Distributed Circular Formation Stabilization of Unicycles
Part II: Arbitrary Information Flow Graph

Mohamed I. El-Hawwary and Manfredi Maggiore

Abstract—In part I of this paper we presented a solution
to the circular formation stabilization problem of kinematic
unicycles when the information flow graph is undirected. This
paper extends the results of part I in two directions. First,
we present a control law that solves the circular formation
stabilization problem when the information flow is described
by an arbitrary directed graph with a globally reachable node.
Second, we generalize our results to the case when the unicycles
are dynamic.

I. Introduction

This paper is a continuation of part I, where we designed
a control law solving the circular formation control problem
(CFCP) for a group of n unicycles under the assumption that
the information flow graph is undirected, and it has a globally
reachable node. In this paper, we provide two extensions.
First, we develop a solution to CFCP for arbitrary directed
information flow graphs with a globally reachable node.
Second, we show that all our results can be straightforwardly
extended to the setting of dynamic unicycles.

Consider again the system of n kinematic unicycles, $n \geq
2$, where the i's unicycle model is given by

$$\begin{aligned}
\dot{x}_1^i &= u_1^i \cos x_3^i \\
\dot{x}_2^i &= u_1^i \sin x_3^i \\
\dot{x}_3^i &= u_2^i
\end{aligned} \tag{1}$$

with state $x^i = (x_1^i, x_2^i, x_3^i) \in \mathbb{R}^2 \times S^1$. The state space of the
system is $\mathcal{X} = (\mathbb{R}^2 \times S^1)^n$, and we let $\chi = \mathrm{col}(x^1, \cdots, x^n)$,
and $x_3 = \mathrm{col}(x_3^1, \cdots, x_3^n)$. This system can be written in the
control affine form $\dot{\chi} = g(\chi)u$.

We refer the reader to part I of this paper for the definitions
of the information flow digraph $\mathcal{G}$ and of the notion of
globally reachable node. We recall the problem we wish to
solve.

Circular Formation Control Problem (CFCP). Consider
the n-unicycles in (1). For a given information flow digraph
$\mathcal{G}$ with a globally reachable node, and a desired formation
specification—expressed by a vector of angles $\alpha \in S^n$, design
a distributed control law which asymptotically stabilizes the
set

$$\begin{aligned}
\Gamma &= \Gamma_1 \cap \Gamma_2 \\
&= \{\chi : L(x_3 - \alpha) = 0, c^{i+1}(x^{i+1}) = c^i(x^i), 1 \leq i \leq n\},
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
\Gamma_1 &= \{\chi : c^{i+1}(x^{i+1}) = c^i(x^i), i \in \{1, \ldots, n\}\} \\
\Gamma_2 &= \{\chi : L(x_3 - \alpha) = 0 \mod 2\pi\},
\end{aligned} \tag{3}$$

and $c^i(x^i)$ is defined as

$$c^i(x^i) = (x_1^i - r \sin x_3^i, x_2^i + r \cos x_3^i) \tag{4}$$

Additionally, the linear velocities u_1^i and angular velocities
u_2^i of the unicycles should be bounded away from zero on Γ
and the unicycles should have a *common asymptotic centre
of rotation*.

In this paper, as in part I, the indices $i \in \{1, \ldots, n\}$ are
evaluated modulo n so that, for instance, $n + 1$ is identified
with 1.

We derive the solution to this problem for arbitrary infor-
mation flow graphs in two stages: in Section II, we address
the case of circulant graphs (i.e., graphs whose Laplacian is
a circulant matrix), and in Section III we further generalize
the result to general digraphs. Moreover, in Section IV we
show that all our results have a straightforward extension to
the case of dynamic unicycles.

