2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

About structure preserving feedback of controlled contact systems

Héctor Ramírez Estay * †, Bernhard Maschke * and Daniel Sbárbaro †

* LAGEP, UMR CNRS 5007
Université Claude Bernard Lyon I
Villeurbanne, France
{ramirez,maschke}@lagep.univ-lyon1.fr

† Departamento de Ingeniería Eléctrica
Universidad de Concepción
Concepción, Chile
{hectrami,sbarbaro}@udec.cl

Abstract— **The conditions for structure preserving feedback of controlled contact system are studied. It is shown that only a constant feedback preserves the canonical contact form, hence a structure preserving feedback implies a contact system with respect to a new contact form. A necessary condition is stated as a matching equation in the feedback, the contact vector fields, the canonical contact form and the closed-loop contact form. Furthermore, for the case of strict contact vector fields a set of solutions is characterized for a particular class of feedback and the relation with classical results on feedback control of Hamiltonian control systems is established. The control synthesis is briefly addressed and illustrated on a simple example.**

I. INTRODUCTION

Geometric control theory has over the past years been oriented to take advantage of the special structure that arises in specific applications. Among those applications are the ones defined by controlled Hamiltonian systems over symplectic manifolds [1]–[3]. It is well known that the symmetries arising from the symplectic structure of Hamiltonian systems can be used to solve a great number of control problems.

A contact structure is the analogue of a symplectic structure for odd-dimensional manifolds. The contact structure arises naturally in the geometric representation of thermodynamic systems [4]–[6]. This kind of representation has been extended to a class of systems called conservative contact systems used to model simple and complex irreversible thermodynamic processes [7]–[9]. Controlled contact systems have recently been proposed in [7]. In this work the contact vector field is equipped with an input map and conjugated inputs and outputs are defined in the sense of port variables and energy balance. Some preliminary results on *constant* interconnection of contact systems where given in [10], and necessary conditions for the stability of the linearization of contact vector fields where presented in [11]. Recently in [12], the framework of conservative controlled contact systems has been used to propose a control design method, where state and co-state variables can be seen as independent variables. However, the closed-loop system is not longer a contact system and hence the structure is not preserved. In [4] a simple example of structure preserving isothermal interaction of two thermodynamic systems is presented in the frame of contact systems.

The authors gratefully acknowledge the support of the French-Chilean CNRS-CONICYT project n°22791 and the first author acknowledge the support of the Chilean CONICYT scholarship program.

Inspired by the work performed on controlled Hamiltonian systems, the aim of this paper is to show the conditions under which it is possible to conserve the geometric structure of a controlled contact system subject to state feedback. The condition for structure preserving feedback is formally stated in the form of a matching equation. It is shown that only a constant feedback preserves the canonical contact form, hence a structure preserving feedback implies a contact system with respect to a new contact form. For strict controlled contact systems we show that, for a particular class of closed-loop contact form, a set of admissible feedback is given as the solution of a quasi-linear partial differential equation. The solutions to the matching equation and the closed-loop contact form have physical interpretation since they may be associated with a thermodynamic potential and cast in the framework of passivity based control [13]. Furthermore, the relation of the closed-loop contact vector field and contact Hamiltonian function with classic results on Hamiltonian control systems is addressed.

The paper is organized as follows, in section II the preliminaries of controlled contact systems are given; section III presents the conditions for feedback equivalence with respect to a different contact form; in section IV the matching condition is characterized for a particular class of feedback and the relation with structure preserving feedback of controlled Hamiltonian system is established; in section V the control synthesis is addressed and a simple example is used to illustrate the results; finally some closing remarks are given in section VI.

II. CONTROLLED CONTACT SYSTEMS

We shall in the following recall briefly the main definitions and properties of the control contact systems considered in this paper and the reader is referred to [7], [14], [15] for details. Consider some $2n+1$-dimensional manifold $\mathcal{M}$ equipped with a *contact form* denoted by θ.

Definition 1: *[14]* A contact structure on $\mathcal{M}$ is determined by a 1-form θ of constant class $(2n+1)$. The pair $(\mathcal{M}, \theta)$ is then called a *contact manifold*, and θ a *contact form*.

According to Darboux's theorem for Pfaffian forms of constant class [14] there exists a set of *canonical coordinates* $(x_0, x, p) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ of $\mathcal{M}$ where the contact form θ is given by

$$\theta = dx_0 - \sum_{i=1}^{n} p_i dx_i$$

where d denotes the exterior derivative. In the following we shall use the Reeb vector field E associated with the contact form θ which is the unique vecor field satisfying

$$i_E\theta = 1 \quad \text{and} \quad i_E d\theta = 0 \tag{1}$$

where i_E denotes the contraction by the vector field E of differential forms. In canonical coordinates the Reeb vector field is expressed as $E = \frac{\partial}{\partial x_0}$. *Contact vector fields* are vector fields which leave the contact distribution invariant.

