2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

# Soft-Constrained Stochastic Nash Games for Weakly Coupled Large-Scale Discrete-time Systems

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*Abstract*— **In this paper, we discuss infinite-horizon soft-constrained stochastic Nash games involving state-dependent noise and deterministic uncertainties in weakly coupled large-scale discrete-time systems. First, we formulate linear quadratic soft-constrained Nash games in which robustness is attained against external disturbance. Then, the conditions for the existence of robust equilibrium are derived based on the solutions of sets of the discrete version of cross-coupled stochastic algebraic Riccati equations (CSAREs). Moreover, various reliable features such as mean square stability are analyzed. After establishing an asymptotic structure along with positive definiteness for CSAREs solutions, we derive the recursive algorithm for solving CSAREs. Finally, we provide a numerical example to verify the efficiency of the proposed method.**

## I. INTRODUCTION

Control of weakly coupled large-scale discrete-time systems has been investigated intensively over the past three decades [1], [2], [3]. If the coupling parameters are sufficiently small, each controller of a subsystem would work independently without considering cooperation with other subsystems. However, if the coupling parameters cannot be ignored, the existence of multiple controllers in weakly coupled large-scale systems become one of the most important issues in control design. Recently, a multiple LQ control problem and Nash games for a class of stochastic discrete-time systems with a stochastic noise has been studied [13], [14]. However, deterministic uncertainty such as modeling errors has not been considered.

It is well known that a single criterion may not be sufficient to accomplish several control purposes. Therefore, various multiobjective linear quadratic (LQ) control schemes have been investigated [4], [5], [6]. Although these results are very elegant in theory, the noncooperative LQ control problem with multiple decision makers for a class of deterministic and stochastic disturbance is an issue that remains to be considered.

The robust equilibrium in indefinite linear quadratic differential games under a deterministic disturbance input for deterministic and stochastic systems have been studied [7], [8], [9], [15]. This concept is well known as the soft-constrained Nash games and these results are based on the steady-state feedback saddle-point solution. Much effort has been concentrated on continuous-time systems, while a good survey of the development in soft-constrained Nash games can be found.

In this paper, we address the soft-constrained stochastic Nash games for a class of stochastic discrete-time systems governed by Itô difference equations with state-dependent noise. As compared with the existing results [15], the discrete-time stochastic case is investigated for the first time. It should be noted that the corresponding cross-coupled Riccati equation is more complicated than the continuous case. The main contributions of this paper are as follows. First, a stochastic soft-constrained Nash strategy is formulated with respect to an infinite horizon case. Then, the results are applied to infinite horizon soft-constrained stochastic Nash games for a class of discrete-time systems. In order to guarantee the existence of strategy sets, the cross-coupled stochastic algebraic Riccati equations (CSAREs) are introduced for the first time. After establishing the asymptotic structure of CSAREs via the Newton-Kantorovich theorem (see [12] for details), a new parameter independent Nash strategy based on the reduced-order solution of SAREs is established. Finally, in order to demonstrate the efficiency of the proposed strategy, a numerical example is provided.

We denote $F_t$ the $\sigma$-algebra generated by $w(k)$, $k \in \mathbf{N}$. Let $L^2(\Omega,\ \Re^n)$ represent the space of $\Re^n$-valued, square integrable random vectors and $l^2_w(\mathbf{N},\ \Re^n)$ the set of non-anticipative square summable stochastic processes $y = \{y(k)\ :\ y(k) \in \Re^n\}_{t\in\mathbf{N}}$. The $l^2$-norm of $y(k) \in l^2_w(\mathbf{N},\ \Re^n)$ is defined by

$$\|y(k)\|^2_{l^2_w(\mathbf{N},\ \Re^n)} := \sum_{k=0}^{\infty} \mathbf{E}[\|y(k)\|^2].$$

## II. PRELIMINARY RESULTS

Consider the following discrete-time stochastic system.

$$x(k+1) = Ax(k)+Bu(k)+[A_p x(k)+B_p u(k)]w(k), \quad \text{(1a)}$$
$$y(k) = Cx(k), \quad \text{(1b)}$$

where $x(k) \in \Re^n$ represents the state vector. $u(k) \in \Re^m$ represents the control input. $y(k) \in \Re^l$ represents the system output. $w(k) \in \Re$ is a one-dimensional sequence of real random process defined in the filtered probability space, which is a wide sense stationary, second-order process with $E[w(k)] = 0$ and $E[w(s)w(k)] = \delta_{st}$ [10], [11]. The following definitions and result are well known.

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This work was supported in part by a Grant-in-Aid for Scientific Research (C)-20500014 from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

*Definition 1:* [10], [11] Consider the stochastic system (1) with $u(k) \equiv 0$. The stochastic system is said to be mean square stable if for any $x(0)$, the corresponding state satisfies $\lim_{k\to\infty} E[\|x(k)\|^2] = 0$. In this case, $(A,\ A_p)$ is stable.

*Definition 2:* [10], [11] Stochastic system (1) is said to be stabilizable in the mean square sense if there exists a state feedback control $u(k) = Kx(k)$ such that for any $x(0)$, the closed-loop system is mean square stable. In this situation, $(A,\ B \mid A_p,\ B_p)$ is stabilizable.

*Definition 3:* [10], [11] Consider the autonomous stochastic system (1) with $u(k) \equiv 0$. $(A,\ A_p \mid C)$ is exactly observable if $y(k) \equiv 0$ for all $k \Rightarrow x(0) = 0$.

*Lemma 1:* [10] Consider the stochastic system (1) with $B_p \equiv 0$. Assume that $(A,\ B \mid A_p\ 0)$ is stabilizable and $(A,\ A_p \mid C)$ is exactly observable. If the following SARE admits a positive definite solution $X$, $(I + BB^TX)^{-1}A$ is Hurwitz, that is it has all its eigenvalues inside the unit circle.

$$-X + C^TC + A_p^TXA_p + A^TX(I + BB^TX)^{-1}A = 0.$$

It should be noted that the above-mentioned SARE is identical as (11) in [10].