The results of this paper do not rely on the passivity theory
reviewed in part I. A more general framework is needed when
the information flow graph is directed, as we now explain.
Recall the storage function used in part I,

$$V(\chi) = c(\chi)^\top L_{(2)} c(\chi) = \frac{1}{2} c(\chi)^\top (L_{(2)} + L_{(2)}^\top) c(\chi),$$

where L is the Laplacian of $\mathcal{G}$, and $L_{(2)} = L \otimes I_2$.
We have established in part I that since $\mathcal{G}$ has a globally
reachable node, L has one eigenvalue at zero, and all its
other eigenvalues have positive real part. This fact, however,
does not imply that $L_{(2)} + L_{(2)}^\top$ is positive semidefinite
when L, and hence $L_{(2)}$, is not symmetric. If $L_{(2)} + L_{(2)}^\top$
is not positive semidefinite, the passivity analysis of part I
cannot be applied. A second obstacle to the extension of the
passivity-based design of part I is the fact that, even if $V(\chi)$
is positive semidefinite and $V^{-1}(0) = \Gamma_1$, the passive output
associated to the storage V,

$$h(\chi) = -r\mathcal{R}(x_3)(L_{(2)} + L_{(2)}^\top) c(\chi)$$

violates the information flow constraint, and thus it cannot be
used in a passivity-based feedback to generate a distributed
controller. To illustrate, consider the feedback transforma-
tion (12) of part I:

$$u = \beta_1 \bar{u} + \beta_2 \tilde{u} \tag{5}$$

where $\beta_1 = \text{blockdiag}\{[1 \ 1/r]^\top, \ldots, [1 \ 1/r]^\top\}$, $\beta_2 = \text{blockdiag}\{[0 \ 1]^\top, \ldots, [0 \ 1]^\top\}$, $\bar{u}$ is a feedback used to enforce Γ-detectability, and $\tilde{u}$ is a PBF stabilizing the set Γ_1 in (3). The simplest PBF, $\tilde{u}(\chi) = -h(\chi)$, has the property that $\tilde{u}^i$ is affected by rows $2i-1$ and $2i$ of $(L_{(2)} + L_{(2)}^\top)$. These rows are different from the corresponding rows of $L_{(2)}$, unless L is symmetric, and therefore $\tilde{u}(\chi)$ violates the information flow constraint embodied in L.

The considerations above suggest that in order to generate distributed control laws solving CFCP, we should replace the PBF $\tilde{u} = -Kh(\chi)$ in equation (21) of part I by a suitable distributed feedback that asymptotically stabilizes the set Γ_1 in (3). Recalling that $\bar{u}$ in (19) of part I was designed to stabilize Γ relative to Γ_1, the problem then is to understand whether the asymptotic stability of Γ relative to Γ_1 and the asymptotic stability of Γ_1 imply the asymptotic stability of Γ. This question is an instance of the general *reduction problem* for a dynamical system $\Sigma : \dot{x} = f(x)$: *Consider two closed sets Γ and Γ_1, with $\Gamma \subset \Gamma_1$, which are positively invariant for Σ; suppose that Γ is stable, attractive, or asymptotically stable relative to Γ_1. When is it that Γ is, respectively, stable, attractive, or asymptotically stable for Σ?*

The answer to this question is contained in the next result, taken from [1] (see [2] for the full version of the paper).

Theorem I.1 (Reduction principle for asymptotic stability, Theorem III.2 in [1])**.** *Let Γ and Γ_1, $\Gamma \subset \Gamma_1 \subset \mathcal{X}$, be two closed positively invariant sets. Then, Γ is [globally] asymptotically stable if the following conditions hold:*

(i) Γ is [globally] asymptotically stable relative to Γ_1,

(ii) Γ_1 is locally stable near Γ,

(iii) Γ_1 is locally attractive near Γ [Γ_1 is globally attractive],

(iv) if Γ is unbounded, then Σ is locally uniformly bounded near Γ,

(v) [all trajectories of Σ are bounded.]

Seibert and Florio in [3] proved an analogous result for the case when Γ is compact. We refer the reader to [1] for the definitions of local stability and attractivity near a set. In this context, it suffices to say that the asymptotic stability of Γ_1 implies conditions (i)-(ii) of Theorem I.1.

By invoking this reduction principle, we propose the following design strategy to solve CFCP for general information flow graphs.

Step 1. Using the feedback transformation (5) and the feedback $\bar{u}$ in equation (19) of part I, we show that Γ is asymptotically stable relative to Γ_1 when $\tilde{u} = 0$.