Definition 2: A (smooth) vector field X on the contact manifold $\mathcal{M}$ is a contact vector field with respect to a contact form θ if and only if there exists a smooth function $\rho \in C^\infty(\mathcal{M})$ such that

$$L_X\theta = \rho\theta, \tag{2}$$

where $L_X\cdot$ denotes the Lie derivative with respect to the vector field X.

It may be shown that the vector space of contact vector fields and the space of smooth real functions are isomorphic [14] which is stated in the following proposition.

Proposition 3: [7] The map $\Phi(X) = i_X\theta$ defines an isomorphism from the vector space of contact vector fields in the space of smooth real functions on the contact manifold.

The function $K = \Phi(X)$ is called *contact Hamiltonian* generating the contact vector field denoted by $X = \Phi^{-1}(K)$, where Φ^{-1} is the inverse isomorphism. Finally the function ρ of equation (2) is given by $\rho = i_E dK$ where E is the Reeb vector field. A contact system on $\mathcal{M}$ is then defined by $\frac{d}{dt}\begin{bmatrix} x_0 \\ x \\ p \end{bmatrix} = X$ which, in any set of canonical coordinates, is expressed by the dynamic equation

$$\frac{d}{dt}\begin{bmatrix} x_0 \\ x \\ p \end{bmatrix} = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -p^\top \\ 0 & 0 & -I_n \\ p & I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial K}{\partial x_0} \\ \frac{\partial K}{\partial x} \\ \frac{\partial K}{\partial p} \end{bmatrix} \tag{3}$$

where I_n denotes the identity matrix of order n. Controlled contact systems are defined by contact Hamiltonians which depend not only on the state variables (x_0, x, p) but also on a time dependent input function $u(t) \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$ [7]. In this paper we shall consider the particular case when the controlled contact system is affine in the input

$$X = X_{K_0} + X_{K_c}u \tag{4}$$

where $K_0 \in C^\infty(\mathcal{M})$ is the *internal contact Hamiltonian* and $K_c \in C^\infty(\mathcal{M})$ is the *interaction (or control) contact Hamiltonian* and where X_{K_0} and X_{K_c} are contact vector fields with respect to the canonical contact form θ. Now we study the problem of structure preserving feedback of a controlled contact system, i.e. which class of state feedback $u = \alpha(x_0, x, p)$, with $\alpha \in C^\infty(\mathcal{M})$, generates a closed-loop vector field X that is again a contact vector field with respect to the contact form θ, which is given in the following proposition.

Proposition 4: Consider the controlled contact system (4) with the condition that K_c vanishes on a submanifold of measure 0 (that is, is fully actuated) and the feedback $u = \alpha(x_0, x, p)$ being a smooth function of the state variables. The closed loop vector field X is a contact vector field with respect to the canonical contact form θ if and only if $\alpha(x_0, x, p) = \alpha_{cte}$ is constant.

Proof: Recall Cartan's formula: $L_X\cdot = i_X d \cdot + d i_X\cdot$. Then one may compute, using (4),

$$\begin{aligned} L_X\theta &= L_{X_{K_0} + \alpha X_{K_c}}\theta \\ &= i_{(X_{K_0} + \alpha X_{K_c})}d\theta + d i_{(X_{K_0} + \alpha X_{K_c})}\theta \\ &= i_{X_{K_0}}d\theta + \alpha i_{X_{K_c}}d\theta + d\left(K_0 + \alpha K_c\right) \\ &= \left(i_{X_{K_0}}d\theta + dK_0\right) + \alpha\left(i_{X_{K_C}}d\theta + dK_c\right) + K_c d\alpha \\ &= L_{X_{K_0}}\theta + \alpha L_{X_{K_c}}\theta + K_c d\alpha \\ &= \left(\rho_0 + \alpha\rho_c\right)\theta + K_c d\alpha \end{aligned} \tag{5}$$

where $\rho_0 = i_E dK_0$ and $\rho_c = i_E dK_c$. Hence the vector field $X = X_{K_0} + X_{K_c}\alpha$ is a contact vector field if and only if there exists a function $\phi \in C^\infty(\mathcal{M})$ such that: $K_c d\alpha = \phi\theta$. Using the definition of θ in local coordinates, $\theta = dx_0 - \sum_{i=1}^{n} p_i dx_i$, we may write

$$K_c\Big(\frac{\partial\alpha}{\partial x^0}dx^0 + \sum_{k=1}^{n}\frac{\partial\alpha}{\partial x^k}dx^k + \sum_{k=1}^{n}\frac{\partial\alpha}{\partial p_k}dp_k\Big) = \phi\Big(dx^0 - \sum_{k=1}^{n}p_k dx^k\Big),$$

which by the assumption of smoothness of the functions and under the condition that K_c vanishes on a submanifold of measure 0 leads to $\frac{\partial\alpha}{\partial x^k} = -\frac{\partial\alpha}{\partial x^0}p_k$ and $\frac{\partial\alpha}{\partial p_k} = 0$, which implies that α is constant. ∎

