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A General Formula for the Stabilization of Event-Based Controlled Systems

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***Abstract*— In this paper, a universal formula is proposed for event-based stabilization of general nonlinear systems affine in the control. The feedback is derived from the original one proposed by Sontag. Under the assumption of the existence of a smooth Control Lyapunov Function, it enables smooth (except at the origin) global asymptotic stabilization of the origin while ensuring that the sampling interval do not contract to zero. Indeed, for any initial condition within any given closed set the minimal sampling interval is proved to be strictly positive. Under homogeneity assumptions the control can be proved to be smooth anywhere and the sampling intervals bounded below for any initial condition.**

I. INTRODUCTION

The classical so-called discrete time framework of controlled systems consists in sampling the system uniformly in time with a constant sampling period T and in computing and updating the control law every time instants $t_k = k \cdot T$. This field, denoted as the time-triggered case (or the synchronous case in sense that all the signal measurements are synchronous), has been widely investigated for linear control systems (see [1] and the references therein), even in the case of delays, sampling jitter and measurement loss that can be seen as a kind of asynchronicity [2]. In the case of nonlinear control systems, one way to address a discrete-time feedback is to implement a continuous time control algorithm with a sufficiently small sampling period [3]. However, the hardware used to sample and hold the plant measurements or compute the feedback control action may make it impossible to reduce the sampling period to a level that guarantees acceptable closed-loop performance. Other way to tackle this problem is the application of sampled-data control algorithms based on an approximate discrete-time models of the process [4] which is not a trivial task. Another proposed approach consist to modify a continuous time stabilizing control using for instance Sontag's general formula to obtain a redesigned control suitable for sampled-data implementation [5].

To overcome these drawbacks, event-based control - also called event-triggered control - has been recently proposed. In this control strategy, the control task is executed after the occurrence of an external event, generated by an event mechanism. Thus in this scheme, the term *sampling time* denotes a time interval between two consecutive events (*e.g.* level crossings of the measure). Hence, two successive sampling instants may not be equidistant in time. Let us first consider general nonlinear systems of the form:

$$\dot{x} = f(x,u) \tag{1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^p$, and f a Lipschitz function vanishing at the origin. For sake of simplicity, we only consider in this paper null stabilization with initial time instant $t_0 = 0$. By *event-based feedback*, it is usually meant a set of the two following functions:

- an *event function* $e : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ that indicates if one needs (when $e \leq 0$) or not (when $e > 0$) to recompute the control. In its more general form, the event function e takes the current state x as input and a memory m of its value last time e became negative. A *memoryless* version is also possible, that is an event function $e : \mathcal{X} \to \mathbb{R}$ that only requires the current value of the state. There is a priori no constraint on the regularity of e.
- a *feedback function* k. We talk about *static* event-based feedback when $k : \mathcal{X} \to \mathcal{U}$. The time (then $k : \mathcal{X} \times \mathbb{R}^+ \to \mathcal{U}$) or simply the sampling index (then $k : \mathcal{X} \times \mathbb{N} \to \mathcal{U}$) can be added to build a *dynamic* event-based feedback.

Additionally, e can depend upon the time (then $e : \mathcal{X} \times \mathbb{R}^+ \to \{0,1\}$) or simply upon the sampling index (then $e : \mathcal{X} \times \mathbb{N} \to \{0,1\}$) and one then talks about *dynamic* event-based feedback. Classical sampled feedback of period T can be seen as a dynamic event-based feedback with $e(x,t) \leq 0$ if and only if $t/T \in \mathbb{N}$.

Typical event-detection mechanisms are functions on the variation of the state (or at least the output) of the system, like in [6], [7], [8], [9], [10]. Although the event-triggered control is well-motivated and allows to relax the periodicity for computations of the control law, only few works report theoretical results about the stability, convergence and performance of event-triggered control systems. In [11] for instance, it is proved that such an approach reduces the number of sampling instants for the same final performance. Recent works deal with the problem of scheduling the control task for continuous-time linear systems [11], [12], [13], [14] and discrete-time linear system [15] where stability and some robustness proprieties such as ISS and $\mathcal{L}_\infty$-performance are exploited. Furthermore, in [15] a Model Predictive Control scheme is used where the event-triggered policies are used for relaxing the computationally demanding algorithms. Some of the above contributions do not need the memory of the last sample, the event function e is *memoryless*, that is, it can simply be formulated as a function of $\mathcal{X}$.

An alternative approach consists in taking e related to the variation of a Lyapunov function - and consequently to the state too - between the current state and the previous sample, like in [16], [17], or in taking e related to the time derivative of the Lyapunov function. Convergence and stability in the nonlinear case is studied in [18], [19], [20]. The main contribution from

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the above mentioned works is the existence of a minimal time between consecutive executions of the control task guaranteeing desired levels of performances in the absence of accumulation points. However, in these works is assumed the Input-to-State Stability (ISS) property of the system which is a very strong assumption. Moreover, these techniques are developed for two classes of nonlinear control systems, namely, state-dependent homogeneous systems and polynomial systems.