## III. PROBLEM FORMULATION

Consider the stochastic linear discrete-time systems with deterministic uncertainty and state-dependent noises, which involve $N$ players

$$\begin{aligned} x(k+1) = A_\varepsilon x(k) + E_\varepsilon v(k) + \sum_{j=1}^{N} B_{j\varepsilon} u_j(k) \\ + A_{p\varepsilon} x(k) w(k),\ x(0) = x^0, \end{aligned} \tag{2}$$

where

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} & \cdots & \varepsilon A_{1N} \\ \varepsilon A_{21} & A_{22} & \cdots & \varepsilon A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & A_{NN} \end{bmatrix},$$

$$E_\varepsilon := \begin{bmatrix} E_{11} & \varepsilon E_{12} & \cdots & \varepsilon E_{1N} \\ \varepsilon E_{21} & E_{22} & \cdots & \varepsilon E_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon E_{N1} & \varepsilon E_{N2} & \cdots & E_{NN} \end{bmatrix},$$

$$A_{p\varepsilon} := \begin{bmatrix} A_{p11} & \varepsilon A_{p12} & \cdots & \varepsilon A_{p1N} \\ \varepsilon A_{p21} & A_{p22} & \cdots & \varepsilon A_{p2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{pN1} & \varepsilon A_{pN2} & \cdots & A_{pNN} \end{bmatrix},$$

$$B_{j\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{1j}} B_{1j} \\ \varepsilon^{1-\delta_{2j}} B_{2j} \\ \vdots \\ \varepsilon^{1-\delta_{Nj}} B_{Nj} \end{bmatrix},\ x(k) := \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_N(k) \end{bmatrix}.$$

$x_i(k) \in \Re^{n_i}$, $i = 1,\ \dots,N$ represents the state vector. $u_i(k) \in \Re^{m_i}$, $i = 1,\ \dots,N$ represents the control input of the $i$-th player. $v(k) \in \Re^{n_v}$ represents the external disturbance. Here, $\varepsilon$ denotes a relatively small positive coupling parameter that relates the linear system with the other subsystems [1].

It is noteworthy that in this study, the strategies $u_i^*(k)$ are restricted as linear feedback strategies such as $u_i(k) := G_{i\varepsilon}x(k)$. We consider the formulation of the objective functions of the players in order to express a desire for robustness.

$$\begin{aligned} &\bar{J}_i(u_1,\ \dots,u_N) \\ &:= \sup_{v(k)\in l_w^2(\mathbf{N},\ \Re^{n_v})} J_i(u_1,\ \dots,u_N,\ v), \end{aligned} \tag{3}$$

where

$$\begin{aligned} &J_i(u_1,\ \dots,u_N,\ v) \\ &= \sum_{k=0}^{\infty} \mathbf{E}[x^T(k)C_{i\varepsilon}^TC_{i\varepsilon}x(k) + u_i^T(k)u_i(k) - v^T(k)v(k)], \\ &C_{i\varepsilon} := \left[\ \varepsilon^{1-\delta_{1i}}C_{1i} \quad \varepsilon^{1-\delta_{2i}}C_{2i} \quad \cdots \quad \varepsilon^{1-\delta_{Ni}}C_{Ni}\ \right]. \end{aligned}$$

*Definition 4:* [8], [9], [15] The strategy set $(u_1^*,\ \dots,u_N^*)$, $u_i^*(k) := G_{i\varepsilon}^*x(k)$ is a soft-constrained stochastic Nash equilibrium strategy set if for each $i = 1,\ \dots,N$, the following inequality holds:

$$\begin{aligned} &\bar{J}_i(u_1^*,\ \dots,u_N^*) \\ &\le \bar{J}_i(u_1^*,\ \dots,u_{i-1}^*,\ u_i,\ u_{i+1}^*,\ \dots,u_N^*). \end{aligned} \tag{4}$$

In the next section, we address the one-player case as a preliminary result.

## IV. MAIN RESULTS

### A. ONE-PLAYER CASE

First, a one-player case is discussed. The result obtained for that particular case will be used as the basis for the derivation of the results for the general $N$-player case.

Consider a linear time-invariant stochastic stabilizable system

$$\begin{aligned} x(k+1) = A_\varepsilon x(k) + E_\varepsilon v(k) + B_{1\varepsilon}u(k) \\ + A_{p\varepsilon}x(k)w(k),\ x(0) = x^0, \end{aligned} \tag{5a}$$

$$z(k) = \begin{bmatrix} C_\varepsilon x(k) \\ D_\varepsilon u(k) \end{bmatrix},\ D_\varepsilon^TD_\varepsilon = I_{m_1},\ C_\varepsilon^TC_\varepsilon = Q_\varepsilon. \tag{5b}$$

The cost function is given below.

$$J(u,\ v) := \sum_{k=0}^{\infty} \mathbf{E}[\|z(k)\|^2 - \|v(k)\|^2]. \tag{6}$$

*Definition 5:* [7] A strategy pair $(u^*,\ v^*) \in \Gamma_u \times \Gamma_v$ is in saddle-point equilibrium if

$$J(u^*,\ v) \le J(u^*,\ v^*) \le J(u,\ v^*) \tag{7}$$

for all $(u^*,\ v) \in \Gamma_u \times \Gamma_v$ and $(u,\ v^*) \in \Gamma_u \times \Gamma_v$, where $\Gamma_u \times \Gamma_v$ means a product vector space.

The following theorem generalizes the existing results of [8], [9], [15], which is a very important result in deterministic soft-constrained Nash games, to a discrete version.

[1]In general, $\varepsilon$ is an arbitrary sign for weakly coupled systems. In this paper, since the sign of the coefficient matrix can be changed without loss of generality, it is assumed that $\varepsilon$ has a positive sign [15].

*Theorem 1:* Suppose that the stochastic algebraic Riccati equation (SARE)

$$P_\varepsilon = Q_\varepsilon + A_{p\varepsilon}^T P_\varepsilon A_{p\varepsilon} + A_\varepsilon^T P_\varepsilon \Lambda_\varepsilon^{-1} A_\varepsilon, \tag{8}$$

has the solution $P_\varepsilon^* \geq 0$, where $\Lambda_\varepsilon := I_{\bar{n}} + S_{1\varepsilon} P_\varepsilon$ and $I_{\bar{n}} - E_\varepsilon P_\varepsilon E_\varepsilon^T > 0$, $S_{1\varepsilon} := B_{1\varepsilon} B_{1\varepsilon}^T - E_\varepsilon E_\varepsilon^T$, $\bar{n} = \sum_{j=1}^N n_j$.