In summary the only feedback control of an affine controlled contact system which leads to a contact vector field equipped with contact form θ is the constant control. The resulting contact system is the sum of two contact vector fields with global contact Hamiltonian $K_0 + K_C\alpha_{cte}$. It is interesting to note that this is precisely the case in the definition of the interconnection of port-contact systems as defined in [10]. It is also interesting to note that this result differs from the case of feedback control of input-output Hamiltonian systems where the feedback leading to a closed-loop Hamiltonian system are characterized as the composition of any function with the control Hamiltonian functions [2], [3].

III. Feedback equivalence with a contact vector field with respect to a different contact form.

Proposition 4 shows that by using non constant state feedback of a controlled contact vector field it is not possible to obtain a contact vector field with respect to *the same contact form*. In this section the feedback conditions under which the closed-loop contact vector field X in (4), is again a contact vector field with respect to a *new contact form* are studied.

A. Matching conditions for feedback equivalence to closed-loop contact structure

In the following we shall derive the conditions for the existence of a new contact form θ_d for which X is a contact vector field. Therefore we shall consider the equivalent condition of the existence of a function $\rho_d \in C^\infty(\mathcal{M})$ such that

$$L_X\theta_d = \rho_d\theta_d. \tag{6}$$

Denote $K_d = i_X\theta_d$ the contact Hamiltonian generating X with respect to the contact form θ_d, then $\rho_d = i_{E_d}dK_d$ where E_d denotes the Reeb vector field associated with θ_d. Using the result in (5) applied to θ_d, one has

$$\begin{aligned}
&L_X\theta_d = L_{X_{K_0}+\alpha X_{K_c}}\theta_d\\
&= i_{(X_{K_0}+\alpha X_{K_c})}d\theta_d + d(i_{(X_{K_0})+\alpha X_{K_c})}\theta_d\\
&= (i_{X_{K_0}}d\theta_d) + \alpha(i_{X_{K_c}}d\theta_d) + d(i_{X_{K_0}}\theta_d) + d(\alpha i_{X_{K_c}}\theta_d)\\
&= (i_{X_{K_0}}d\theta_d) + \alpha(i_{X_{K_c}}d\theta_d) + d(i_{X_{K_0}}\theta_d) + \alpha d(i_{X_{K_c}}\theta_d)\\
&+ (i_{X_{K_c}}\theta_d)d\alpha\\
&= \big[(i_{X_{K_0}}d\theta_d) + d(i_{X_{K_0}}\theta_d)\big] + \alpha\big[(i_{X_{K_C}}d\theta_d) + d(i_{X_{K_c}}\theta_d)\big]\\
&+ (i_{X_{K_C}}\theta_d)d\alpha\\
&= L_{X_{K_0}}\theta_d + \alpha L_{X_{K_c}}\theta_d + (i_{X_{K_C}}\theta_d)d\alpha
\end{aligned} \tag{7}$$

which leads by subtraction with (6), to the following problem formulation.

Problem 5: Under which conditions there exist a contact form θ_d, a function $\rho_d \in C^\infty(\mathcal{M})$ and a feedback $u = \alpha \in C^\infty(\mathcal{M})$ in (4) such that the following matching equation is satisfied

$$\rho_d\theta_d = L_{X_{K_0}}\theta_d + \alpha L_{X_{K_c}}\theta_d + (i_{X_{K_C}}\theta_d)d\alpha. \tag{8}$$

In this section, and in order to simplify the matching equation (8), the following assumption is made.

Assumption 6: The controlled contact Hamiltonian and the closed-loop contact Hamiltonian are strict contact Hamiltonians, i.e., $\rho_d = \rho_0 = \rho_c = 0$.