The solution of (1) with event-based feedback (e, k) starting in $x_0 \in \mathcal{X}$ at $t = 0$ is then defined as the solution of the differential system (when it exists, a discussion follows on that subject):

$$\dot{x} = f(x, k(m)) \quad (2)$$

$$m(x) = \begin{cases} x & \text{if } e(x,m) \leq 0, x \neq 0 \\ m & \text{elsewhere} \end{cases} \quad (3)$$

$$\text{with:} \quad x(0) = x_0 \text{ and } m(0) = x(0) \quad (4)$$

If f is assumed to be Lipschitz, and events are punctual, a unique solution in the Caratheodory sense always exists without any smoothness assumption on k similarly to [21] when . However, this solution may not exists for all $t \geq 0$ as shown in item 3 of section II. Let $t \to x(t, x_0)$ denote this solution. Given an event function e, and a feedback k defined as above, for any initial condition $x(t = 0) = x_0$ it fully defines a *sampling set* $T_{e,k,x_0} := \{t_0, t_1, t_2, \dots\}$ as the set of time instant $t_0 = 0$, t_1, etc. (called *sampling instants*) at which e is negative. The duration between two successive sampling instants will be called *inter-sampling duration*. The event-based closed-loop solution is therefore defined *at least* for all positive t in $[0, \sup(T_{e,k,x_0})[$. This interval is closed if $\sup(T_{e,k,x_0}) \in T_{e,k,x_0}$. To illustrate this we give in the next section different examples of possible phenomena. This will introduce new notions and definitions given in section III. Section IV is dedicated to the main theorem that extends Sontag's universal formula for smooth feedback stabilization to event-based stabilization.

Notations: In the following, $\mathcal{B}(d, x)$ will stand for the ball of radius d centered at x and $\mathcal{B}(d)$ for the ball of radius d centered at the origin. $x(t; x_0, t_0, u)$ will denote the solution of a differential system starting in x_0 at t_0 with control u. For sake of simplicity, u will be omitted when trivial and $x(t; x_0)$ will stand for $x(t; x_0, 0)$.

II. WHAT CAN HAPPEN WITH EVENT-BASED CONTROL ?

To illustrate different phenomena that can arise with event-based feedback systems, we consider the simple linear integrator $\dot{x} = u$. All event functions considered in these examples are assumed to be memoryless that is to be just functions of $\mathcal{X}$. Between two sampling instants t_i and t_{i+1}, u remains constant so that: $x_{i+1} = x_i + (t_{i+1} - t_i) \cdot u$, x_i denoting the value of the state when the i^{th} event occurs. With the following different feedback laws and event functions and initial conditions, it gives:

1) $k(x) = -x$, $e(x) = 0$ when $|x| = \exp(-\kappa)$, $\kappa \in \mathbb{Z}$ and initial condition $x_0 = 0$. Then

$$T_{e,k,x_0} := \{0\}$$

and the trajectory $x(t) = 0$ is defined for all $t \in [0, +\infty[$.

2) $k(x) = -x$, $e(x) = 0$ when $|x| = \exp(-\kappa)$, $\kappa \in \mathbb{Z}$ and initial condition $x_0 = 1$. Assuming that at time t_i of the sampling set T_{e,k,x_0}, the state of the system is $x_i = \pm\exp(-\kappa_i)$, then at the next sampling instant t_{i+1}, the state of the system becomes $x_{i+1} = \pm\exp(-\kappa_{i+1}) = \pm\exp(-\kappa_i - 1)$. The sampling is therefore periodic of period $1 - \exp(-1)$:

$$T_{e,k,x_0} := \{j \cdot (1 - \exp(-1)), \, j \in \mathbb{N}\}$$

The trajectory is well-defined for all $t \in [0, +\infty[$.

3) $k(x) = -x$, $e(x) = 0$ when $|x| = \exp(-\kappa \cdot |\kappa|)$, $\kappa \in \mathbb{Z}$ and initial condition $x_0 = 1$. In that case, one can calculate that $t_i = i - \sum_{j=1}^{i} \exp(-2j + 1)$. And when i tends to infinity, t_i tends to $\bar{t} := \frac{\exp(-1)}{1-\exp(-2)}$. The trajectory is then well-defined *only* for all $t \in [0, \bar{t}[$.