For any $u(k)$ and $v(k)$ which make the closed-loop system asymptotically mean-square stable, the strategy pair

$$u^*(k) = -B_{1\varepsilon}^T P_\varepsilon \Lambda_\varepsilon^{-1} A_\varepsilon x(k), \tag{9a}$$
$$v^*(k) = E_\varepsilon^T P_\varepsilon \Lambda_\varepsilon^{-1} A_\varepsilon x(k) \tag{9b}$$

is in saddle-point equilibrium if the closed-loop system is asymptotically mean-square stable. $J(u^*,\ v^*) = x^T(0) P_\varepsilon^* x(0)$.

*Proof:* Since this can be proved as an extension of the existing results [7], it is omitted. ■

It should be noted that the saddle point solution that is derived may not be a unique saddle point.

### B. SOFT-CONSTRAINED STOCHASTIC NASH GAMES

Consider the infinite-horizon discrete-time soft-constrained stochastic Nash games, we have the following.

*Theorem 2:* Assume that $(A_\varepsilon,\ A_{p\varepsilon} \mid C_\varepsilon)$ is exactly observable. Then, we have the following.

**(i)** The game has equal upper and lower value if, and only if, the SARE (8) admits a positive definite solution satisfying $I_{\bar{n}} - E_\varepsilon P_\varepsilon E_\varepsilon^T > 0$.

**(ii)** If the SARE (8) admits a positive definite solution satisfying $I_{\bar{n}} - E_\varepsilon P_\varepsilon E_\varepsilon^T > 0$, then it admits a minimal solution. Then, the finite value of the game is $\mathbf{E}[x^T(0) P_\varepsilon x(0)]$.

**(iii)** The upper value of the game is finite if, and only if, the upper and lower values are equal.

**(iv)** If $P_\varepsilon > 0$ exists with the conditions (ii), the closed-loop stochastic system with strategies (9) is mean square stable. In other words, the matrix defined below is Hurwitz.

$$\begin{aligned}\boldsymbol{A}_{\alpha\varepsilon} &:= (I_{\bar{n}} - B_{1\varepsilon} B_{1\varepsilon}^T P_\varepsilon \Lambda_\varepsilon^{-1}) A_\varepsilon \\ &= (I_{\bar{n}} - E_\varepsilon E_\varepsilon^T P_\varepsilon) \Lambda_\varepsilon^{-1} A_\varepsilon.\end{aligned} \tag{10}$$

This implies that the linear stochastic system

$$x(k+1) = \boldsymbol{A}_{\alpha\varepsilon} x(k) + E_\varepsilon v(k) + A_{p\varepsilon} x(k) w(k) \tag{11}$$

is bounded-input-bounded state stable.

**(v)** If $P_\varepsilon > 0$ exists with the conditions (ii), the following feedback matrix is Hurwitz:

$$\boldsymbol{A}_{\beta\varepsilon} := (I_{\bar{n}} - S_{1\varepsilon} P_\varepsilon \Lambda_\varepsilon^{-1}) A_\varepsilon = \Lambda_\varepsilon^{-1} A_\varepsilon. \tag{12}$$

*Proof:* Parts (i)–(iii) follow from the existing results in [7] by extending to stochastic case.

To prove part (iv), we use a similar argument in [7] that is based on LQ theory. Toward this end, we first note that boundedness of the upper value implies, with $v(k) \equiv 0$, that

$$\begin{aligned}&x^T(k) Q_\varepsilon x(k) + u^{*T}(k) u^*(k) \to 0 \ \text{ as } \ k \to \infty \\ \Leftrightarrow\ &\sqrt{Q}_\varepsilon x(k) \to 0 \ \text{ and } \ B_{1\varepsilon}^T P_\varepsilon \Lambda_\varepsilon^{-1} A_\varepsilon x(k) \to 0, \\ &\sqrt{Q}_\varepsilon x(k+1) \to 0 \ \Leftrightarrow \sqrt{Q}_\varepsilon \boldsymbol{A}_{\alpha\varepsilon} x(k) \to 0 \\ \Leftrightarrow\ &\sqrt{Q}_\varepsilon A_\varepsilon x(k) \to 0,\ \ldots,\ \sqrt{Q}_\varepsilon A_\varepsilon x(k+n-1) \to 0 \\ \Leftrightarrow\ &\cdots \ \Leftrightarrow \ \sqrt{Q}_\varepsilon A_\varepsilon^{n-1} x(k) \to 0.\end{aligned}$$

However, $\sqrt{Q}_\varepsilon A_\varepsilon^i x(k) \to 0$, $i = 0,\ \ldots,\ n-1$ implies by detectability that $x(k) \to 0$ and hence $\boldsymbol{A}_{\alpha\varepsilon}$ is Hurwitz.

Finally, we prove the part (v). The closed-loop system is given as $x(k+1) = \boldsymbol{A}_{\beta\varepsilon} x(k) + A_{p\varepsilon} x(k) w(k)$. On the other hand, let us consider the SARE $-P_\varepsilon + Q_\varepsilon + A_{p\varepsilon}^T P_\varepsilon A_{p\varepsilon} + A_\varepsilon^T P_\varepsilon \boldsymbol{A}_{\beta\varepsilon} = 0$. Hence by using Lemma 1 and the similar technique in [7], [10], the stability of $\boldsymbol{A}_{\beta\varepsilon}$ can be proved. ■