This assumption implies that X, and respectively X_{K_0} and X_{K_c}, leave invariant the contact form itself θ_d (respectively θ). In canonical coordinates this means that they do not depend on the coordinate x_0 associated with the Reeb vectorfield. This is not a restrictive assumption since for contact systems arising from the modelling of physical systems, the contact Hamiltonian indeed do not depend on the x_0 coordinate representing the energy (or more generally a thermodynamic potential) [7]. Under this assumption the matching equation (8) is reduced to a relation on the feedback α and the closed-loop contact structure θ_d

$$L_{X_{K_0}}\theta_d + \alpha L_{X_{K_c}}\theta_d + (i_{X_{K_C}}\theta_d)d\alpha = 0. \tag{9}$$

B. Matching to a contact form obtained by adding an exact form

In the following, in order to ease the solution of this matching equation, we shall restrict the closed-loop contact form θ_d as follows.

Assumption 7: The 1-form θ_d is defined as

$$\theta_d = \theta + dF, \tag{10}$$

with $F \in C^\infty(\mathcal{M})$ satisfying $i_EdF = 0$.

Note that the condition $i_EdF = 0$ is equivalent in canonical coordinates to assume that the function F depends only on (x, p) and not on x_0.

Proposition 8: The 1-form defined by (10) is a contact form.

Proof: Recall that θ_d is a contact form if it is a Pfaffian form of class $2n+1$, satisfying [14],

$$\theta_d \wedge (d\theta_d)^n \neq 0, \tag{11}$$

$$\theta_d \wedge (d\theta_d)^{n+1} = 0. \tag{12}$$

Consider first the inequality (11). Note that using $d^2F = 0$ one has that

$$\begin{aligned}
\theta_d \wedge (d\theta_d)^n &= (\theta + dF) \wedge (d(\theta + dF))^n\\
&= (\theta + dF) \wedge (d\theta)^n
\end{aligned}$$

Now proceed by contradiction and assume that $\theta_d \wedge (d\theta_d)^n = 0$. Then, using the fact that i_E is a $\wedge$ antiderivation and the properties (1) of the Reeb vector field:

$$\begin{aligned}
&i_E\left[\theta_d \wedge (d\theta_d)^n\right]\\
&\quad= i_E\left[(\theta + dF) \wedge (d\theta)^n\right]\\
&\quad= i_E(\theta + dF) \wedge (d\theta)^n + (-1)\,(\theta + dF) \wedge i_E\left((d\theta)^n\right)\\
&\quad= (1 + i_EdF) \wedge (d\theta)^n
\end{aligned}$$

and $i_EdF = 0$, implies that $(d\theta)^n = 0$ which is contradicting the fact that θ is of class $2n+1$. To check (12) notice that $(d\theta)^{n+1}$ is full rank, hence $dF \wedge (d\theta)^{n+1} = 0$ no matter the choice of F and

$$\begin{aligned}
\theta_d \wedge (d\theta_d)^{n+1} &= \theta \wedge (d\theta)^{n+1} + dF \wedge (d\theta)^{n+1}\\
&= \theta \wedge (d\theta)^{n+1} = 0
\end{aligned}$$

■

Notice that it has been assumed that F satisfies $i_EdF = 0$. However, from the proof of Proposition 8 it is clear that it is only required that $i_EdF \neq 1$. In this sense the assumption $i_EdF = 0$ is restrictive, but is justified since it has a clear physical interpretation. The closed-loop contact form (10) is thus given by

$$\begin{aligned}
\theta_d = \theta + dF &= \Big(dx_0 - \sum_{i=1}^{n} p_i dx_i\Big) + dF,\\
&= d(x_0 + F) - \sum_{i=1}^{n} p_i dx_i.
\end{aligned}$$

The closed-loop contact structure can be seen as the original one changed in the direction of the Reeb vector field (the x_0 coordinate). Recall that the contact structure appears in the differential-geometric representation of thermodynamic systems [4], [5], where x_0 is associated with a thermodynamic potential, such as the energy U or the entropy S. Hence θ_d can be interpreted as changing the *energy* of the system. In [4, pp. 273] it is shown that a similar contact structure is obtained through the isothermal interaction of two different thermodynamic systems, each one defined with respect to a different contact structure. From a control perspective it is possible to think that the system defined by the internal contact Hamiltonian is interacting with a control system, defined by the controller, through the control vector field. The resulting contact structure θ_d is then the result of the interaction of system and controller. The relation with passivity based control [16] is also quite evident. In that case the aim is to add a certain function to the open-loop storage (energy) function such that the resulting closed-loop storage function is a Lyapunov function for the controlled system. From a geometric point of view, the fact that the geometry of a system is preserved after feedback with respect to a new geometric structure is not uncommon. For instance in the case of port-controlled Hamiltonian systems (PCHS) [17], the interconnection and damping assignment passivity based control (IDA-PBC) method [18], renders after feedback a PCHS with respect to a set of new structure matrices and storage function. Let us express the matching equation (9) with θ_d defined by (10) in terms of a matching equation in the function F and the feedback α. The Lie derivatives in (9) may be developed as

$$L_{X_{K_0}}(\theta + dF) = L_{X_{K_0}}\theta + L_{X_{K_0}}dF = \rho\theta + L_{X_{K_0}}dF$$

with

$$L_{X_{K_0}}dF = i_{X_{K_0}}d(dF) + d(i_{X_{K_0}}dF) = d\left(X_{K_0}(F)\right).$$