4) $k(x) = -x^3$, $e(x) = 0$ when $|x| = \exp(-\kappa)$, $\kappa \in \mathbb{Z}$ and initial condition $x_0 = 1$. In that case,

$$t_{i+1} - t_i = \frac{x_{i+1} - x_i}{-x_i^3} = \exp(2i) \cdot [1 - \exp(-1)]$$

and when i tends to infinity, $t_{i+1} - t_i$ also tends to infinity. Although the inter-sampling duration tends to infinity, the trajectory is well-defined for all $t \in [0, +\infty[$ as in case 2.

5) $k(x) = -x \cdot (1 - \exp(-1)) \cdot (1 - \log|x|)$ if $|x| \leq 1$, $k(x) = -x \cdot (1 - \exp(-1))$ elsewhere, and $k(0) = 0$ by continuity, $e(x) = 0$ when $|x| = \exp(-\kappa)$, $\kappa \in \mathbb{Z}$ and initial condition $x_0 = 1$. Here, one can prove that $t_{i+1} - t_i = \frac{1}{i+1}$ and therefore $t_{i+1} - t_i$ tends to zero when i tends to infinity as in case 3 but the t_i's do not converge to a finite limit.

Consider now the unstable system $\dot{x} = (x + u)^3$. The control u being constant between each sampling instant, the solution is the one of a Bernoulli differential system whose solution is:

$$x_{i+1} = \frac{x_i + u}{\sqrt{1 - 2(t_{i+1} - t_i) \cdot (x_i + u)^2}} - u$$

Then, taking

6) $k(x) = -2x$, $e(x) = 0$ when $|x| = \exp(-\kappa)$, $\kappa \in \mathbb{Z}$ and initial condition $x_0 = 1$. Then the inter-sampling duration is:

$$t_{i+1} - t_i = \frac{\exp(2i)}{2} \cdot \left[1 - \frac{1}{(2 - \exp(-1))^2}\right]$$

and when i tends to infinity, $t_{i+1} - t_i$ also tends to infinity. However, the origin of the closed loop system can be proved to be asymptotically stable and the trajectories well-defined on $[0, +\infty[$ for all initial condition.

In cases 1 to 6, the system can trivially be proved to be globally null-asymptotically stable taking $V(x) := \frac{1}{2}x^2$. Cases 1 and 2 show that the sampling set is clearly initial condition dependent. Case 2 to 5 show that for the same system and initial condition, the sampling can be periodic, contractile or expansile (with a finite or infinite limit) depending upon the event function or the feedback. Case 6 shows the inconsistency of the Shannon criteria in the event based paradigm and in particular that the inter-sampling duration can infinitely increase even when insuring the stability of the closed loop of an open-loop unstable system.

III. PRELIMINARY DEFINITIONS FOR EVENT-BASED SYSTEMS

Usually, the event is of null measure, in the sense that the control is recomputed only at distinct x (in the countable sense). However, taking $e(x) = 0$ for all $x \in \mathcal{X}$ would mean that one recomputes the control at each x and therefore that one applies a classical continuous-time feedback. On the sets of non null measure where $e(x) = 0$, the solution is understood in the classical sense (with all possible solution existence problems if the field is discontinuous). Elsewhere, the solution can be intended in the Caratheodory sense. To go further on that, we define:

Definition 3.1 (Well-defined event-based control): An event-based control (k, e) will be said **well-defined** if and only if for any initial condition x_0 at $t = 0$, the solution $t \to x(t; x_0)$ exists for all $t \geq 0$.

Property 3.1 (Minimal Sampling Interval - MSI): An event-based control (k, e) will be said to fulfill the **Minimal Sampling Interval property (MSI)** if and only if for any initial condition x_0 at $t = 0$, there exists a non zero minimal sampling interval $\underline{\tau}(x_0)$ defined by:

$$\underline{\tau}(x_0) := \inf_{i \in \mathbb{N}, t_i \in \mathcal{T}(x_0)} t_{i+1} - t_i > 0$$

In that case, the control is piecewise constant between each time sample and we define:

- x_i, $i \in \mathcal{I} \subset \mathbb{N}$ with $x_0 := x(t = 0)$ as the series of successive values of the state at which e is negative for a given initial condition x_0
- t_i, $i \in \mathcal{I} \subset \mathbb{N}$ with $t_0 := 0$ as the corresponding series of time instants

The aim of definition 3.1 is to exclude solutions with sampling intervals converging to zero at some time (case 3 of section II) or to infinity (case 5 of section II). It quite trivially follows:

Theorem 3.2: An MSI event-based control is well-defined.

