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Multi-Objective Decision-Making Problems for Discrete-Time Stochastic Systems with State- and Disturbance-Dependent Noise

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Abstract— **In this paper, we consider three types of infinite-horizon multi-objective decision-making problems for a class of discrete-time linear stochastic systems with state- and disturbance-dependent noise. First, the H_2/H_∞ control problem with multiple decision makers is considered. Second, in order to improve the transient response, the linear quadratic control under the Pareto solution is investigated. Finally, the soft-constrained stochastic Nash games are formulated in which robustness is attained against disturbance input. The decision strategies for the three types of problem are derived. It is found that the conditions for the existences of these strategies are related to the solutions of cross-coupled stochastic algebraic Riccati equations (CSAREs). We develop some new algorithms based on linear matrix inequality (LMI) to solve the CSAREs. Numerical example is provided to verify the efficiency of the proposed decision strategies.**

I. INTRODUCTION

In recent years, stochastic H_2/H_∞ control problems for a class of discrete-time systems has been studied by several researchers (see, e.g., [1], [2], [3], [4], [5]). The finite horizon mixed H_2/H_∞ control problem was investigated in [1]. These results were extended to the infinite horizon mixed H_2/H_∞ control for discrete-time stochastic systems with state and disturbance dependent noise [2]. In [3], the H_2/H_∞ control for discrete-time Markovian jump linear systems has also been considered. However, to the best of our knowledge, control problems involving multiple decision makers for discrete-time stochastic systems with state and disturbance dependent noise have not been investigated up to now. It is obvious that the control problems involving multiple decision makers have found a wide range of applications in practice, especially, in large-scale systems. Therefore, it is important to investigate the control problems involving multiple decision makers for discrete-time stochastic systems with state and disturbance dependent noise.

In this paper, we consider three types of infinite-horizon multi-objective decision-making problems for a class of discrete-time linear stochastic systems with state- and disturbance-dependent noise. First, the H_2/H_∞ control problem with multiple decision makers is considered. Second, in order to improve the transient response, the linear quadratic control under the Pareto solution is investigated. Finally, the soft-constrained stochastic Nash games are formulated in which robustness is attained against disturbance input [8], [9], [10], [12]. It is worth pointing out that all the proposed control strategies are established based on the solutions of some cross-coupled stochastic algebraic Riccati equations (CSAREs). Some new algorithms based on linear matrix inequality (LMI) are developed to solve the CSAREs. Finally, numerical example is given to verify the efficiency of the proposed control strategies.

Notation: The notations used in this paper are fairly standard. $\mathbf{E}[\cdot]$ denotes the expectation operator. The l^2-norm of $y(k) \in l_w^2(\mathbf{N},\ \Re^n)$ is defined by $\|y(k)\|^2_{l_w^2(\mathbf{N},\ \Re^n)} := \sum_{k=0}^{\infty} \mathbf{E}[\|y(k)\|^2]$.

II. PRELIMINARY RESULTS

Consider the following discrete-time stochastic system.

$$x(k+1) = Ax(k)+Bu(k)+[A_p x(k)+B_p u(k)]w(k), \quad (1a)$$
$$y(k) = Cx(k), \quad (1b)$$

where $x(k) \in \Re^n$ represents the state vector. $u(k) \in \Re^m$ represents the control input. $y(k) \in \Re^l$ represents the system output. $w(k) \in \Re$ is a one-dimensional sequence of real random process defined in the filtered probability space, which is a wide sense stationary, second-order process with $E[w(k)] = 0$ and $E[w(s)w(k)] = \delta_{st}$ [2], [6].

The following result is a special case for $N = 1$, $r = 1$ of Theorem 4.1 in [13].

Proposition 1: If $(A,\ A_p \mid C,\ C_p)$ is detectable, then the following are equivalent:

(i) the pair $(A,\ A_p)$ is stable;

(ii) the affine equation $-Z + A^T ZA + A_p^T ZA_p + C^T C + C_p^T C_p = 0$ has a solution $Z = Z^T \geq 0$.

The following lemma plays a key technical role in this paper [2], [6], [7].

Let us consider the following stochastic linear quadratic (LQ) control problem subject to (1):

$$\text{minimize } J(u) := \sum_{k=0}^{\infty} \mathbf{E}[x^T(k)Qx(k) + u^T(k)Ru(k)],$$
$$Q = Q^T \geq 0,\ R = R^T > 0. \quad (2)$$

Lemma 1: Assume that for any $u(k)$, the closed-loop system is mean square stable. Suppose that the following stochastic algebraic Riccati equation (SARE) has a solution $P = P^*$.

$$-P + A^T PA + A_p^T PA_p + Q - \boldsymbol{L}^T \boldsymbol{R}^{-1} \boldsymbol{L} = 0, \quad (3)$$

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where $\boldsymbol{R} := R + B^T PB + B_p^T PB_p$ and $\boldsymbol{L} := B^T PA + B_p^T PA_p$.

Then, an optimal feedback control is given by

$$u^*(k) = K^* x(k) = -\boldsymbol{R}^{-1}\boldsymbol{L}x(k), \quad (4)$$

where $J(u^*) = x^T(0)P^*x(0)$, and the feedback gain K^* can be obtained by solving the following semidefinite programming (SDP).

Moreover, P^* is a maximal solution, which is the unique optimal solution.

$$\text{maximize } \ \mathbf{Tr}\,[P], \quad (5a)$$

$$\text{subject to}$$

$$\begin{bmatrix} -P+A^TPA+A_p^TPA_p+Q & \boldsymbol{L}^T \\ \boldsymbol{L} & \boldsymbol{R} \end{bmatrix} \geq 0. \quad (5b)$$

To this end, we consider the following system

$$x(k+1) = Ax(k)+Bv(k)+\Big[A_px(k)+B_pv(k)\Big]w(k), \quad (6a)$$

$$z(k) = Cx(k),\ x(0) = x^0, \quad (6b)$$

where $v(k) \in \Re^{n_v}$ represents the external disturbance. $z(k) \in \Re^{n_z}$ represents the controlled output.