Recalling that $\rho_d = \rho_0 = \rho_c = 0$, we have that $L_{X_{K_0}}\theta_d = d\left(X_{K_0}(F)\right)$ and $L_{X_{K_c}}\theta_d = d\left(X_{K_c}(F)\right)$. Furthermore $i_{X_{K_C}}\theta_d = i_{X_{K_C}}(\theta + dF) = K_c + X_{K_C}(F)$. Hence, the matching equation (9) becomes

$$d\left(X_{K_0}(F)\right) + \alpha d\left(X_{K_c}(F)\right) + \left[K_c + X_{K_C}(F)\right]d\alpha = 0. \tag{13}$$

Since $X = X_{K_0} + X_{K_c}\alpha$, it follows that

$$d(X(F)) = d(X_{K_0}(F)) + \alpha d(X_{K_c}(F)) + X_{K_c}(F)d\alpha.$$

Equation (13) may finally be rewritten as the following matching equation in the feedback α and the function F

$$d\left(X(F)\right) + K_c d\alpha = 0. \tag{14}$$

Remark 9: Notice that if $d\alpha = 0$ (i.e. α is constant), then (13) is satisfied if $d\left(X(F)\right) = 0$, or equivalently if $X(F)$ is constant. This in turn is satisfied if $dF \in \operatorname{ann}\left(\operatorname{Span}\{X_{K_0}, X_{K_c}\}\right)$, i.e $X(F) = 0$. Two special cases may be identified, namely when $dF = 0$ i.e. $\theta_d = \theta$ (Proposition 4) and when F is an invariant of X.

C. Closed-loop contact Hamiltonian function and vector field

The closed-loop contact Hamiltonian function is given by the contraction of the closed-loop contact Hamiltonian vector field and the closed-loop contact form

$$K = i_X\theta_d.$$

Computing this last expression yields

$$\begin{aligned} K &= i_{X_{K_0}}(\theta + dF) + \alpha i_{X_{K_c}}(\theta + dF) \\ &= K_0 + i_{X_{K_0}}dF + \alpha(K_c + i_{X_{K_c}}dF) \\ &= K_0 + X_{K_0}(F) + \alpha(K_c + X_{K_c}(F)). \end{aligned} \tag{15}$$

Now, if X is a contact vector field with respect to θ_d (i.e. $L_X\theta_d = \rho_d\theta_d$), then it is generated by $K = i_X\theta_d$ with respect to the canonical coordinates of the contact form θ_d. Hence, X may be equivalently defined as

$$X = X_{K_0} + \alpha X_{K_c} = \hat{X}_K \tag{16}$$

where $\hat{X}_K$ denotes the contact vector field generated by K with respect to the contact form θ_d.

IV. Decoupling the matching equation

In order to solve the matching equation (13) we shall consider a particular class of feedback and thereby reduce the problem to a condition on the function F. Therefore observe that by taking the exterior derivative of (14) we get $dK_c \wedge d\alpha = 0$. This leads to consider a candidate feedback function of the interaction contact Hamiltonian function K_c

$$\alpha = \varphi \circ K_c. \tag{17}$$

where $\varphi \in C^\infty(\mathbb{R})$, $\varphi : \mathbb{R} \to \mathbb{R}$. Note that this control law solves the equation $dK_c \wedge d(\varphi \circ K_c) = dK_c \wedge (\varphi' \circ K_c)dK_c = 0$, where $\varphi'(\lambda) = \frac{d}{d\lambda}(\varphi(\lambda))$. Replacing this control law in (13), the equation is reduced to a matching equation in F and $\varphi \circ K_c$

$$\begin{aligned} d(X_{K_0}(F)) &+ (\varphi \circ K_c)d(X_{K_c}(F)) \\ &+ (X_{K_c}(F) + K_c)(\varphi' \circ K_c)dK_c = 0. \end{aligned}$$

This equation may be written as

$$d\left[X_{K_0}(F) + (\varphi \circ K_c)X_{K_c}(F)\right] + K_c(\varphi' \circ K_c)dK_c = 0.$$

By defining $\Psi(\lambda) = \int_0^\lambda \chi(\varphi' \circ \chi)d\chi$, the previous expression may be rewritten as

$$d\left[X_{K_0}(F) + (\varphi \circ K_c)X_{K_c}(F) + \Psi \circ K_c\right] = 0.$$