Proof of Theorem 3.2: The proof is trivial since if the event-based control is MSI, then $\mathcal{T}(x_0)$ is either finite or countably infinite and the t_i are isolated in $\mathbb{R}^+$. The solution $t \to x(t; x_0)$ hence exists for all $t \geq 0$ in the Caratheodory sense. ■

This minimal sampling period is useful for implementation purpose but also when the feedback k is discontinuous for robustness purpose [22]. However, it would be more suitable to have such a bound less depending upon initial condition:

Property 3.3 (Semi-uniformly MSI event-based control): An event-based control (k, e) will be said **semi-uniformly MSI** if and only if for any $\delta > 0$:

$$\underline{\tau}(\delta) := \inf_{i \in \mathbb{N}, t_i \in \mathcal{T}(x_0), x_0 \in \mathcal{B}(\delta)} t_{i+1} - t_i > 0$$

Property 3.4 (Uniformly MSI event-based control): An event-based control (k, e) will be said **uniformly MSI** if and only if:

$$\underline{\tau} := \inf_{i \in \mathbb{N}, t_i \in \mathcal{T}(x_0), x_0 \in \mathcal{X}} t_{i+1} - t_i > 0$$

Properties 3.1 to 3.3 can be specified using the qualifying term "global" when $\mathcal{X} = \mathbb{R}^n$ in opposition to the term "local" that was omitted above for sake of simplicity. Now that the above notions for event-based controlled systems are appropriately defined, notions like stability, asymptotic stability and stabilizability naturally follow since they rely on the resulting trajectory. The question that arises then is: does a universal formula for uniformly discrete event-based feedback stabilization exist similarly to the continuous time case ? This is the purpose of the next section.

IV. A UNIVERSAL FORMULA FOR EVENT-BASED STABILIZATION

In the sequel, the analysis is restricted to systems affine in the control:

$$\dot{x} = f(x) + g(x)u = f(x) + \sum_i g_i(x)u_i \tag{5}$$

where f and g are smooth functions with f vanishing at the origin. We assume that a Control Lyapunov Function (CLF) exists for system (5), that is a smooth and positive definite function $V : \mathcal{X} \to \mathbb{R}$ so that for each $x \neq 0$ there is some $u \in \mathcal{U}$ such that:

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u < 0 \tag{6}$$

In addition, one may require that the CLF V fulfills the *small control property* [23], that is that for each $\varepsilon > 0$ there is some $\delta > 0$ such that for any x in the ball $\mathcal{B}(\delta)\backslash\{0\}$, there is some u with $\|u\| \leq \varepsilon$ such that (6) holds. Then, it is known that it is possible to design a smooth feedback control that asymptotically stabilizes the system. This is known as the Sontag's universal formula:

Theorem 4.1 (Sontag's universal formula): If there exists a CLF for system (5), then the feedback $k : \mathcal{X} \to \mathcal{U}$, smooth on $\mathcal{X}\backslash\{0\}$ is such that:

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)k(x) < 0, \quad x \in \mathcal{X}\backslash\{0\} \tag{7}$$

for k defined by:

$$k_i(x) := -b_i(x)\phi(a(x), \beta(x)), \quad i \in \{1, \ldots, p\}$$

where $a(x) := \frac{\partial V}{\partial x} f(x)$, $b(x) := \frac{\partial V}{\partial x} g(x)$, $\beta(x) := \|b(x)\|^2$ and

$$\phi(y_1, y_2) := \begin{cases} \frac{y_1 + \sqrt{y_1^2 + y_2 q(y_2)}}{y_2} & \text{if } y_2 \neq 0 \\ 0 & \text{if } y_2 = 0 \end{cases} \tag{8}$$

with $q : \mathbb{R} \to \mathbb{R}$ is any real analytic function such that $q(0) = 0$ and $y_2 q(y_2) > 0$ whenever $y_2 \neq 0$.

Moreover, if the CLF satisfies the so called *small control property*, then taking $q(y_2) := y_2$, the control is continuous at the origin [23].

The main purpose of this paper is to establish that a universal formula also exists in the event-based context up to a slight modification of the original formula proposed by Sontag:

Theorem 4.2: If there exists a CLF for system (5), then the event-based feedback (e, k) defined below is semi-uniformly MSI, smooth on $\mathcal{X}\backslash\{0\}$, and such that:

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)k(m) < 0, \quad x \in \mathcal{X}\backslash\{0\} \tag{9}$$

where m is defined by (3) and:

$$k_i(x) := -b_i(x)\gamma(x) \tag{10}$$

$$e(x, m) := -a(x) - b(x)k(m) - \sigma\sqrt{a(x)^2 + \beta(x)\alpha(x)} \tag{11}$$

where $a(x)$, $b(x)$ and $\beta(x)$ are as in Theorem 4.1, $\alpha : \mathcal{X} \to \mathbb{R}^+$ is any smooth function strictly positive on $\mathcal{S} := \{x \in \mathcal{X} | \beta(x) \neq 0\}$, $\sigma \in]0, 1[$, and

$$\gamma(x) := \begin{cases} \frac{a(x) + \sqrt{a(x)^2 + \beta(x)\alpha(x)}}{\beta(x)} & \text{if } x \in \mathcal{S} \\ 0 & \text{if } x \notin \mathcal{S} \end{cases} \tag{12}$$