Definition 1: [2] Suppose that for any given $0 < T \in \mathbf{N}$, there exists a unique solution $x(k,\ 0,\ v) \in l_w^2(\mathbf{N}_{T+1},\ \Re^n)$ of (1) with initial value $x(0) = 0$. In the system (6), if the disturbance input $v(k) \in l_w^2(\mathbf{N},\ \Re^{n_v})$ and the controlled output $z(k) \in l_w^2(\mathbf{N},\ \Re^{n_z})$, then the perturbed operator $L : l_w^2(\mathbf{N},\ \Re^{n_v}) \to l_w^2(\mathbf{N},\ \Re^{n_z})$ is defined by

$$Lv(k) := Cx(k,\ 0,\ v),\ \forall v(k) \in l_w^2(\mathbf{N},\ \Re^{n_v}),\ x(0) = 0 \quad (7)$$

with its norm

$$\|L\|^2 := \sup_{\substack{v(k)\,\in\, l_w^2(\mathbf{N},\ \Re^{n_v}),\\ v(k)\neq 0,\ x^0=0}} \frac{\|z(k)\|^2_{l_w^2(\mathbf{N},\ \Re^{n_z})}}{\|v(k)\|^2_{l_w^2(\mathbf{N},\ \Re^{n_v})}} = \sup_{\substack{v(k)\,\in\, l_w^2(\mathbf{N},\ \Re^{n_v}),\\ v(k)\neq 0,\ x^0=0}} \frac{\mathbf{E}[\|Cx(k)\|^2_{l_w^2(\mathbf{N},\ \Re^{n_z})}]}{\mathbf{E}[\|v(k)\|^2_{l_w^2(\mathbf{N},\ \Re^{n_v})}]}. \quad (8)$$

Definition 2: [2] The system (6) is said to be internally stable if it is mean square stable in the absence of $v(k)$.

The following lemma can be viewed as the discrete version.

Lemma 2: [2] If the stochastic system (6) is internally stable and $\|L\| < \gamma$ for given $\gamma > 0$, then there exists a stabilizing solution $P \leq 0$ to the following SARE

$$-P + A^TPA + A_p^TPA_p - Q - \boldsymbol{L}^T\boldsymbol{R}_\gamma^{-1}\boldsymbol{L} = 0, \quad (9)$$

where $\boldsymbol{R}_\gamma := \gamma I_{n_v} + B^TPB + B_p^TPB_p$, $(A + BF_\gamma,\ A_p + B_pF_\gamma)$ is stable with

$$F_\gamma = -\boldsymbol{R}_\gamma^{-1}\boldsymbol{L}. \quad (10)$$

Conversely, if (8) is internally stable and (9) has a stabilizing solution $P \leq 0$, then $\|L\| < \gamma$.

III. H_2/H_∞ CONTROL WITH MULTIPLE DECISION MAKERS

A. PROBLEM FORMULATION

Consider the stochastic linear discrete-time system with state-dependent noises, which involve N-decision makers

$$x(k+1) = Ax(k) + Bv(k) + \sum_{j=1}^{N} B_ju_j(k) + \Big[A_px(k) + B_pv(k)\Big]w(k),\ x(0) = x^0, \quad (11a)$$

$$z_i(k) = \begin{bmatrix} C_ix(k) \\ D_iu_i(k) \end{bmatrix},\ z(k) = \begin{bmatrix} Cx(k) \\ D_1u_1(k) \\ \vdots \\ D_Nu_N(k) \end{bmatrix}, \quad (11b)$$

where $D_i^TD_i = I_{m_i}$ $u_i(k) \in l_w^2(\mathbf{N},\ \Re^{m_i})$, $i = 1,\ \dots,N$ represents the i-th control input.

Given a disturbance attenuation level $\gamma > 0$, define performance functions

$$J_0(u_1,\ \dots,u_N,\ v) := \sum_{k=0}^{\infty}\mathbf{E}[\gamma^2\|v(k)\|^2 - \|z(k)\|^2] \quad (12)$$

and

$$J_i(u_1,\ \dots,u_N,\ v) := \sum_{k=0}^{\infty}\mathbf{E}[\|z_i(k)\|^2],\ i = 1,\ \dots,N. \quad (13)$$

The infinite horizon stochastic H_2/H_∞ control with multiple decision makers of system (11) is stated as follows:

Given $\gamma > 0$, find if possible strategies $u_i^*(k) \in l_w^2(\mathbf{N},\ \Re^{m_i})$, $i = 1,\ \dots,N$ such that

i) $u_i^*(k)$ stabilizes system (11) internally.

ii) $\|L_{u_i^*}\|^2$

$$= \sup_{\substack{v(k)\,\in\, l_w^2(\mathbf{N},\ \Re^{n_v}),\\ v(k)\neq 0,\ x^0=0}} \frac{\sum_{k=0}^{\infty}\mathbf{E}\left[\|Cx(k)\|^2 + \sum_{j=1}^{N}\|u_j^*(k)\|^2\right]}{\sum_{k=0}^{\infty}\mathbf{E}[\|v(k)\|^2]} < \gamma^2. \quad (14)$$

iii) When the worst case disturbance $v^*(k) \in l_w^2(\mathbf{N},\ \Re^{n_v})$, if exists, is implemented in (11), $u_i^*(k)$ minimizes the output energy

$$J_i(u_1,\ \dots,u_N,\ v^*) := \sum_{k=0}^{\infty}\mathbf{E}[\|z_i(k)\|^2] = \sum_{k=0}^{\infty}\mathbf{E}[\|C_ix(k)\|^2 + \|u_i(k)\|^2],\ i = 1,\ \dots,N. \quad (15)$$