*Theorem 3:* Assume that for all $u_i(k)$, $i = 1,\ \ldots,\ N$ and $v(k)$, the closed-loop system is asymptotically mean-square stable. Suppose that $N$ real symmetric matrices $P_{i\varepsilon}^* \geq 0$ and $N$ real matrices $G_{i\varepsilon}^*$ exist such that

$$\begin{aligned}\boldsymbol{F}_i &:= \boldsymbol{F}_i(P_{i\varepsilon},\ G_{1\varepsilon},\ \ldots,\ G_{(i-1)\varepsilon},\ G_{(i+1)\varepsilon},\ \ldots,\ G_{N\varepsilon}) \\ &= -P_{i\varepsilon} + Q_{i\varepsilon} + A_{p\varepsilon}^T P_{i\varepsilon} A_{p\varepsilon} \\ &\quad + \hat{\boldsymbol{A}}_{-i\varepsilon}^T P_{i\varepsilon} \Lambda_{i\varepsilon}^{-1} \hat{\boldsymbol{A}}_{-i\varepsilon} = 0,\end{aligned} \tag{13a}$$
$$u_i^*(k) = -B_{i\varepsilon}^T P_{i\varepsilon} \Lambda_{i\varepsilon}^{-1} \hat{\boldsymbol{A}}_{-i\varepsilon} x(k), \tag{13b}$$
$$v^*(k) = E_\varepsilon^T P_{i\varepsilon} \Lambda_{i\varepsilon}^{-1} \hat{\boldsymbol{A}}_{-i\varepsilon} x(k),\ i = 1,\ \ldots,\ N, \tag{13c}$$

where $Q_{i\varepsilon} := C_{i\varepsilon}^T C_{i\varepsilon}$, $\Lambda_{i\varepsilon} := I_{\bar{n}} + S_{i\varepsilon} P_{i\varepsilon}$, $S_{i\varepsilon} := B_{i\varepsilon} B_{i\varepsilon}^T - E_\varepsilon E_\varepsilon^T$, $\hat{\boldsymbol{A}}_{-i\varepsilon} := A_\varepsilon + \sum_{j=1,\ j\neq i}^N B_{j\varepsilon} G_{j\varepsilon}$.

Then, $(G_{1\varepsilon}^*,\ \ldots,\ G_{N\varepsilon}^*)$, and this strategy set denotes the soft-constrained stochastic Nash equilibrium. Furthermore, $\bar{J}_i(G_{1\varepsilon}^* x,\ \ldots,\ G_{N\varepsilon}^* x) = x^T(0) P_{i\varepsilon}^* x(0)$.

*Proof:* Now, let us consider the following problem in which the cost function (14) is minimal at $u(k) = G_\varepsilon x(k) = G_\varepsilon^* x(k)$.

$$\phi(G_\varepsilon) := \sup_{v(k) \in l_w^2(\mathbf{N},\ \Re^{n_v})} \hat{J}(u,\ v), \tag{14}$$

where

$$\begin{aligned}&\hat{J}(u,\ v) = J_i(u_1^*,\ \ldots,\ u_{i-1}^*,\ u,\ u_{i+1}^*,\ u_N^*,\ v) \\ &= \sum_{k=0}^\infty \mathbf{E}[x^T(k) Q_{i\varepsilon} x(k) + u^T(k) u(k) - v^T(k) v(k)]\end{aligned}$$

and $x(k)$ follows from

$$\begin{aligned}x(k+1) &= \hat{\boldsymbol{A}}_{-i\varepsilon} x(k) + E_\varepsilon v(k) + \hat{B}_\varepsilon u(k) \\ &\quad + A_{p\varepsilon} x(k) w(k),\ x(0) = x^0.\end{aligned} \tag{15}$$

Note that the function $\phi$ coincides with function $J$ in Theorem 3. Applying Theorem 3 to this minimization problem as $P_{i\varepsilon} \Rightarrow P_\varepsilon$, $\hat{\boldsymbol{A}}_{-i\varepsilon} \Rightarrow A_\varepsilon$, $\hat{B} \Rightarrow B_1$ and $x^T(k) Q_{i\varepsilon} x(k) \Rightarrow \|z(k)\|$ yields the fact that the function $\phi$ is minimal at

$$G_\varepsilon^* \Rightarrow G_{i\varepsilon}^*. \tag{16}$$

Moreover, the minimal value is $x^T(0) P_{i\varepsilon}^* x(0)$. ■

## V. WEAKLY-COUPLED LARGE-SCALE STOCHASTIC SYSTEMS

In order to obtain Nash strategies, the asymptotic structure of CSAREs (9) is established. The following analysis requires a basic assumption [10].

*Assumption 1:* $(A_{ii},\ B_{ii}\ |\ A_{pii})$, $i = 1,\ \dots,N$ are stabilizable and $(A_{ii},\ A_{pii}\ |\ C_{ii})$, $i=1,\ \dots,N$ are exactly observable.

Since $A_{\varepsilon}$, $E_{\varepsilon}$, $B_{i\varepsilon}$, $A_{p\varepsilon}$ and $C_{i\varepsilon}$ include $\varepsilon$, the solutions $P_{i\varepsilon}$ of the CSAREs (13a) should contain $\varepsilon$. On the basis of this fact, the solution of CSAREs (9) is assumed to have the following structure [13], [14], [15].

$$P_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}}P_{i1} & \varepsilon P_{i12} & \cdots & \varepsilon P_{i1N} \\ \varepsilon P_{i12}^T & \varepsilon^{1-\delta_{i2}}P_{i2} & \cdots & \varepsilon P_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^T & \varepsilon P_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}}P_{iN} \end{bmatrix}, \quad (17a)$$

$$G_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}}G_{i1} & \cdots & \varepsilon^{1-\delta_{ii}}G_{ii} & \cdots & \varepsilon^{1-\delta_{iN}}G_{iN} \end{bmatrix}. \quad (17b)$$

By substituting matrices $A_{\varepsilon}$, $E_{\varepsilon}$, $B_{i\varepsilon}$, $A_{p\varepsilon}$, $C_{i\varepsilon}$ and $P_{i\varepsilon}$ into CSAREs (13a), setting $\varepsilon=0$, and partitioning CSAREs (13a), the following reduced-order stochastic algebraic Riccati equations (SAREs) are obtained; here, $\bar{P}_{ii},\ i=1,\ \dots,N$ are the limiting solutions of the CSAREs (9) as $\varepsilon \to +0$.

$$\bar{P}_{ii} = Q_{ii} + A_{pii}^T\bar{P}_{ii}A_{pii} + A_{ii}^T\bar{P}_{ii}\Lambda_{ii}^{-1}A_{ii},$$
$$i=1,\ \dots,N, \quad (18)$$

where $\Lambda_{ii} := I_{n_i} + S_{ii}\bar{P}_{ii}$, $S_{ii} = B_{ii}B_{ii}^T - E_{ii}E_{ii}^T$, $Q_{ii} := C_{ii}^TC_{ii}$ and $I_{n_i} - E_{ii}E_{ii}^T\bar{P}_{ii} > 0$.