As in Theorem 4.1, if the CLF satisfies the so called *small control* *property*, then the control is continuous at the origin. Moreover, if there exists some smooth function $w : \mathcal{X} \to \mathbb{R}^+$ strictly positive on $\mathcal{S}$ and such that $w(x)\beta(x) - a(x) \geq 0$ on $\mathcal{S}$, then the control is smooth on $\mathcal{X}$ with the choice:

$$\alpha(x) := w(x)^2\beta(x) - 2w(x)a(x) \tag{13}$$

Before giving the proof of Theorem 4.2, let us explain the ideas behind the construction of feedback (10). In the event function, $a(x) + b(x)k(m)$ is the time derivative of the Lyapunov function V and therefore the event function detects when the Lyapunov function V stops to be *enough* decreasing. In the control, the term $a(x)$ outside the square root is here to compensate the autonomous evolution of the CLF. The term $a(x)$ inside the square-root is linked to the one outside by smoothness considerations. In the term $\beta(x)\alpha(x)$, $\beta(x)$ is added for smoothness reasons in connection with the $\beta(x)$ at the denominator and $\alpha(x)$ plays a fundamental role by tuning how fast the CLF must decrease when an event occurs. This is the term that enables to avoid too close successive events.

We next focus on homogeneous systems that gave rise to an important literature for general nonlinear systems (see for instance [24], [25] and the references therein) and more recently for event-based approaches (mainly in [20], [16], [18]). However, in these event-based contributions, ISS is assumed contrary to the proposed result. We shortly recall some definitions given for general nonlinear system (see the cited references for more detailed definitions and properties):

Definition 4.1: For n positive real numbers r_i, $i \in \{1, \ldots, n\}$, $d > -\min_i r_i$, all $\lambda > 0$, and $\Lambda^r = \operatorname{diag}(\lambda^{r_1}, \lambda^{r_2}, \ldots, \lambda^{r_n})$,

1) a *function* $V : \mathcal{X} \to \mathbb{R}$ is *homogeneous of degree* d if for all $x \in \mathcal{X}$,
$$V(\Lambda^r x) = \lambda^d V(x)$$
2) a *function* $e : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is *homogeneous of degree* d if for all $x, m \in \mathcal{X}$,
$$e(\Lambda^r x, \Lambda^r m) = \lambda^d e(x, m)$$
3) a *vector field* $h : \mathcal{X} \to \mathcal{X}$ is *homogeneous of degree* d if for all $x \in \mathcal{X}$,
$$h(\Lambda^r x) = \lambda^d \Lambda^r h(x)$$
4) a *controlled system* of the form (1) with feedback $u = k(x)$ is *homogeneous of degree* d if k is such that $x \to f(x, k(x))$ is homogeneous of degree d
5) an *event-based controlled system* of the form (2-4) is *homogeneous of degree* (d_1, d_2) if k is such that $(x, m) \to f(x, k(m))$ and $(x, m) \to e(x, m)$ are respectively homogeneous of degree d_1 and d_2.
6) the notation $\|\cdot\|_{\Lambda^r}$ will denote the homogeneous p-norm, that is:
$$\|x\|_{\Lambda^r} = \left(\sum_{i=1}^{n} |x_i|^{\frac{p}{r_i}} \right)^{\frac{1}{p}}$$
with p sufficiently large so that the norm is smooth except at the origin.

Property 4.3: Consider now an event-based controlled dynamical system homogeneous of degree (d_1, d_2) then as long as the trajectory exists and is unique, the solution is such that:

$$x(t; \Lambda^r x_0) = \Lambda^r x(\lambda^{d_1} t; x_0) \tag{14}$$

Proof of Property 4.3: Between events, (14) holds by homogeneity of $f(x, k(m))$. e being homogeneous of degree d_2, $e(\Lambda^r x, \Lambda^r m) = \lambda^{d_2} e(x, m)$. λ being strictly positive, $e(\Lambda^r x, \Lambda^r m)$ and $e(x, m)$ have the same sign, therefore events along the trajectories of $t \to x(t; \Lambda^r x_0)$ appear for the same t as for $t \to \Lambda^r x(\lambda^{d_1} t; x_0)$. ■

For homogeneous systems, Theorem 4.2 becomes:

Theorem 4.4: Assume that f and each g_i are homogeneous respectively of degree d_f and d_g, identical for all g_i, $i \in \{1, \ldots, p\}$ with $d_g < d_f$. Assume in addition that the CLF V is homogeneous of degree d_V, then the following feedback proposed in [26], that corresponds to (10) with α as in (13) and $w(x) = \|x\|_{\Lambda^r}^{d_f - 2d_g - d_V}$,

$$k_i(x) := -\nu b_i(x) \|x\|_{\Lambda^r}^{d_f - 2d_g - d_V} \tag{15}$$

with $\nu > 0$ sufficiently large and e as in (11) is such that:

1) the event-based controlled system is homogeneous of degree $(d_f, d_V + d_f)$
2) the event-based control is smooth and uniformly MSI
3) the CLF is strictly decreasing for all $x \in \mathcal{X} \setminus \{0\}$

The end of the section is dedicated to the proofs of Theorems 4.2 and 4.4.