Furthermore, by integration by parts of $\Psi(\lambda)$ it is possible to rewrite the previous expression as

$$d\left[X_{K_0}(F) + (\varphi \circ K_c)X_{K_c}(F) + K_c(\varphi \circ K_c) - \Phi \circ K_c\right] = 0.$$

where $\Phi(\lambda) = \int_0^\lambda(\varphi \circ \chi)d\chi$. This means that there is a constant $c_F \in \mathbb{R}$ such that

$$X_{K_0}(F) + (\varphi \circ K_c)[K_c + X_{K_c}(F)] - \Phi \circ K_c = c_F. \tag{18}$$

Note that the matching condition is parametrized by $\varphi \circ K_c$ and F. The previous development leads to the following characterization of the closed-loop contact Hamiltonian function and vector field in terms of the selected feedback.

Proposition 10: Define $\tilde{\mathcal{X}}$ a $2n$-dimensional manifold, $(x,p) \in \tilde{\mathcal{X}}$. If $K_0, K_c, F \in \mathcal{C}^\infty(\tilde{\mathcal{X}})$, $\theta_d = \theta + dF$ and $\alpha = \varphi \circ K_c(x,p)$, where $\varphi \in C^\infty(\mathbb{R})$, $\varphi : \mathbb{R} \to \mathbb{R}$, then $X = X_{K_0} + \alpha X_{Kc}$ is invariant with respect to θ_d if and only if (18) is satisfied. Furthermore, $X = \hat{X}_K$ with $K = K_0 + \Phi \circ K_c$.

Proof: $\mathbb{R}^{2n+1} \simeq \mathcal{M} \xrightarrow{K_c} \mathbb{R} \xrightarrow{\varphi} \mathbb{R}$, thus

$$\alpha = \varphi \circ K_c(x_0, x, p),$$

but $K_c \in \mathcal{X}$ is a strict contact Hamiltonian (i.e., $K_c = K_c(x,p)$), hence $\alpha(x,p) = \varphi \circ K_c(x,p)$. Replacing the control law in the expression of the closed-loop contact Hamiltonian (15), and since $F(x,p)$ and $\varphi \circ K_c$ verify (18), we obtain $K = K_0 + \Phi \circ K_c + c_F$, and since the constant c_F does not change the closed-loop vector field we may write $K = K_0 + \Phi \circ K_c$. ■

The closed-loop contact vector field is invariant with respect to the new contact form θ_d,

$$\hat{X}_K = \hat{X}_{K_0+\Phi\circ K_c} = X_{K_0} + \alpha X_{K_c}.$$

The closed-loop contact Hamiltonian is equal to the internal contact Hamiltonian plus the composition of the function Φ and the interaction contact Hamiltonian K_c. Furthermore, the feedback is exactly the derivative of the function Φ, $\varphi = \frac{d}{d\lambda}(\Phi(\lambda))$. This result is similar to the one obtained when investigating structure preserving feedback of Hamiltonian systems [2]. In the case of Hamiltonian control systems, a feedback of the form $\alpha = \frac{d}{d\lambda}(C(\lambda))$, where $C(x,p)$ is the interaction Hamiltonian function and $P \circ C$ some potential control function, generates a closed-loop Hamiltonian function $H = H_0 + P \circ C$. The Hamiltonian closed-loop vector field is $X_H = X_{H_0+P\circ C}$. The difference between control of Hamiltonian systems and contact systems is that in the first the geometry of the system is not changed in closed-loop. Proposition 4 shows that applying feedback on a contact system necessary changes its contact form, i.e., the geometry of the system is changed in closed-loop. This introduces, unlike control of Hamiltonian systems, the matching condition (18) due to the change in the contact form.

V. Control synthesis

It is clear that the key step in finding a structure preserving feedback for a control contact system is the existence of solutions of (18). In this section the existence of solutions to this equation is analysed from a control synthesis perspective. Without loose of generality it is assumed that $c_F = 0$. Consider the closed-loop contact Hamiltonian function

$$K = K_0 + \Phi \circ K_c$$

Consider K as a control design parameter $K = K_d$, where K_d is some desired closed-loop contact Hamiltonian function with some prescribed properties. From the previous equation,

$$\Phi \circ K_c = K_d - K_0, \tag{19}$$

hence $\Phi \circ K_c$ may be seen as a function that is added to the internal contact Hamiltonian in order to shape the closed-loop contact Hamiltonian K_d. Note that $\Phi \circ K_c$ is not a completely free design parameter since it is a composite function with K_c. Equation (18) may be written as

$$(X_{K_0} + (\varphi \circ K_c)X_{K_c})(F) + (\varphi \circ K_c)K_c - \Phi \circ K_c = 0.$$