It should be noted that under Assumption 1, there exists a unique positive semi-definite stabilizing solution $\bar{P}_{ii}$. The limiting behavior of $P_{i\varepsilon}$ when $\varepsilon \to +0$ is described by the following theorem.

*Theorem 4:* Under Assumption 1, suppose that the following matrix is nonsingular.

$$\boldsymbol{J} := \prod_{j=1}^{N} \bar{J}_{jj}^1, \quad (19)$$

where

$$\bar{J}_{ii}^1 := \left.\frac{\partial \boldsymbol{F}_i}{\partial P_{i\varepsilon}}\right|_{\varepsilon=0} = -I_{\bar{n}}\otimes I_{\bar{n}} + A_{p0}^T\otimes A_{p0}^T$$
$$+(\Lambda_{i0}^{-1}\hat{\boldsymbol{A}}_{-i0})^T\otimes \hat{\boldsymbol{A}}_{-i0}^T$$
$$-(\Lambda_{i0}^{-1}\hat{\boldsymbol{A}}_{-i0})^T\otimes[\hat{\boldsymbol{A}}_{-i0}^T\bar{P}_{i0}\Lambda_{i0}^{-1}(B_{i0}B_{i0}^T - E_0E_0^T)],$$
$$\bar{P}_{i0} := \textbf{block diag}\begin{pmatrix} 0 & \cdots & 0 & \bar{P}_{ii} & 0 & \cdots & 0 \end{pmatrix},$$
$$\Lambda_{i0} := I_{\bar{n}} + (B_{i0}B_{i0}^T - E_0E_0^T)\bar{P}_{i0},$$
$$\hat{\boldsymbol{A}}_{-i0} := A_0 + \sum_{j=1,\ j\neq i}^{N} B_{j0}\bar{G}_{j0},$$
$$\bar{G}_{i0} := \begin{bmatrix} 0 & \cdots & 0 & \bar{G}_{ii} & 0 & \cdots & 0 \end{bmatrix},$$
$$A_0 := \textbf{block diag}\begin{pmatrix} A_{11} & \cdots & A_{NN} \end{pmatrix},$$
$$E_0 := \textbf{block diag}\begin{pmatrix} E_{11} & \cdots & E_{NN} \end{pmatrix},$$
$$A_{p0} := \textbf{block diag}\begin{pmatrix} A_{p11} & \cdots & A_{pNN} \end{pmatrix},$$
$$B_{i0} := \begin{bmatrix} 0 & \cdots & 0 & B_{ii}^T & 0 & \cdots & 0 \end{bmatrix}^T,\ i=1,\ \dots,N.$$

Then, there exists a small constant $\sigma^*$ such that for all $\varepsilon \in (0,\ \sigma^*)$, CSAREs (9) admit the positive definite solution $P_{i\varepsilon}$ and feedback gain $G_{i\varepsilon}$ that can be expressed as

$$P_{i\varepsilon} = \bar{P}_{i0} + O(\varepsilon), \quad (20a)$$
$$G_{i\varepsilon} = \bar{G}_{i0} + O(\varepsilon),\ i=1,\ \dots,N. \quad (20b)$$

In order to prove Theorem 4, the following lemmas will be used. It may be noted that it is easy to prove this lemma by using the formula given in [16].

*Lemma 2:* Let matrices $X$, $A$, $B$, $C$, $D$ and $Q$ with appropriate dimension be given. If $R := Q + CXD$ is square and invertible, then

$$\frac{\partial}{\partial \mathrm{vec}X}\mathrm{vec}[AR^{-1}B] = -(DR^{-1}B)^T\otimes(AR^{-1}C).$$

By using Lemma 2, let us prove Theorem 4 corresponding to the asymptotic structure of solutions.

*Proof:* The asymptotic structure of (20) can be obtained by applying the Newton-Kantorovich theorem [12]. Now, by using Lemma 1, let us define Newton's method as follows:

$$\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} - [\boldsymbol{J}_{\varepsilon}^{(n)}]^{-1}\boldsymbol{f}(\boldsymbol{x}^{(n)}), \quad (21)$$