If the above $(u_1^*,\ \dots,u_N^*,\ v^*)$ exist, we say that the infinite horizon stochastic H_2/H_∞ control with multiple decision makers is solvable. Obviously, $(u_1^*,\ \dots,u_N^*,\ v*)$ are the

Nash equilibria of the two functionals (12) and (13), which satisfy

$$J_0(u_1^*,\ \dots,u_N^*,\ v^*) \le J_0(u_1^*,\ \dots,u_N^*,\ v), \quad (16a)$$

$$\begin{aligned} &J_i(u_1^*,\ \dots,u_N^*,\ v^*) \\ &\le J_i(u_1^*,\ \dots,u_{i-1}^*,\ u_i,\ u_{i+1}^*,\ \dots,u_N^*,\ v^*), \\ &i=1,\ \dots,N. \end{aligned} \quad (16b)$$

These equilibria are based on the Nash solutions and applied to the cases of multiple decision makers.

B. SOLUTION TO THE MULTI-OBJECTIVE MIXED H_2/H_∞ PROBLEM

In this section, we shall present a solution to the stochastic H_2/H_∞ control with multiple decision makers by solving a cross-coupled stochastic algebraic Riccati equations (CSAREs).

Theorem 1: For the discrete-time stochastic perturbed systems (11), suppose that the following CSAREs have solutions $(X,\ Y_1,\ \dots,Y_N,\ F,\ K_1,\ \dots,K_N)$ with $X<0$ and $Y_i>0$, $i=1,\ \dots,N$.

$$\begin{aligned} &-X+\bar{\boldsymbol{A}}^T X\bar{\boldsymbol{A}}+A_p^T XA_p \\ &\quad -Q-\sum_{j=1}^N K_j^T K_j-\bar{\boldsymbol{L}}^T\bar{\boldsymbol{R}}_\gamma^{-1}\bar{\boldsymbol{L}}=0, \end{aligned} \quad (17a)$$

$$F=-\bar{\boldsymbol{R}}_\gamma^{-1}\bar{\boldsymbol{L}}, \quad (17b)$$

$$\begin{aligned} &-Y_i+\bar{\boldsymbol{A}}_{-i}^T Y_i\bar{\boldsymbol{A}}_{-i}+(A_p+B_pF)^T Y_i(A_p+B_pF) \\ &\quad +C_i^T C_i-\hat{\boldsymbol{L}}_{-i}^T\hat{\boldsymbol{R}}_i^{-1}\hat{\boldsymbol{L}}_{-i}=0, \end{aligned} \quad (17c)$$

$$K_i=-\hat{\boldsymbol{R}}_i^{-1}\hat{\boldsymbol{L}}_{-i},\ i=1,\ \dots,N, \quad (17d)$$

where $Q:=C^TC$, $\bar{\boldsymbol{A}}:=A+\sum_{j=1}^N B_jK_j$, $\bar{\boldsymbol{L}}:=B^TX\bar{\boldsymbol{A}}+B_p^TXA_p$, $\bar{\boldsymbol{R}}_\gamma:=\gamma I_{n_v}+B^TXB+B_p^TXB_p$, $\bar{\boldsymbol{A}}_{-i}:=A+BF+\sum_{j=1,\ j\neq i}^N B_jK_j$ $\hat{\boldsymbol{L}}_{-i}:=B^TY_i\bar{\boldsymbol{A}}_{-i}$ and $\hat{\boldsymbol{R}}_i:=I_{m_i}+B_i^TY_iB_i$.

Define the set $(u_1^*,\ \dots,u_N^*)$ by

$$u_i^*(k):=K_i^*x(k)=-\hat{\boldsymbol{R}}_i^{-1}\hat{\boldsymbol{L}}_{-i}x(k),\ i=1,\ ..,N. \quad (18)$$

Then, this strategy set denotes the finite horizon H_2/H_∞ control.

Proof: Set $Z=-X$ and the equation (17a) yields:

$$\begin{aligned} &-Z+\bar{\boldsymbol{A}}^T Z\bar{\boldsymbol{A}}+A_p^T ZA_p+Q+\sum_{j=1}^N K_j^T K_j \\ &\quad +\check{\boldsymbol{L}}^T\check{\boldsymbol{R}}_\gamma^{-1}\check{\boldsymbol{L}}=0, \end{aligned} \quad (19)$$

where $\check{\boldsymbol{L}}=B^TZ\bar{\boldsymbol{A}}+B_p^TZA_p=-\bar{\boldsymbol{L}}$, $\check{\boldsymbol{R}}_\gamma=\gamma^2I_{n_v}-B^TZB-B_p^TZB_p>0$.

We rewrite (19) in the form

$$-Z+\bar{\boldsymbol{A}}^T Z\bar{\boldsymbol{A}}+A_p^T ZA_p+\boldsymbol{C}^T\boldsymbol{C}+C_p^TC_p=0, \quad (20)$$

where $\boldsymbol{C}=\begin{pmatrix} U \\ \frac{1}{\sqrt{2}}\check{\boldsymbol{R}}_\gamma^{\frac{1}{2}}\check{F} \\ O_{n_v\times n}\end{pmatrix}$, $C_p=\begin{pmatrix} O_{\rho\times n} \\ O_{n_v\times n} \\ \frac{1}{\sqrt{2}}\check{\boldsymbol{R}}_\gamma^{\frac{1}{2}}\check{F}\end{pmatrix}$ where $\rho=\mathbf{range}(Q+\sum_{j=1}^N K_j^TK_j)$, $U\in\Re^{\rho\times n}$ is obtained from the factorization $U^TU=Q+\sum_{j=1}^N K_j^TK_j$ and $\check{F}=\check{\boldsymbol{R}}_\gamma^{-1}\check{\boldsymbol{L}}=-\bar{\boldsymbol{R}}_\gamma^{-1}\bar{\boldsymbol{L}}=F$.