Since K_0 and K_c are given, and $\Phi \circ K_c$ (and hence also $\varphi \circ K_c$) has been assigned, the previous equation represents a quasi-linear PDE in F. We may express this in the canonical coordinates as

$$\begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial p} \end{bmatrix}^\top \begin{bmatrix} -\frac{\partial K_0}{\partial p} - (\varphi \circ K_c)\frac{\partial K_c}{\partial p} \\ \frac{\partial K_0}{\partial x} + (\varphi \circ K_c)\frac{\partial K_c}{\partial x} \end{bmatrix} + (\varphi \circ K_c)K_c - \Phi \circ K_c = 0. \tag{20}$$

The solutions of this equation represent all possible functions F for a specified Φ. Hence, by fixing the desired closed-loop contact Hamiltonian, equation (20) determines how the closed-loop contact form θ_d and the Reeb vector field (the coordinate associated to the energy) are shaped. A different approach is to assign the desired contact form θ_d, and hence the desired function F, and solve the previous equation in Φ and φ. Since $\varphi = \frac{d}{d\lambda}(\Phi(\lambda))$, equation (20) may be rewritten as a differential equation in Φ. In order to illustrate the synthesis a simple example is presented.

A. An illustrative example

Consider a single adiabatic compartment that can interact with the environment only through a controlled entropy source. The dynamic equation of the system is given by

$$\dot{S} = u \tag{21}$$

where S is the entropy in the compartment and u is a controlled entropy flow. This system is characterized by the vector of thermodynamic variables (U, S, T), where $U(S)$ is the internal energy and $T(S)$ is the temperature that depends on the entropy and may be modelled as $T = e^S$ [19]. The canonical contact form of this system is

$$\theta = dU - TdS.$$

On the submanifold where $\theta = 0$, the canonical variables (x_0, x, p) are identical to (U, S, T). The internal and interaction contact Hamiltonians are defined ([12]) by $K_0 = 0$, and $K_c = T(x) - p$. From a control perspective it is of interest to shape θ in order to obtain the following closed-loop contact form

$$\theta_d = d(x_0 + F) - pdx = dx_{0_d} - pdx,$$

i.e., the energy of the closed-loop system is changed in some prescribed way $x_{0_d} = U_d(x,p)$, where U_d is a desired closed-loop energy profile. For this purpose consider the choice $F = U_a(x,p)$, such that $U_d(x,p) = U(x) + U_a(x,p)$ and F represents an added internal energy. Furthermore, define the desired closed-loop contact Hamiltonian as $K_d = c(T(x)-p)^2$, with c a design parameter. Since $K_0 = 0$, from (19) we have that $\Phi = c(T(x)-p)^2$ and $\varphi = 2c(T(x)-p)$. Replacing this functions in (20) we obtain the following PDE

$$\frac{\partial U_a}{\partial x} + T(x)\frac{\partial U_a}{\partial p} = -\frac{1}{2}(T(x) - p).$$

The solution to this equation is

$$U_a(x,p) = -\frac{1}{2}(T(x) - p)x + \mathcal{U}_a(T(x) - p),$$

where $\mathcal{U}_a(T(x) - p)$ is an arbitrary function (degree of freedom). Hence, the selected closed-loop contact Hamiltonian function is equipped with the closed-loop contact form $\theta_d = dU_d(x,p) - pdx$. Furthermore, note that the x-coordinate of the closed-loop contact vector field is given by

$$\dot{x} = 2c(T(x) - p).$$

Hence, be choosing $c = -\frac{k}{2}$, with $k > 0$ a positive control gain, the system is asymptotically stable with Lyapunov function $V = (T(x) - p)^2$ for some temperature profile defined by $p = T_d(x^*)$, where $x^* = S^*$ is a desired entropy value.