where

$$\boldsymbol{f}(\boldsymbol{x}^{(n)}) := \Big[\ \mathrm{vec}\boldsymbol{F}_1^{(n)} \quad \cdots \quad \mathrm{vec}\boldsymbol{F}_N^{(n)}$$
$$\mathrm{vec}\boldsymbol{G}_1^{(n)} \quad \cdots \quad \mathrm{vec}\boldsymbol{G}_N^{(n)}\ \Big],$$
$$\boldsymbol{x}^{(n)} := \Big[\ [\mathrm{vec}P_{1\varepsilon}^{(n)}]^T \quad \cdots \quad [\mathrm{vec}P_{N\varepsilon}^{(n)}]^T$$
$$[\mathrm{vec}G_{1\varepsilon}^{(n)}]^T \quad \cdots \quad [\mathrm{vec}G_{N\varepsilon}^{(n)}]^T\ \Big]^T,$$
$$\boldsymbol{x}^{(0)} := \Big[\ [\mathrm{vec}\bar{P}_{10}]^T \quad \cdots \quad [\mathrm{vec}\bar{P}_{N0}]^T$$
$$[\mathrm{vec}\bar{G}_{10}]^T \quad \cdots \quad [\mathrm{vec}\bar{G}_{N0}]^T\ \Big]^T,$$
$$\boldsymbol{J}_{\varepsilon}^{(n)} := \boldsymbol{J}_{\varepsilon}(P_{1\varepsilon}^{(n)},\ \dots,P_{N\varepsilon}^{(n)},\ G_{1\varepsilon}^{(n)},\ \dots,G_{N\varepsilon}^{(n)})$$
$$\boldsymbol{J}_{\varepsilon}(P_{1\varepsilon},\ \dots,P_{N\varepsilon},\ G_{1\varepsilon},\ \dots,G_{N\varepsilon})$$
$$= \begin{bmatrix} \boldsymbol{J}_{11}^1 & \cdots & 0 & 0 & \cdots & \boldsymbol{J}_{1N}^2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{J}_{NN}^1 & \boldsymbol{J}_{N1}^2 & \cdots & 0 \\ \boldsymbol{J}_{11}^3 & \cdots & 0 & \boldsymbol{J}_{11}^4 & \cdots & \boldsymbol{J}_{1N}^4 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{J}_{NN}^3 & \boldsymbol{J}_{N1}^4 & \cdots & \boldsymbol{J}_{NN}^4 \end{bmatrix},$$
$$\boldsymbol{J}_{ij}^1 := \frac{\partial \mathrm{vec}\boldsymbol{F}_i}{\partial \mathrm{vec}P_{j\varepsilon}},\ \boldsymbol{J}_{ij}^2 := \frac{\partial \mathrm{vec}\boldsymbol{F}_i}{\partial \mathrm{vec}G_{j\varepsilon}},$$
$$\boldsymbol{J}_{ij}^3 := \frac{\partial \mathrm{vec}\boldsymbol{G}_i}{\partial \mathrm{vec}P_{j\varepsilon}},\ \boldsymbol{J}_{ij}^4 := \frac{\partial \mathrm{vec}\boldsymbol{G}_i}{\partial \mathrm{vec}G_{j\varepsilon}},$$
$$\boldsymbol{J}_{ij}^1 = 0,\ \boldsymbol{J}_{ij}^3 = 0,\ i\neq j,\ \boldsymbol{J}_{ii}^2 = 0,$$
$$\boldsymbol{J}_{ij}^2 = O(\varepsilon),\ \boldsymbol{J}_{ij}^4 = O(\varepsilon),\ i\neq j,$$
$$\boldsymbol{J}_{ii}^4 = I_{m_i}\otimes I_{m_i},\ i,\ j=1,\ \dots,N,$$
$$\boldsymbol{F}_i^{(n)}$$
$$:= \boldsymbol{F}_i(P_{i\varepsilon}^{(n)},\ G_{1\varepsilon}^{(n)},\ \dots,G_{(i-1)\varepsilon}^{(n)},G_{(i+1)\varepsilon}^{(n)},\ \dots,G_{N\varepsilon}^{(n)}),$$
$$\boldsymbol{G}_i^{(n)} := \boldsymbol{G}_i(P_{i\varepsilon}^{(n)},\ G_{1\varepsilon}^{(n)},\ \dots,G_{N\varepsilon}^{(n)}),$$

$$\begin{aligned}\boldsymbol{G}_i &:= \boldsymbol{G}_i(P_{i\varepsilon},\ G_{1\varepsilon},\ \dots ,G_{N\varepsilon})\\ &= G_{i\varepsilon} + B_{i\varepsilon}^T P_{i\varepsilon}\Lambda_{i\varepsilon}^{-1}\hat{\boldsymbol{A}}_{-i\varepsilon} = 0.\end{aligned}$$

It is easy to verify that the following equation holds.

$$\begin{aligned}&\boldsymbol{J}_{\varepsilon}^{(0)}(P_{1\varepsilon}^{(0)},\ \dots ,P_{N\varepsilon}^{(0)},\ G_{1\varepsilon}^{(0)},\ \dots ,G_{N\varepsilon}^{(0)})\\ &= \boldsymbol{J}_{\varepsilon}^{(0)}(\bar{P}_{10},\ \dots ,\bar{P}_{N0},\ \bar{G}_{10},\ \dots ,\bar{G}_{N0}) = \boldsymbol{J} + O(\varepsilon),\end{aligned} \tag{22}$$

where

$$\boldsymbol{J} := \begin{bmatrix} \bar{J}_{11}^1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \bar{J}_{NN}^1 & 0 & \cdots & 0 \\ \bar{J}_{11}^3 & \cdots & 0 & \boldsymbol{J}_{11}^4 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \bar{J}_{NN}^3 & 0 & \cdots & \boldsymbol{J}_{NN}^4 \end{bmatrix},$$

$$\begin{aligned}\bar{J}_{ii}^3 := \left.\frac{\partial \boldsymbol{G}_i}{\partial P_{i\varepsilon}}\right|_{\varepsilon=0} &= (\Lambda_{i0}^{-1}\hat{\boldsymbol{A}}_{-i0})^T \otimes B_{i0}^T\\ &-(\Lambda_{i0}^{-1}\hat{\boldsymbol{A}}_{-i0})^T \otimes [B_{i0}^T\bar{P}_{i0}\Lambda_{i0}^{-1}(B_0B_0^T - E_0E_0^T)].\end{aligned}$$

By using the assumption that $\boldsymbol{J}$ is nonsingular, the matrix $\boldsymbol{J}_{\varepsilon}^{(0)}$ in (11) is invertible for sufficiently small $\varepsilon$. Using the Newton-Kantorovich theorem, the error estimate is given by

$$\|\boldsymbol{x} - \boldsymbol{x}^{(n)}\| = O(\varepsilon^{2^n}),\ n = 0,\ 1,\ \dots , \tag{23}$$

where $\boldsymbol{x} := \begin{bmatrix} [\mathrm{vec}P_{1\varepsilon}]^T & \cdots & [\mathrm{vec}G_{N\varepsilon}]^T \end{bmatrix}^T$.

Substituting $n = 0$ into (23), equation (20) holds. ■

### A. AN APPROXIMATE STRATEGY

A new parameter independent Nash strategy can be obtained by neglecting the term of $O(\varepsilon)$ of the full order strategy (13b). Moreover, it can be constructed by solving reduced-order SAREs (18). If $\varepsilon$ is very small or unknown, it is obvious that Nash strategy (13b) can be approximated as

$$\bar{u}_i(k) = \bar{G}_{i0}x(k) = \bar{G}_{ii}x_i(k) = -B_{ii}^T\bar{P}_{ii}\Lambda_{ii}^{-1}A_{ii}x_i(k). \tag{24}$$

It should be noted that the parameter-independent soft-constrained Nash strategies (24) can be constructed without information of the small parameter. The main result of this paper is as follows.