Let us prove that under the considered assumptions the system $(\bar{\boldsymbol{A}},\ A_p\mid\boldsymbol{C},\ C_p)$ is detectable. To this end, we take $H=(O_{n\times\rho}\quad\sqrt{2}B\check{\boldsymbol{R}}_\gamma^{-\frac{1}{2}}\quad\sqrt{2}B_p\check{\boldsymbol{R}}_\gamma^{-\frac{1}{2}})$ and we obtain $\bar{\boldsymbol{A}}+H\boldsymbol{C}=\bar{\boldsymbol{A}}+BF$ and $A_p+HC_p=A_p+B_pF$. So, we may conclude that $(\bar{\boldsymbol{A}}+H\boldsymbol{C},A_p+HC_p)$ is stable which shows that $(\bar{\boldsymbol{A}},\ A_p\mid\boldsymbol{C},\ C_p)$ is detectable. Applying the implication (ii)→(i) from Proposition 1 to the equation (20) we deduce that the pair $(\bar{\boldsymbol{A}},\ A_p)$ is stable. Thus we have proved that the strategies $u_i^*(k)$ introduced by (18) achieved the internal stability of the system obtained from (11). This means that the first task of mixed H_2/H_∞ control problem under consideration is fulfilled.

Using again the stability of the pair $(\bar{\boldsymbol{A}}+BF,\ A_p+B_pF)$ we deduce via (17a) that $P=X$ is the stabilizing solution of the Riccati equation of type (9) associated to the system (11) where $u_i(k)$ are replaced by $u_i^*(k)$.

Now, let us consider the following problem in which the cost function (21) is minimal at $K_i=K_i^*$.

$$\phi(F):=\sup_{v(k)\in l_w^2(\mathbf{N},\ \Re^{n_v})}\sum_{k=0}^\infty\mathbf{E}[\gamma^2\|v(k)\|^2-\|\hat{z}(k)\|^2], \quad (21)$$

$$\hat{z}(k)=\bar{\boldsymbol{C}}x(k)=\begin{bmatrix} C^T & (D_1K_1^*)^T & \cdots & (D_NK_N^*)^T\end{bmatrix}^Tx(k),$$

where $x(k)$ follows from

$$\begin{aligned} x(k+1)=&\left(A+\sum_{j=1}^N B_jK_j^*\right)x(k)+Bv(k) \\ &+\Big[A_px(k)+B_pv(k)\Big]w(k),\ x(0)=x^0. \end{aligned} \quad (22)$$

Note that the function ϕ coincides with function J_0 in Lemma 2. Applying Lemma 2 to this optimization problem as $X\Rightarrow P$, $\bar{\boldsymbol{A}}\Rightarrow A$ and $\bar{\boldsymbol{C}}\Rightarrow C$, yields the fact that the function ϕ is minimal at

$$F_\gamma^*=-\boldsymbol{R}_\gamma^{-1}\boldsymbol{L}\Rightarrow F^*=-\bar{\boldsymbol{R}}_\gamma^{-1}\bar{\boldsymbol{L}}. \quad (23)$$

On the other hand, consider the following LQ problem.

$$\psi(K_i):=\min_{u_i(k)\in l_w^2(\mathbf{N},\ \Re^{m_i})}\sum_{k=0}^\infty\mathbf{E}\|z_i(k)\|^2 \quad (24)$$

and $x(k)$ follows from

$$\begin{aligned} x(k+1)=&\left(A+BF^*+\sum_{j=1,\ j\neq i}^N B_jK_j^*\right)x(k)+B_iu_i(k) \\ &+[A_p+B_pF^*]x(k)w(k),\ x(0)=x^0. \end{aligned} \quad (25)$$

The function ψ coincides with function J_i in Lemma 1. Applying Lemma 1 to this optimization problem as $Y_i\Rightarrow P$, $\bar{\boldsymbol{A}}_{-i}\Rightarrow A$ and $A_p+B_pF^*\Rightarrow A_p$ yields the fact that the function ψ is minimal at

$$K^*=-\boldsymbol{R}^{-1}\boldsymbol{L}\Rightarrow K_i^*=-\hat{\boldsymbol{R}}_i^{-1}\hat{\boldsymbol{L}}. \quad (26)$$

So $(u_1^*,\ \dots,u_N^*,\ v^*)$ solve the finite horizon H_2/H_∞ control problem of stochastic system (11). ■

C. NUMERICAL ALGORITHM VIA SDP

The iterative procedure for solving problems is considered. The algorithm is given below.

Step 1. As the initialization procedure, solve the following two SDPs independently.

$$\text{maximize}\ \ \mathbf{Tr}\ [Y_i^{(0)}], \tag{27a}$$

$$\text{subject to} \begin{bmatrix} \Phi^{(0)} & \hat{\boldsymbol{L}}_{-i}^{(0)T} \\ \hat{\boldsymbol{L}}_{-i}^{(0)} & \hat{\boldsymbol{R}}_i^{(0)} \end{bmatrix} \geq 0,\ i = 1,\ \dots, N, \tag{27b}$$

where $\Phi^{(0)} := -Y_i^{(0)} + A^T Y_i^{(0)} A + A_p^T Y_i^{(0)} A_p + C_i^T C_i$, $\hat{\boldsymbol{L}}_{-i}^{(0)} := B_i^T Y_i^{(0)} A$ and $\hat{\boldsymbol{R}}^{(0)} := I_{m_i} + B_i^T Y_i^{(0)} B_i$.