Proof of Theorem 4.2: We begin the proof by establishing that γ is smooth on $\mathcal{X} \setminus \{0\}$. For this, consider the algebraic equation:

$$F(x, p) := \beta(x)p^2 - 2a(x)p - \alpha(x) = 0 \tag{16}$$

Note first that $p = \gamma(x)$ is a solution of (16) for all $x \in \mathcal{X}$. It is easy to prove that the partial derivative of F with respect to p is always strictly positive on $\mathcal{X} \setminus \{0\}$:

$$\frac{\partial F}{\partial p} := 2\beta(x)p - 2a(x)$$

Indeed, when $\beta(x) = 0$, equation (6) gives $\frac{\partial F}{\partial p} = -2a(x) > 0$ and when $\beta(x) \neq 0$, equation (12) gives $\frac{\partial F}{\partial p} = 2\sqrt{a(x)^2 + \beta(x)\alpha(x)} > 0$, $\frac{\partial F}{\partial p}$ never vanishes at each point of the form $\{(x, \gamma(x)) | x \in \mathcal{X} \setminus \{0\}\}$. Furthermore, F is smooth with respect to x and p since so are functions a, β and α. Therefore, using the implicit function theorem, γ is smooth on $\mathcal{X} \setminus \{0\}$.

The decrease of the CLF is trivial to prove. Indeed, for each x_i, $i \in \mathbb{N}$:

$$\begin{aligned} \frac{dV}{dt}(x_i) &= \frac{\partial V}{\partial x}(x_i) f(x_i) + \frac{\partial V}{\partial x}(x_i) g(x_i) k(x_i) \\ &= -\sqrt{a(x_i)^2 + \beta(x_i)\alpha(x_i)} \\ &< 0 \text{ for all } x \neq 0 \end{aligned}$$

With the updated control, the event function becomes strictly positive: $e(x_i, x_i) = (1 - \sigma)\sqrt{a(x_i)^2 + \beta(x_i)\alpha(x_i)} > 0$. Therefore, by smoothness of the Lyapunov function V, f and g, it clearly follows that $\frac{dV}{dt}(x(t; x_i, t_i)) < 0$ for all $t \in [t_i, t_{i+1}[$, that is until the next event occurs. t_{i+1} is necessarily bounded since, if not, the Lyapunov function V should converge to a constant value where $\frac{dV}{dt} = 0$. The event function precisely prevents this phenomena detecting $\frac{dV}{dt}$ is *close to vanish* and updates the control if it happens.

To prove that the event-based control is MSI, we have to prove that for any initial condition in a a priori given set, the sampling intervals are bounded below. First of all, notice that events occur only when $\dot{V}$ vanishes or when $x = 0$ and therefore from equation (9) there is no event on the set $\{x \in \mathcal{X} | \beta(x) = 0\} \cup \{0\}$. We therefore restrict the study to the set $\mathcal{S} \setminus \{0\}$ where α is strictly positive by assumption. Let us rewrite the time derivative of the CLF along the trajectories:

$$\begin{aligned}\frac{dV}{dt}(x) &= a(x) + b(x)k(m) \\ &= -\sqrt{a(x)^2+\beta(x)\alpha(x)}+b(x)(k(m)-k(x))\end{aligned} \quad (17)$$

where, defining for $m \in \mathcal{S}$, $\vartheta_m := V(m)$ and the set $\mathcal{V}_{\vartheta_m} := \{x \in \mathcal{X} | V(x) \leq \vartheta_m\}$, x belongs to $\mathcal{V}_{\vartheta_x} \subset \mathcal{V}_{\vartheta_m}$. Note that although m must belong to $\mathcal{S}$, this is not necessarily the case for x. First see that for $t = t_i$, $x = m$ and therefore, since α is strictly positive on $\mathcal{S}$ and a necessarily non zero on the frontier of $\mathcal{S}$ (except possibly at the origin):

$$\begin{aligned}\frac{dV}{dt}(x) &= -\sqrt{a(m)^2 + \beta(m)\alpha(m)} \\ &\leq -\inf_{\substack{m \in \mathcal{S} \\ \text{s.t. } V(m)=\vartheta_m}} \sqrt{a(m)^2 + \beta(m)\alpha(m)} =: -\chi(\vartheta_m) < 0\end{aligned}$$