VI. CONCLUSION AND FUTURE WORK

The conditions for structure preserving feedback of controlled contact system of the class $X_K = X_{K_0} + X_{K_c}u$ have been studied. It has been shown that in order to preserve the canonical contact form θ only a constant feedback is allowed (Proposition 4). However, the closed-loop system can still be a contact system with respect to a new 1-form θ_d. The conditions have been formally stated in terms of the definition of a contact system and a matching equation. For the case of strict controlled contact systems, i.e., contact systems whose contact Hamiltonian function does not depend on the x_0 coordinate, the matching equation is reduced to a relation in the 1-form θ_d and the state feedback. Moreover, for a particular class of feedback it has been shown that it is possible to render the closed-loop system a contact system with respect to the closed-loop contact form $\theta_d = \theta + dF$, where the function F is the solution to a quasi-linear PDE (Propositions 8 and 10). The closed-loop contact form θ_d changes the canonical contact form θ in the direction of the x_0 coordinate. From a control perspective this means that the energy of the system is changed, and hence the feedback may be interpreted in the frame of passive and dissipative control. Furthermore, the results may be considered as an extension of classical results on structure preserving feedback of Hamiltonian control systems. A control synthesis method that uses the desired closed-loop contact Hamiltonian function as design parameter has been suggested and illustrated on a simple thermodynamic system. The control design may also be performed assigning the closed-loop contact form and solving the matching equation for the control law. Future work will study the control synthesis and specialize these results to conservative controlled contact system arising from the modelling of physical processes.

ACKNOWLEDGMENT

The authors wish to thank Prof. Arjan van der Schaft for helpful discussions and comments.

REFERENCES

[1] R. Brockett, "Control theory and analytical mechanics," in *Geometric Control Theory*, C. Martin and R. Hermann, Eds. Brookline, MA, USA: Math Sci Press, 1977, pp. 1–46.

[2] A. van der Schaft, "On feedback control of Hamiltonian systems," in *Theory and Applications of Nonlinear Control Systems*, C. I. Byrnes and A. Lindquist, Eds. New York, NY, USA: Elsevier North-Holland, Inc., 1986, pp. 273–290.

[3] ——, *System theory and mechanics*. New York, NY, USA: Springer-Verlag New York, Inc., 1989, pp. 426–452.

[4] R. Hermann, *Geometry, Physics and Systems*. Marcel Dekker, 1973.

[5] R. Mrugala, J. Nulton, J. Schon, and P. Salamon, "Contact structure in thermodynamic theory," *Reports in Mathematical Physics*, vol. 29, pp. 109–121, 1991.

[6] M. Grmela, "Reciprocity relations in thermodynamics," *Physica A*, vol. 309, pp. 304–328, 2002.

[7] D. Eberard, B. M. Maschke, and A. J. van der Schaft, "An extension of Hamiltonian systems to the thermodynamic phase space: Towards a geometry of nonreversible processes," *Reports on Mathematical Physics*, vol. 60, pp. 175–198, 2007.

[8] A. Favache, D. Dochain, and B. Maschke, "An entropy-based formulation of irreversible processes based on contact structures," *Chemical Engineering Science*, vol. 65, pp. 5204–5216, 2010.

[9] H. Ramírez, B. Maschke, and D. Sbarbaro, "On the Hamiltonian formulation of the CSTR," *Proceedings of The 49th IEEE Conference on Decision and Control, Atlanta, USA, December*, 2010.

[10] D. Eberard, "Extension des systemes Hamiltoniens a ports aux systemes irréversibles: une approche par la géométrie de contact," Ph.D. dissertation, Université Claude Bernhard (Lyon 1), 2006.

[11] A. Favache, V. D. Santos, D. Dochain, and B. Maschke, "Some properties of conservative control systems," *IEEE Transaction on Automatic Control*, vol. 54, no. 10, pp. 2341–2351, 2009.

[12] H. Ramírez, B. Maschke, and D. Sbarbaro, "Lyapunov based control using contact structures," *Proceedings of the 18th World Congress of the International Federation of Automatic Control (IFAC), Milan, Italy, August*, 2011.

[13] A. van der Schaft, "L2-gain analysis of nonlinear systems and nonlinear state-feedback h∞ control," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 770–784, 1992.

[14] P. Libermann and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics*. D. Reidel Publishing Company, 1987.

[15] V. I. Arnold, *Mathematical methods of classical mechanics*. Springer-Verlag, New York, 1978.

[16] A. van der Schaft, *L2-Gain and Passivity Techniques in Nonlinear Control*, 2nd ed. Secaucus, NJ, USA: Springer-Verlag New York, Inc., 2000.

[17] B. Maschke and A. van der Schaft, "Port-controlled Hamiltonian systems: modelling origins and systemtheoretic properties," *Proceedings of the International Symposium on Nonlinear Control Systems Design, NOLCOS'92, Bordeaux, France, June*, 1992.

[18] R. Ortega, A. van der Schaft, B. Maschke, and G. Escobar, "Interconnection and damping assignment passivity based control of port-controlled Hamiltonian systems," *Automatica*, vol. 38, pp. 585–596, 2002.

[19] F. Couenne, C. Jallut, B. Maschke, P. Breedveld, and M. Tayakout, "Bond graph modelling for chemical reactors," *Mathematical and Computer Modelling of Dynamical Systems*, vol. 12, pp. 159–174, 2006.