Step 2. We design a distributed feedback $\tilde{u}(\chi)$ which asymptotically stabilizes Γ_1 and guarantees that the closed-loop system is LUB near Γ. Moreover, we show that, with this feedback, the unicycles have a common asymptotic centre of rotation.

Step 3. By invoking the reduction theorem for asymptotic stability above, we conclude that the feedback in question solves CFCP.

II. SOLUTION OF CFCP FOR CIRCULANT DIGRAPHS

In this section we apply the reduction-based set stabilizing procedure previously outlined to solve CFCP when the information flow digraph Laplacian is circulant, i.e., it takes the form (see [4])

$$L = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \\ l_n & l_1 & \cdots & l_{n-1} \\ \vdots & \vdots & & \vdots \\ l_2 & l_3 & \cdots & l_1 \end{bmatrix}$$

In the development that follows we will need the next lemma.

Lemma II.1. *If the Laplacian L of a digraph $\mathcal{G}$ with a globally reachable node is circulant, then the matrix $L + L^\top$ is positive semidefinite with a simple eigenvalue at 0 with geometric multiplicity 1.*

Proof: If L is circulant, then $L^\top$ is also the Laplacian of a graph with the same node set, which we denote $\mathcal{G}^\top$. Therefore, $L + L^\top$ is the Laplacian of a graph with the same nodes as those of $\mathcal{G}$, and whose arcs are the arcs of $\mathcal{G}$ and those of $\mathcal{G}^\top$. Such graph, therefore, has a globally reachable node, and its Laplacian $L + L^\top$ has one eigenvalue at zero with geometric multiplicity 1 and $n-1$ positive eigenvalues. ∎

Remark. Lemma II.1 is not applicable to digraphs with non-circulant Laplacians because if L is the Laplacian of $\mathcal{G}$, in general it is not true that $L^\top$ is the Laplacian of a digraph.

Step 1: Asymptotic stabilization of Γ relative to Γ_1

Consider system (1) with feedback transformation (5). Let $\bar{u}$ be defined as in part I,

$$\bar{u}^i = v - v_1 \sin(L^i(x_3 - \alpha)), \ i = 1, \ldots, n, \quad (6)$$

and let $\tilde{u} = 0$. In the proof of Lemma V.2 of part I it was shown that the derivative of the function $W(x_3) = \sum_{i=1}^n [1 - \cos(L^i(x_3 - \alpha))]$ along solutions of the closed-loop system is given by

$$\dot{W} = -v_1 S(x_3)^\top L S(x_3) = -\frac{v_1}{2} S(x_3)^\top (L + L^\top) S(x_3),$$

where $S(x_3) = \text{col}(\sin(L^1(x_3 - \alpha)), \ldots, \sin(L^n(x_3 - \alpha)))$. Since, by Lemma II.1, $L + L^\top$ is positive semidefinite and has one eigenvalue at zero with geometric multiplicity one, the proof of Lemma V.2 in part I is applicable in this context, and it shows that the set Γ is asymptotically stable relative to Γ_1.

Step 2: Stabilization of Γ_1 and LUB property

Referring to the feedback transformation (5), let $\bar{u}$ be defined as in (6), and let

$$\tilde{u}(\chi) = K\mathcal{R}(x_3)\varphi(L_{(2)}c(\chi)),$$

where $K > 0$ and $\varphi(y) = \phi(y)y$, with $\phi : \mathbb{R}^{2n} \to (0, +\infty)$ a locally Lipschitz function such that $\sup_{y \in \mathbb{R}^{2n}} \|\phi(y)y\| <$

$v/(2Kr)$. Since

$$\dot{x}_3^i = \frac{\bar{u}^i}{r} + \tilde{u}^i$$
$$= \frac{v}{r} - \frac{v_1}{r}\sin(L^i(x_3 - \alpha)) +$$
$$K[0\ 0\ \cdots\ \cos x_3^i\ \sin x_3^i\ \cdots\ 0\ 0]\phi(L_{(2)}c(\chi))L_{(2)}c(\chi),$$

our choice of ϕ guarantees that $\|\dot{x}_3\| \geq \mu > 0$ for some $\mu > 0$. Next, the dynamics of the centres of rotation are given by

$$\dot{c} = -rK\mathcal{R}(x_3(t))^\top \tilde{u}(\chi)$$
$$= -rK\phi(L_{(2)}c(\chi))\mathcal{R}(x_3)^\top \mathcal{R}(x_3)L_{(2)}c(\chi)$$
$$= -rK\phi(L_{(2)}c(\chi))R(x_3(t))L_{(2)}c(\chi).$$