$$\text{maximize}\ \ \mathbf{Tr}\ [X^{(0)}], \tag{28a}$$

$$\text{subject to} \begin{bmatrix} \Psi^{(0)} & \bar{\boldsymbol{L}}^{(0)T} \\ \bar{\boldsymbol{L}}^{(0)} & \bar{\boldsymbol{R}}_\gamma^{(0)} \end{bmatrix} \geq 0, \tag{28b}$$

where $\Psi^{(0)} := -X^{(0)} + A^T X^{(0)} A + A_p^T X^{(0)} A_p - Q$, $\bar{\boldsymbol{L}}^{(0)} := B^T X^{(0)} A + B_p^T X^{(0)} A_p$ and $\bar{\boldsymbol{R}}_\gamma^{(0)} := \gamma I_{n_v} + B^T X^{(0)} B + B_p^T X^{(0)} B_p$.

Step 2. Set $K_i^{(0)} = -[\hat{\boldsymbol{R}}^{(0)}]^{-1} \hat{\boldsymbol{L}}_{-i}^{(0)}$ and $X^{(0)} := -[\bar{\boldsymbol{R}}_\gamma^{(0)}]^{-1} \bar{\boldsymbol{L}}^{(0)}$.

Step 3. Solve the following two SDPs independently.

$$\text{maximize}\ \ \mathbf{Tr}\ [Y_i^{(k+1)}], \tag{29a}$$

$$\text{subject to} \begin{bmatrix} \Phi^{(k)} & \hat{\boldsymbol{L}}_{-i}^{(k)T} \\ \hat{\boldsymbol{L}}_{-i}^{(k)} & \hat{\boldsymbol{R}}_i^{(k)} \end{bmatrix} \geq 0,\ i = 1,\ \dots, N, \tag{29b}$$

where $\Phi^{(k)} := -Y_i^{(k+1)} + \bar{\boldsymbol{A}}_{-i}^{(k)T} Y_i^{(k+1)} \bar{\boldsymbol{A}}_{-i}^{(k)} + (A_p + B_p F^{(k)})^T Y_i^{(k+1)} (A_p + B_p F^{(k)}) + C_i^T C_i$, $\bar{\boldsymbol{A}}_{-i}^{(k)} := A + BF^{(k)} + \sum_{j=1,\ j \neq i}^N B_j K_j^{(k)}$ $\hat{\boldsymbol{L}}_{-i}^{(k)} := B^T Y_i^{(k+1)} \bar{\boldsymbol{A}}_{-i}^{(k)}$ and $\hat{\boldsymbol{R}}^{(k)} := I_{m_i} + B_i^T Y_i^{(k+1)} B_i$.

$$\text{maximize}\ \ \mathbf{Tr}\ [X^{(k)}], \tag{30a}$$

$$\text{subject to} \begin{bmatrix} \Psi^{(k)} & \bar{\boldsymbol{L}}^{(k)T} \\ \bar{\boldsymbol{L}}^{(k)} & \bar{\boldsymbol{R}}_\gamma^{(k)} \end{bmatrix} \geq 0, \tag{30b}$$

where $\Psi^{(k)} := -X^{(k+1)} + \bar{\boldsymbol{A}}^{(k)T} X^{(k+1)} \bar{\boldsymbol{A}}^{(k)} + A_p^T X^{(k+1)} A_p - Q$, $\bar{\boldsymbol{A}}^{(k)} := A + \sum_{j=1}^N B_j K_j^{(k)}$, $\bar{\boldsymbol{L}}^{(k)} := B^T X^{(k+1)} \left(A + \sum_{j=1}^N B_j K_j^{(k)}\right) + B_p^T X^{(k+1)} A_p$, $\bar{\boldsymbol{R}}_\gamma^{(k)} := \gamma I_{n_v} + B^T X^{(k+1)} B + B_p^T X^{(k+1)} B_p$.

Step 4. Set $K_i^{(k+1)}$ as follows.

$$K_i^{(k+1)} = -[\hat{\boldsymbol{R}}^{(k)}]^{-1} \hat{\boldsymbol{L}}_{-i}^{(k)},\ F^{(k+1)} = -[\bar{\boldsymbol{R}}_\gamma^{(k)}]^{-1} \bar{\boldsymbol{L}}^{(k)}. \tag{31}$$

Step 5. If the algorithm converges, then $X^{(k)} \to X$, $Y_i^{(k)} \to Y_i$ as $k \to \infty$, where Y_i is the solution of CSAREs (17c), STOP. That is, stop if any norm of the error of difference between $Y_i^{(k)}$ and Y_i is less than a pre-specified precision. Otherwise, increment $k \to k+1$ and go to Step 3. If the algorithm does not converge, declare the algorithm fails.

It should be noted that convergence of the above algorithm cannot be guaranteed. Particularly, it may be noted that when A_p matrices with a certain structure, and/or small problems are considered, the convergence will be attained. However, we found the proposed algorithm to work well in practice.

IV. PARETO/H_∞ STRATEGY

In this section, we consider the cooperative game theory and we modify the formulation introduced in the previous section to make it compatible with our specific problem. Assume a team of N players with the stochastic system described in (11a). Each player wants to optimize its own cost described in (15). As the definition of Pareto efficient solution [11], let us combine the individual cost functions in (15) into a team cost function according to the following.

$$J(u_1,\ \dots, u_N,\ v^*) := \sum_{k=0}^{\infty} \sum_{j=1}^{N} \rho_j \mathbf{E}[\|z_j(k)\|^2]$$
$$= \sum_{k=0}^{\infty} \sum_{j=1}^{N} \rho_j \mathbf{E}[\|C_i x(k)\|^2 + \|u_i(k)\|^2],$$
$$\sum_{j=1}^{N} \rho_j = 1,\ 0 < \rho_i < 1,\ i = 1,\ \dots, N. \tag{32}$$

A Pareto solution is a set $(u_1,\ \dots, u_N)$, which minimizes $J(u_1,\ \dots, u_N,\ v^*)$. From the above problem, we obtain the following necessary optimality conditions.