Considering now the second time derivative of the CLF:

$$\ddot{V}(x) = \left(\frac{\partial a}{\partial x}(x) + k(m)^T \frac{\partial b^T}{\partial x}(x)\right)(f(x) + g(x)k(m))$$

By continuity of all the functions involved, both terms can be bounded for all $x \in \mathcal{V}_{\vartheta_m}$ by the following upper bounds $\varrho_1(\vartheta_m)$ and $\varrho_2(\vartheta_m)$:

$$\begin{aligned}\varrho_1(\vartheta_m) &:= \sup_{\substack{m \in \mathcal{S} \text{ s.t. } V(m)=\vartheta_m \\ x \in \mathcal{V}_{\vartheta_m}}} \left\| \frac{\partial a}{\partial x}(x) + k(m)^T \frac{\partial b^T}{\partial x}(x) \right\| \\ \varrho_2(\vartheta_m) &:= \sup_{\substack{m \in \mathcal{S} \text{ s.t. } V(m)=\vartheta_m \\ x \in \mathcal{V}_{\vartheta_m}}} \|f(x) + g(x)k(m)\|\end{aligned}$$

Therefore, $\dot{V}$ is strictly negative at any event instant t_i and can not vanish until the time $\underline{\tau}(\vartheta_m)$ is elapsed and this minimal sampling is only depending on the level ϑ_m of the CLF in m:

$$\underline{\tau}(\vartheta_m) \geq \frac{\chi(\vartheta_m)}{\varrho_1(\vartheta_m)\varrho_2(\vartheta_m)} > 0 \quad (18)$$

which ends the proof, the event-based feedback (10-11) is semi-uniformly MSI.

To prove the continuity of k at the origin, we only need to consider the points where $\beta(x) \neq 0$ since we already have $k(x) = 0$ if $\beta(x) = 0$. Considering first the subset where $a(x) > 0$, we have:

$$\begin{aligned}\|k(x)\| &\leq \frac{|a(x)| + \sqrt{a(x)^2 + \beta(x)\alpha(x)}}{\sqrt{\beta(x)}} \\ &\leq \frac{2\,|a(x)|}{\sqrt{\beta(x)}} + \sqrt{\alpha(x)}\end{aligned}$$

With the small control property, for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $x \in \mathcal{B}(\delta) \setminus \{0\}$, there exists some u with $\|u\| \leq \varepsilon$ such that $a(x) + b(x)u < 0$. Therefore $|a(x)| < \sqrt{\beta(x)}\varepsilon$. The continuity of α at the origin where it vanishes yields that for the same ε there is a $\delta' > 0$ such that for all $x \in \mathcal{B}(\delta') \setminus \{0\}$, $\sqrt{\alpha(x)} \leq \varepsilon$. Therefore, for any $x \in \mathcal{B}(\min(\delta, \delta')) \setminus \{0\}$:

$$\|k(x)\| \leq 3\varepsilon$$

Now, if $a(x) \leq 0$, using the triangular inequality:

$$0 \leq a(x) + \sqrt{a(x)^2 + \beta(x)\alpha(x)} \leq \sqrt{\beta(x)\alpha(x)}$$

And therefore using again the continuity of α:

$$\|k(x)\| \leq \sqrt{\alpha(x)} \leq \varepsilon$$

which ends the proof of the continuity.

Finally, with α as in (13), the control becomes $k_i(x) = -b_i(x)w(x)$ which is obviously smooth on $\mathcal{X}$. ∎

Proof of Theorem 4.4: Take ν such that:

$$\nu > \sup_{x \in \left\{x \in \mathcal{X} | \|x\|_{\Lambda^r}=1, a(x) \geq 0\right\}} \frac{a(x)}{\beta(x)\,\|x\|_{\Lambda^r}^{d_f - 2d_g - d_V}} \quad (19)$$

As in [26], k is homogeneous of degree $d_f - d_g$ and the system is therefore homogeneous of degree d_f. In addition e is homogeneous of degree $d_V + d_f$. Item 1 therefore holds. Thanks to Theorem 4.2, item 3 also holds and k is smooth on $\mathcal{X}$. To finish the proof, remains to establish that the event-based feedback is uniformly MSI. For this, we invoke the homogeneity of the Lyapunov level sets $\mathcal{V}_{\vartheta_m}$ together with (14), it follows that for all $\vartheta_m > 0$:

$$\underline{\tau}(\vartheta_m) = \underline{\tau}(1)$$

∎

V. EXAMPLES

Consider the linear time-invariant system:

$$\dot{x} = Ax + Bu \quad (20)$$

Take P, a positive definite matrix solution of the Riccati equation $PA + A^T P - 4\varepsilon PBB^T P = -P$ (that can be proved to exist as soon as (A, B) is a stabilizable pair). Then $V(x) := x^T Px$ is a CLF for system (20) since for all $x \neq 0$, $u = -2\varepsilon B^T Px$ renders $\dot{V}$ strictly negative. Since $a(x) = x^T(PA + A^T P)x$, $b(x) = 2x^T PB$ and $\beta(x) = 4x^T PBB^T Px$, the Riccati equation gives:

$$\varepsilon\beta(x) - a(x) = x^T Px = V(x) \geq 0 \quad \forall x \neq 0$$

Therefore, taking $w(x) = \varepsilon$, and $\alpha(x)$ according to (13) the control is smooth everywhere and linear:

$$k(x) = -\varepsilon b(x)^T$$

We next consider the nonlinear system proposed in [27]:

$$\begin{aligned}\dot{x}_1 &= -x_1^5 + x_1 x_2 \\ \dot{x}_2 &= x_2^2 + u\end{aligned} \quad (21)$$

which admits $V(x) = \frac{1}{2}\|x\|^2$ as CLF and $a(x) = -x_1^6 + x_1^2 x_2 + x_2^3$, $b(x) = x_2$. Taking

$$\alpha(x) = \beta(x) + a(x)^2$$

and $\sigma = 0.1$, it gives the trajectories of the states, control and event function represented in Figure 1. Now, taking $w(x) = \frac{1}{2}(1 + x_2)^2$,

$$w(x)\beta(x) - a(x) = \begin{pmatrix} x_1^3 & x_2 \end{pmatrix} \begin{pmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1^3 \\ x_2 \end{pmatrix} + \frac{1}{2}x_2^4$$

is strictly positive for all $x \neq 0$. Therefore with $\alpha(x)$ as in (13), the control is smooth. The resulting trajectories with $\sigma = 0.1$ are represented in Figure 2.

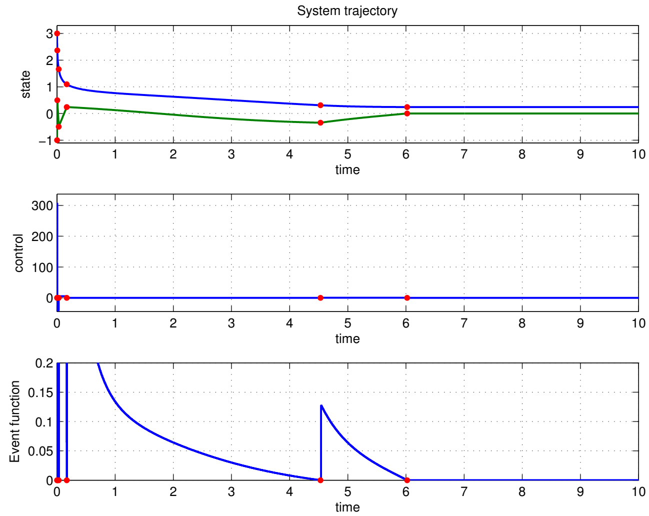

Fig. 1. System (21) under event-based control with $\alpha(x) = \beta(x) + a(x)^2$. The red dots represent the events instants.

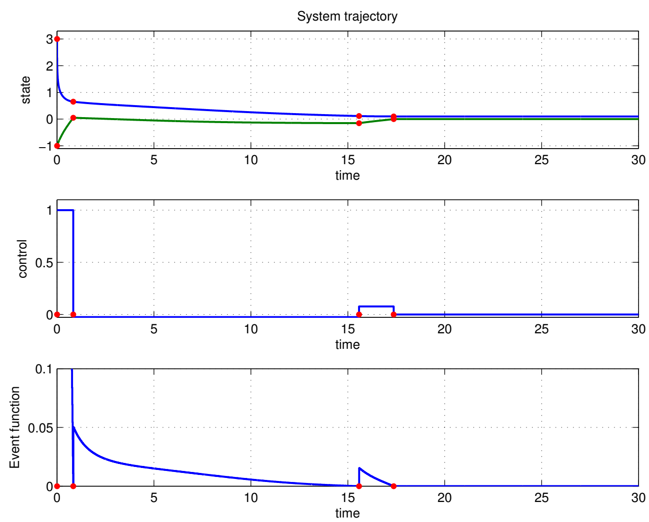

Fig. 2. System (21) under smooth event-based control. The red dots represents the events instants.

VI. CONCLUSION

In this paper, we proposed an extension of the universal formula for smooth feedback stabilization to event-based controlled systems. A modification of the original formula is necessary to ensure that there is a minimal sampling interval between two consecutive events avoiding phenomena like accumulation points. As in the original work, if the Control Lyapunov Function fulfills the small control property, then the control is continuous at the origin. With additional homogeneity assumptions, the control can be proved to be smooth everywhere and the minimal sampling intervals bounded below for all initial conditions.

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