions on Automatic Control*, 52 (6):1099–1103, 2007.

[11] E. Hernandez-Varga, R. Middleton, P. Colaneri, and F. Blanchini. Discrete-time control for switched positive systems with application to mitigating viral escape. *International Journal of Robust and Nonlinear Control*, DOI: 10.1002/rnc.1628, 2010.

[12] E. Hernandez-Varga, R. Middleton, and P. Colaneri. Optimal and MPC Switching Strategies for Mitigating Viral Mutation and Escape, to appear in Proceedings of the 2011 IFAC World Congress, Milano, Italy, 2011.

[13] D. Liberzon. *Switching in Systems and Control*, Birkhäuser, 2003.

[14] F. Knorn, O. Mason, and R.N. Shorten. On linear co–positive Lyapunov functions for sets of linear positive systems. *Automatica*, 45, Issue 8:1943–1947, August 2009.

[15] S. L. Liu, S. Y. Wang, Sensitivity Analysis of Nonnegative Irreducible Matrices, *Applied Mathematics Letters*, Vol. 12, pp. 121-124, 1999.

[16] O. Mason and R.N. Shorten. On common quadratic Lyapunov functions for stable discrete-time LTI systems. *IMA J. of Applied Mathematics*, 69:271–283, 2004.

[17] O. Mason and R.N. Shorten. On the simultaneous diagonal stability of a pair of positive linear systems. *Linear Algebra and its Appl.*, 413:13–23, 2006.

[18] M. Nagumo. Uber die lage der integralkurven gewöhnlicher differentialgleichungen. *Proc. Phys-Math. Soc. Japan*, 24(3):272–559, 1942.

[19] N. Rouche, P. Habets, and M. Laloy. *Stability theory by Liapunov's direct method*, volume 22. Springer-Verlag, New York, 1977, Applied Mathematical Sciences, xii+396 pp. ISBN 0-387-90258-9.

[20] Z. Sun and S. S. Ge. *Switched Linear Systems Control and Design, Springer*, 2005, DOI: 10.1007/1-84628-131-8.

[21] V.I. Utkin. *Sliding Modes in Control and Optimization*, Springer-Verlag, 1992.

[22] M.E. Valcher and P. Santesso. Reachability properties of single-input continuous-time positive switched systems, *IEEE Transactions on Automatic Control*, vol. 55, no.5, pp. 1117-1130, 2010.

[23] X. Xu and P. Antsaklis. Stabilization of second-order LTI switched systems. *International J. of Control*, vol.. 73, no. 14, pp. 1261-1279, 2000.

[24] A. Zappavigna, P. Colaneri, J.C. Jeromel and R. Middleton. Stabilization of continuous-time switched linear positive systems. *Proceedings of the 2010 American Control Conference*, pp. 3275-3280, Baltimore, MD, 2010.