*Theorem 5:* The parameter-independent soft-constrained Nash strategies (24) results in the following relation.

$$\begin{aligned}&\bar{J}_i(\bar{u}_1(k),\ \dots ,\bar{u}_N(k)) - \bar{J}_i(u_1^*(k),\ \dots ,u_N^*(k))\\ &= O(\varepsilon),\end{aligned} \tag{25}$$

where

$$\bar{J}_i(\bar{u}_1(k),\ \dots ,\bar{u}_N(k)) := \mathbf{Tr}[L_{i\varepsilon}], \tag{26a}$$

$$\bar{J}_i(u_1^*(k),\ \dots ,u_N^*(k)) := \mathbf{Tr}[P_{i\varepsilon}], \tag{26b}$$

$$L_{i\varepsilon} = \bar{\boldsymbol{Q}}_{i\varepsilon} + A_{p\varepsilon}^T L_{i\varepsilon}A_{p\varepsilon} + \bar{\boldsymbol{A}}_{\varepsilon}^T L_{i\varepsilon}\Psi_{i\varepsilon}^{-1}\bar{\boldsymbol{A}}_{\varepsilon}, \tag{26c}$$

with $\Psi_{i\varepsilon} := I_{\bar{n}} - E_{\varepsilon}E_{\varepsilon}^T L_{i\varepsilon}$ $\bar{\boldsymbol{A}}_{\varepsilon} := A_{\varepsilon} + \sum_{j=1}^N B_{j\varepsilon}\bar{G}_{i0}$, $\bar{\boldsymbol{Q}}_{i\varepsilon} := Q_{i\varepsilon} + \bar{G}_{i0}^T\bar{G}_{i0}$.

The following lemma that is based on Theorem 1 is needed to prove the Theorem 5 and the result can be proved by means of the saddle point equilibrium solution.

*Lemma 3:* Let us consider the following optimization problem

$$x(k+1) = A_{\varepsilon}x(k) + E_{\varepsilon}v(k) + A_{p\varepsilon}x(k)w(k), \tag{27a}$$

$$\max_{v(k)\in l_w^2(\mathbf{N},\ \Re^{n_v})} \tilde{J}(v), \tag{27b}$$

where

$$\tilde{J}(v) = \sum_{k=0}^{\infty}\mathbf{E}[x^T(k)Q_{\varepsilon}x(k) - v^T(k)v(k)],\ Q_{\varepsilon} = Q_{\varepsilon}^T \geq 0.$$

If the stochastic system (27) is internally stable, then there exists a stabilizing solution $X_{\varepsilon} \geq 0$ to the following SARE

$$X_{\varepsilon} = Q_{\varepsilon} + A_{p\varepsilon}^T X_{\varepsilon}A_{p\varepsilon} + A_{\varepsilon}^T X_{\varepsilon}\Psi_{\varepsilon}^{-1}A_{\varepsilon}, \tag{28}$$

where $\Psi_{\varepsilon} := I_{\bar{n}} - E_{\varepsilon}E_{\varepsilon}^T X_{\varepsilon}$ and $(A_{\varepsilon} + E_{\varepsilon}F_{\varepsilon},\ A_{p\varepsilon})$ is stable with

$$v^*(k) := F_{\varepsilon}^*x(k) = E_{\varepsilon}^T X_{\varepsilon}\Psi_{\varepsilon}^{-1}A_{\varepsilon}x(k). \tag{29}$$

Then we have $\max_{v(k)\in l_w^2(\mathbf{N},\ \Re^{n_v})} \tilde{J}(v) = \tilde{J}(v^*) = x^T(0)X_{\varepsilon}x(0)$.

As compared with the existing result of the stochastic bounded real lemma [10], it is worth pointing out that the SARE and the novel proof are given.

*Proof:* First, by using the result of Lemma 3, it is easy to found that $L_{i\varepsilon}$ satisfies the SARE (26c). Without loss of generality, suppose that the SARE (26c) has the following asymptotic structure.

$$L_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}}L_{i1} & \varepsilon L_{i12} & \cdots & \varepsilon L_{i1N} \\ \varepsilon L_{i12}^T & \varepsilon^{1-\delta_{i2}}L_{i2} & \cdots & \varepsilon L_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon L_{i1N}^T & \varepsilon L_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}}L_{iN} \end{bmatrix}.$$

By using the equality $L_{i\varepsilon}B_{j\varepsilon} = O(\varepsilon)$, $i \neq j$, and using the result of (20), the parameter independent reduced-order SARE of (26c) can be obtained as follows.

$$\begin{aligned}\bar{L}_{ii} &= Q_{ii} + \bar{G}_{ii}^T\bar{G}_{ii} + A_{pii}^T\bar{L}_{ii}A_{pii} + (A_{ii} + B_{ii}\bar{G}_{ii})^T\\ &\quad\times\bar{L}_{ii}(I_{n_i} - E_{ii}E_{ii}^T\bar{L}_{ii})^{-1}(A_{ii} + B_{ii}\bar{G}_{ii}),\end{aligned} \tag{30}$$

where $\bar{L}_{ii}$ is the limiting solution of the SARE (26c).

On the other hand, the CSARE (13a) can be changed as follows.

$$\begin{aligned}&-P_{i\varepsilon} + Q_{i\varepsilon} + G_{i\varepsilon}^T G_{i\varepsilon} + A_{p\varepsilon}^T P_{i\varepsilon}A_{p\varepsilon}\\ &\quad+\tilde{\boldsymbol{A}}_{i\varepsilon}^T P_{i\varepsilon}(I_{\bar{n}} - E_{\varepsilon}E_{\varepsilon}^T P_{i\varepsilon})\tilde{\boldsymbol{A}}_{i\varepsilon} + O(\varepsilon^2) = 0,\end{aligned} \tag{31}$$

where $\tilde{\boldsymbol{A}}_{i\varepsilon} := A_{\varepsilon} + \sum_{j=1}^N B_{j\varepsilon}G_{j\varepsilon}$.