Theorem 2: For the discrete-time stochastic perturbed systems (11), suppose the following CSAREs have solutions $(X^*,\ Y^*,\ F^*,\ M_1^*,\ \dots, M_N^*)$ with $X^* < 0$ and $Y^* > 0$, $i = 1,\ \dots, N$.

$$-X + \tilde{\boldsymbol{A}}^T X \tilde{\boldsymbol{A}} + A_p^T X A_p$$
$$-Q - \sum_{j=1}^{N} M_j^T M_j - \tilde{\boldsymbol{L}}^T \tilde{\boldsymbol{R}}_\gamma^{-1} \tilde{\boldsymbol{L}} = 0, \tag{33a}$$

$$F = -\tilde{\boldsymbol{R}}_\gamma^{-1} \tilde{\boldsymbol{L}}, \tag{33b}$$

$$\boldsymbol{M}(Y,\ F,\ M_1,\ \dots, M_N)$$
$$:= -Y + \tilde{\boldsymbol{A}}_F^T Y \tilde{\boldsymbol{A}}_F + (A_p + B_p F)^T Y (A_p + B_p F)$$
$$+ \sum_{j=1}^{N} \rho_j [C_j^T C_j + M_j^T M_j] = 0, \tag{33c}$$

$$M_i = -\tilde{\boldsymbol{R}}_i^{-1} \tilde{\boldsymbol{L}}_{-i},\ i = 1,\ \dots, N, \tag{33d}$$

having the additional properties $\tilde{\boldsymbol{R}}_\gamma > 0$ and the pair $(\tilde{\boldsymbol{A}} + BF,\ A_p + B_p F)$ is stable, where $\tilde{\boldsymbol{A}} := A + \sum_{j=1}^N B_j M_j$, $\tilde{\boldsymbol{L}} := B^T X \tilde{\boldsymbol{A}} + B_p^T X A_p$, $\tilde{\boldsymbol{R}}_\gamma := \gamma I_{n_v} + B^T X B + B_p^T X B_p$, $\tilde{\boldsymbol{A}}_F := A + BF + \sum_{j=1}^N B_j M_j$, $\tilde{\boldsymbol{A}}_{-i} := A + BF + \sum_{j=1,\ j \neq i}^N B_j M_j$ $\tilde{\boldsymbol{L}}_{-i} := B_i^T Y \tilde{\boldsymbol{A}}_{-i}$ and $\tilde{\boldsymbol{R}}_i := \rho_i I_{m_i} + B_i^T Y B_i$.

Define the set $(u_1^*,\ \dots, u_N^*)$ by

$$u_i^*(k) := M_i^* x(k) = -\tilde{\boldsymbol{R}}_i^{-1} \tilde{\boldsymbol{L}}_{-i} x(k),\ i = 1,\ ..,N. \tag{34}$$

Then, this strategy set denotes the infinite horizon H_2/Pareto optimal strategy. Moreover, the minimal value is $x^T(0) Y^* x(0)$.

Proof: Proceeding as in the proof of Theorem 1, one can show that the pair $(\tilde{\boldsymbol{A}},\ A_p)$ is stable. That is that the strategies (34) internally stabilizes the system (11). Let us consider the following closed-loop stochastic system

$$x(k+1)=\tilde{\boldsymbol{A}}x(k)+Bv(k)+\left[A_px(k)+B_pv(k)\right]w(k), \quad (35a)$$

$$\tilde{z}(k)=\left[\begin{array}{cccc} C^T & (D_1M_1^*)^T & \cdots & (D_NM_N^*)^T \end{array}\right]^T x(k). \quad (35b)$$

Hence, by using Lemma 2 equations (35a) and (35b) are established.

On the other hand, let us consider the following optimization problem under the Pareto strategy.

$$\text{minimize} \quad J(M_1x, \ \ldots, M_Nx, \ v^*) \quad (36)$$

such that

$$x(k+1)=\tilde{\boldsymbol{A}}_{-i}x(k)+B_iu(k)+\left[A_p+B_pF\right]x(k)w(k). \quad (37)$$

Let us consider the Lagrangian $\mathcal{L}$

$$\mathcal{L}(Y, \ F, \ M_1, \ \ldots, M_N) = \mathbf{Tr}[Y]+\mathbf{Tr}[\boldsymbol{M}(Y, \ F, \ M_1, \ \ldots, M_N)G], \quad (38)$$

where G is a symmetric positive definite matrix of Lagrange multipliers. Using the condition of the Lagrange multipliers approach, the necessary conditions for M_i to be optimal can be found by setting $\partial\mathcal{L}/M_i$ to zero. By using $G>0$, the resulting equations (33d) can be obtained simultaneously for M_i. ■

V. SOFT-CONSTRAINED STOCHASTIC NASH GAMES

In this section, the soft-constrained stochastic Nash games for a class of discrete-time system with state- and disturbance-dependent noise are discussed.

A. ONE-PLAYER CASE

First, a one-player case is considered. The result obtained for that particular case will be used as the basis for the derivation of the results for the general N-player case.

Consider a linear time-invariant stochastic stabilizable system

$$x(k+1) = Ax(k)+Bv(k)+B_1u(k) +A_px(k)w(k), \ x(0)=x^0, \quad (39a)$$

$$z(k) = \left[\begin{array}{c} Cx(k) \\ Du(k) \end{array}\right], \ D^TD=I_m, \ C^TC=Q. \quad (39b)$$

The cost function is given below.

$$J(u, \ v) := \sum_{k=0}^{\infty}\mathbf{E}[\|z(k)\|^2-\|v(k)\|^2]. \quad (40)$$

Definition 3: [8] A strategy pair $(u^*, \ v^*)\in\Gamma_u\times\Gamma_v$ is in saddle-point equilibrium if

$$J(u^*, \ v)\le J(u^*, \ v^*)\le J(u, \ v^*) \quad (41)$$

for all $(u^*, \ v)\in\Gamma_u\times\Gamma_v$ and $(u, \ v^*)\in\Gamma_u\times\Gamma_v$, where $\Gamma_u\times\Gamma_v$ means a product vector space.

The following theorem generalizes the existing results of [9], [10], [12], which is a very important result in deterministic soft-constrained Nash games, to a discrete version.

Theorem 3: Assume that for any $u(k)$ and $v(k)$, the closed-loop system is asymptotically mean-square stable. Suppose that the SARE has the solution $P^*\ge 0$.

$$P = Q+A_p^TPA_p+A^TP\Lambda^{-1}A, \quad (42)$$

where $\Lambda := I_n+(B_1B_1^T-BB^T)P$ and $I_n-BPB^T>0$.

The strategy pair

$$u^*(k) = -B_1^TP\Lambda^{-1}Ax(k), \quad (43a)$$

$$v^*(k) = B^TP\Lambda^{-1}Ax(k) \quad (43b)$$

is in saddle-point equilibrium if it is asymptotically mean-square stable. That is, inequality (37) related to the cost function $J(u, \ v)$ is satisfied. Moreover, $J(u^*, \ v^*) = x^T(0)P^*x(0)$.

Proof: Since this can be proved as an extension of the existing results [8], it is omitted. ■

Theorem 4: Consider the infinite-horizon discrete-time soft-constrained stochastic Nash games. Then we have the following.

(i) The game has equal upper and lower value if, and only if, the SARE admits a positive definite solution satisfying $I_n-BPB^T>0$.

(ii) If the SARE admits a positive definite solution satisfying $I_n-BPB^T>0$, then it admits a minimal solution. Then, the finite value of the game is $\mathbf{E}[x^T(0)Px(0)]$.

(iii) The upper value of the game is finite if, and only if, the upper and lower values are equal.

(iv) If $P>0$ exists, the closed-loop stochastic system with strategies (43) is mean square stable.

(v) The following feedback matrix is Hurwitz:

$$\boldsymbol{A}_\beta := [I_n+(B_1B_1^T-BB^T)P\Lambda^{-1}]A. \quad (44)$$

Proof: Parts (i)–(v) follow from the existing results in [8] by extending to stochastic case. ■

It is noteworthy that in this study, the strategies $u_i^*(k)$ are restricted as linear feedback strategies such as $u_i(k) := G_ix(k)$. We consider the formulation of the objective functions of the players in order to express a desire for robustness.

$$\bar{J}_i(u_1, \ \ldots, u_N) := \sup_{v(k)\in l_w^2(\mathbf{N}, \ \Re^{n_v})} J_i(u_1, \ \ldots, u_N, \ v), \quad (45)$$

where $x(k)$ satisfies the stochastic system (11a) with $B_p\equiv 0$ and

$$J_i(u_1, \ \ldots, u_N, \ v) = \sum_{k=0}^{\infty}\mathbf{E}[x^T(k)C_i^TC_ix(k)+u_i^T(k)u_i(k)-v^T(k)v(k)].$$

Definition 4: [9], [10], [12] The strategy set $(u_1^*, \ \ldots, u_N^*)$, $u_i^*(k) := G_i^*x(k)$ is a soft-constrained stochastic Nash equilibrium strategy set if for each $i=1, \ \ldots, N$, the following inequality holds:

$$\bar{J}_i(u_1^*, \ \ldots, u_N^*) \le \bar{J}_i(u_1^*, \ \ldots, u_{i-1}^*, \ u_i, \ u_{i+1}^*, \ \ldots, u_N^*), \quad (46)$$

for all $x(0)$.

Theorem 5: Assume that for all $u_i(k)$, $i = 1, \dots, N$ and $v(k)$, the closed-loop system is asymptotically mean-square stable. Suppose that N real symmetric matrices $P_i^* \geq 0$ and N real matrices G_i^* exist such that

$$P_i = Q_i + A_p^T P_i A_p + \hat{\boldsymbol{A}}_{-i}^T P_i \Lambda_i^{-1} \hat{\boldsymbol{A}}_{-i}, \quad (47a)$$

$$u_i^*(k) = G_i^* x(k) = -B_i^T P_i \Lambda_i^{-1} \hat{\boldsymbol{A}}_{-i} x(k), \quad (47b)$$

$$v^*(k) = B^T P_i \Lambda_i^{-1} \hat{\boldsymbol{A}}_{-i} x(k), \ i = 1, \dots, N, \quad (47c)$$

where $Q_i := C_i^T C_i$, $\Lambda_i := I_n + (B_i B_i^T - BB^T)P_i$, $\hat{\boldsymbol{A}}_{-i} := A + \sum_{j=1, \ j \neq i}^{N} B_j G_j$.

Then, $(G_1^*, \dots, G_N^*)$, and this strategy set denotes the soft-constrained stochastic Nash equilibrium. Furthermore, $\bar{J}_i(G_1^* x, \dots, G_N^* x) = x^T(0) P_i^* x(0)$.