By comparing CSARE (31) with (30), it is important to note that $\bar{P}_{ii}$ satisfy the SARE (30) by the following equality.

$$\begin{aligned}&(A_{ii} + B_{ii}\bar{G}_{ii})^T\bar{P}_{ii}(I_{n_i} - E_{ii}E_{ii}^T\bar{P}_{ii})^{-1}(A_{ii} + B_{ii}\bar{G}_{ii})\\ &= A_{ii}^T\bar{P}_{ii}\Lambda_{ii}^{-1}A_{ii} - \bar{G}_{ii}^T\bar{G}_{ii}.\end{aligned} \tag{32}$$

Therefore, the limiting solution of the SAREs (13a) and (26c) is identical. Consequently, $L_{i\varepsilon} - P_{i\varepsilon} = O(\varepsilon)$, which implies (25). ■

Table 1. The rate of the cost degradation of exact strategies (13b) and (24).

| $\varepsilon$ | $\bar{J}_1(u_1^*(k),\ u_2^*(k))$ | $\bar{J}_2(u_1^*(k),\ u_2^*(k))$ | $\bar{J}_1(\bar{u}_1(k),\ \bar{u}_2(k))$ | $\bar{J}_2(\bar{u}_1(k),\ \bar{u}_2(k))$ | $\eta_1$ | $\eta_2$ |
|---|---|---|---|---|---|---|
| 1.0000e+2 | 1.6233 | 1.0537 | 1.6234 | 1.0537 | 1.2285 | 2.8687e+1 |
| 1.0000e+3 | 1.6227 | 1.0536 | 1.6227 | 1.0536 | 1.2274 | 2.8685e+1 |
| 1.0000e+4 | 1.6227 | 1.0536 | 1.6227 | 1.0536 | 1.2274 | 2.8685e+1 |
| 1.0000e+5 | 1.6227 | 1.0536 | 1.6227 | 1.0536 | 1.2274 | 2.8685e+1 |
| 1.0000e+6 | 1.6227 | 1.0536 | 1.6227 | 1.0536 | 1.2274 | 2.8666e+1 |

## VI. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the soft-constrained Nash strategy, we present results for a simple example. The system matrices are given as follows.

$$A_\varepsilon = \begin{bmatrix} 1 & \varepsilon \\ 0.1\varepsilon & -0.5 \end{bmatrix},\ A_{p\varepsilon} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix},$$
$$E = \begin{bmatrix} 0.1 & \varepsilon \\ \varepsilon & 0.5 \end{bmatrix}, B_{1\varepsilon} = \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix},\ B_{2\varepsilon} = \begin{bmatrix} \varepsilon \\ 2 \end{bmatrix},$$
$$C_{1\varepsilon} = \begin{bmatrix} 1 & 2\varepsilon \end{bmatrix},\ C_{2\varepsilon} = \begin{bmatrix} \varepsilon & 1 \end{bmatrix}.$$

Small parameter $\varepsilon = 0.001$ is chosen. The soft-constrained stochastic Nash strategies strategies are given in (33a) and (33b). On the other hand, the exact ones are given in (33c) and (33d) by using the Newton's method.

$$\bar{u}_1(k) = \bar{G}_{10}x(k) = \begin{bmatrix} -6.2257\mathrm{e}{-1} & 0 \end{bmatrix} x(k), \tag{33a}$$
$$\bar{u}_2(k) = \bar{G}_{20}x(k) = \begin{bmatrix} 0 & 2.1281\mathrm{e}{-1} \end{bmatrix} x(k), \tag{33b}$$
$$u_1^*(k) = G_{1\varepsilon}^* x(k) = \begin{bmatrix} -6.2257\mathrm{e}{-1} & -6.7860\mathrm{e}{-4} \end{bmatrix} x(k), \tag{33c}$$
$$u_2^*(k) = G_{2\varepsilon}^* x(k) = \begin{bmatrix} 6.7352\mathrm{e}{-5} & 2.1281\mathrm{e}{-1} \end{bmatrix} x(k). \tag{33d}$$

Consequently, solving the proposed parameter independent approach allows us to determine the $O(\varepsilon)$ close to the soft-constrained stochastic Nash strategies. Finally, the closed-loop poles are at $3.7743\mathrm{e}{-1}$ and $-7.4385\mathrm{e}{-2}$. Thus, it means that the closed-loop stochastic systems are mean square stable.

Finally, we evaluate the relation (25) by using the exact strategies (13b) and the parameter independent ones (24). The values of the cost functional for various $\varepsilon$ are given in Table 1, where $\eta_i := |\bar{J}_i(u_1^*(k),\ u_2^*(k)) - \bar{J}_i(\bar{u}_1(k),\ \bar{u}_2(k))|/\varepsilon^2$. It is easy to verify that $|\bar{J}_i(u_1^*(k),\ u_2^*(k)) - \bar{J}_i(\bar{u}_1(k),\ \bar{u}_2(k))| = O(\varepsilon^2)$ because of $\eta_i < \infty$.

Although the right hand side of the relation of (25) is $O(\varepsilon)$, it is observed that $O(\varepsilon^2)$. It may be strictly shown that $|\bar{J}_i(u_1^*(k),\ u_2^*(k)) - \bar{J}_i(\bar{u}_1(k),\ \bar{u}_2(k))| = O(\varepsilon^2)$. However, we note that the question of the complexity status of this feature remains open.

## VII. CONCLUSIONS

The soft-constrained for a class of discrete-time linear systems with stochastic noise and deterministic uncertainty has been solved. First, the soft-constrained Nash strategy set has been formulated by using the CSAREs for the first time. It is worth pointing out that this strategy set is based on the saddle point solution. Second, the weakly-coupled large-scale systems are considered. After establishing the asymptotic structure the numerical algorithm has been discussed. In particular, it has been shown that the exact Nash strategy set can also be computed recursively. Furthermore, the parameter independent strategy set has also been formulated.

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