Proof: Now, let us consider the following problem in which the cost function (45) is minimal at $u(k) = Gx(k) = G^* x(k)$.

$$\phi(G) := \sup_{v(k) \in l_w^2(\mathbf{N}, \Re^{n_v})} \hat{J}(u, \ v), \quad (48)$$

where

$$\hat{J}(u, \ v) = J_i(u_1^*, \dots, u_{i-1}^*, \ u, \ u_{i+1}^*, \ u_N^*, \ v)$$
$$= \sum_{k=0}^{\infty} \mathbf{E}[x^T(k) Q_i x(k) + u^T(k) u(k) - v^T(k) v(k)]$$

and $x(k)$ follows from

$$x(k+1) = \hat{\boldsymbol{A}}_{-i} x(k) + Bv(k) + \hat{B} u(k) + A_p x(k) w(k), \ x(0) = x^0. \quad (49)$$

Note that the function ϕ coincides with function J in Theorem 3. Applying Theorem 3 to this minimization problem as $P_i \Rightarrow P$, $\hat{\boldsymbol{A}}_{-i} \Rightarrow A$, $\hat{B} \Rightarrow B_1$ and $x^T(k) Q_i x(k) \Rightarrow \|z(k)\|$ yields the fact that the function ϕ is minimal at $G^* \Rightarrow G_i^*$. Moreover, the minimal value is $x^T(0) P_i x(0)$. ■

VI. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of our proposed three strategies, we have run a simple numerical example. The system matrices are given as follows.

$$A = \begin{bmatrix} 0 & 1 \\ 0.1 & -0.5 \end{bmatrix}, \ A_p = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}, \ B_p = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \ B = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}.$$

By solving the corresponding CSAREs (17), (33) and (54), we obtain the linear state feedback strategies.

H_2/H_∞ Strategies

$$K_1 = \begin{bmatrix} -1.1201e-02 & -3.5876e-02 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -1.2046e-02 & -3.9329e-02 \end{bmatrix}.$$

Pareto$/H_\infty$ Strategies

$$M_1 = \begin{bmatrix} -1.2911e-02 & -4.5625e-02 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} -1.1696e-02 & -4.1329e-02 \end{bmatrix}.$$

Soft − Constrained Nash Strategies

$$G_1 = \begin{bmatrix} 1.0879e-02 & -3.0387e-02 \end{bmatrix},$$
$$G_2 = \begin{bmatrix} -6.2526e-02 & -2.0916e-01 \end{bmatrix}.$$

It is easy to verify that these strategies satisfy the multi-objective control purpose, respectively. Indeed, although the simulation results are omitted briefly due to the pages limitations, the control performance was monitored by measuring the state trajectories.

VII. CONCLUSIONS

The three types of control problems involving multiple decision makers for the discrete-time linear stochastic system with state- and disturbance-dependent noise have been studied. First, as the extension of the existing H_2/H_∞ control problem, H_2/H_∞ control problem with multiple decision makers is considered. Second, in order to improve the transient response, the linear quadratic control under the Pareto solution is investigated. Finally, the soft-constrained stochastic Nash games are formulated in which robustness is attained against disturbance input. It is worth pointing out that all the proposed control strategies are established based on the solutions of the cross-coupled stochastic algebraic Riccati equations (CSAREs). The new algorithms based on linear matrix inequality (LMI) have been developed to solve the CSAREs. A numerical example has been addressed to demonstrate the validity of the proposed control strategies.

REFERENCES

[1] W. Zhang, Y. Huang and H. Zhang, Stochastic H_2/H_∞ Control for Discrete-Time Systems with State and Disturbance Dependent Noise, *Automatica*, vol. 43, no.3, 2007, pp 513-521.

[2] W. Zhang, Y. Huang and L. Xie, Infinite Horizon Stochastic H_2/H_∞ Control for Discrete-Time Systems with State and Disturbance Dependent Noise, *Automatica*, vol. 44, no. 9, 2008, pp 2306-2316.

[3] O.L.V. Costa and R.P. Marques, Mixed H_2/H_∞ Control of Discrete-Time Markovian Jump Linear Systems, *IEEE Trans. Automatic Control*, vol.43, no.1, 1998, pp 95-100.

[4] R. Muradore and G. Picci, Mixed H_2/H_∞ Control: The Discrete-Time Case, *Systems and Control Letters*, vol.54, no.1, 2005, pp 1-13.

[5] A. Tahmasebi and X. Chen, "Discrete Time Multi-Objective H_2/H_∞ Control," *Proc. 49th Conf. Decision and Control*, pp 3162-3167, Atlanta, Dec. 2010.

[6] Y. Huang, W. Zhang and H. Zhang, "Infinite Horizon LQ Optimal Control for Discrete-Time Stochastic Systems," *Proc. 6th world congress on control and automation*, pp 252-256, Dalian, June 2006.

[7] Oswaldo L. V. Costa and Wanderlei L. de Paulo, Generalized Coupled Algebraic Riccati Equations for Discrete-Time Markov Jump with Multiplicative Noise systems, *European J. Control*, vol. 14, no. 5, 2008, pp 391-408.

[8] T. Basar and P. Bernhard, *H_∞ Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, Boston: Birkhaüser, 1995.

[9] W.A.V.D. Broek, J.C. Engwerda and J.M. Schumacher, Robust Equilibria in Indefinite Linear-Quadratic Differential Games, *J. Optimization Theory and Applications*, vol. 119, no. 3, 2003, pp 565-595.

[10] J.C. Engwerda, A Numerical Algorithm to Find Soft-Constrained Nash Equilibria in Scalar LQ-Games, *Int. J. Control*, vol. 79, no. 6, 2006, pp 592-603.

[11] J.C. Engwerda, *LQ Dynamic Optimization and Differential Games*, Chichester: John Wiley & Sons, 2005.

[12] H. Mukaidani, Soft-Constrained Stochastic Nash Games for Weakly Coupled Large-Scale Systems, *Automatica*, vol. 45, no. 5, 2009, pp 1272-1279.

[13] V. Dragan, T. Morozan, A.M. Stoica, *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems, New York: Springer, 2010.*