The above can be viewed as a time-varying system whose time-dependency is brought about by the signal $x_3(t)$. We use averaging theory to analize this system. Our arguments here are sketched due to space limitations. The averaged system is

$$\dot{c}_{\text{avg}} = -rK\phi(L_{(2)}c_{\text{avg}})\bar{R}L_{(2)}c_{\text{avg}}, \tag{7}$$

where $\bar{R}$ can be shown to be positive definite. Letting $P = \begin{bmatrix} \mathbf{1} & \begin{matrix} 0 \\ I_{n-1} \end{matrix} \end{bmatrix}$, using the coordinate transformations $z = P^{-1}c$, $z_{\text{avg}} = P^{-1}c_{\text{avg}}$, and partitioning $z = (\bar{z}, \tilde{z})$, $z_{\text{avg}} = (\bar{z}_{\text{avg}}, \tilde{z}_{\text{avg}})$, we obtain

$$\dot{\bar{z}} = K\phi(L_{(2)}Pz)A_{12}(t)\tilde{z} \qquad \dot{\bar{z}}_{\text{avg}} = K\phi(L_{(2)}Pz_{\text{avg}})\bar{A}_{12}\tilde{z}_{\text{avg}}$$
$$\dot{\tilde{z}} = K\phi(L_{(2)}Pz)A_{22}(t)\tilde{z} \qquad \dot{\tilde{z}}_{\text{avg}} = K\phi(L_{(2)}Pz_{\text{avg}})\bar{A}_{22}\tilde{z}_{\text{avg}},$$

where the matrix $\bar{A}_{22}$ is Hurwitz. By the definition of P, the terms $L_{(2)}Pz$ and $L_{(2)}Pz_{\text{avg}}$ are linear functions of only $\tilde{z}$ and $\tilde{z}_{\text{avg}}$, respectively. Since the real-valued function $\phi(\cdot)$ is bounded away from zero on any compact set, the origin of the $\tilde{z}_{\text{avg}}$ subsystem is exponentially stable and globally asymptotically stable. By the averaging theorem, for small enough K the linear time-varying system with matrix $KA_{22}(t)$ is globally exponentially stable. This fact implies that for small enough K the origin of the $\tilde{z}$ subsystem is exponentially stable and globally uniformly asymptotically stable. We thus have that the unicycles have a common asymptotic centre of rotation and there exists $M > 0$ such that for all $\chi(0) \in \mathcal{X}$, $\|c(\chi(t))\| \leq M\|L_{(2)}c(\chi(0))\|$, thus proving that the closed-loop system is LUB near Γ.

Step 3: Solution of CFCP

The arguments presented in the previous two steps and the reduction principle for asymptotic stability in Theorem I.1 yield the following result.

Proposition II.2. *Assume that the information flow graph has a circulant Laplacian with a globally reachable node. Let $v > v_1 > 0$ and $\phi : \mathbb{R}^{2n} \to (0, +\infty)$ be a locally Lipschitz function such that $\sup_{y \in \mathbb{R}^{2n}} \|\phi(y)y\| < \infty$. Then, there exists $K^\star > 0$ satisfying $\sup_{y \in \mathbb{R}^{2n}} \|\phi(y)y\| < v/(2K^\star r)$*

such that for all $K \in (0, K^\star)$ the feedback

$$u_1^i = v - v_1 \sin(L^i(x_3 - \alpha))$$
$$u_2^i = \frac{u_1^i}{r} + K\phi(L_{(2)}c(\chi))\Big[\cos x_3^i L_{(2)}^{2i-1}c(\chi) \tag{8}$$
$$+ \sin x_3^i L_{(2)}^{2i}c(\chi)\Big], \quad i = 1, \ldots, n$$

solves CFCP and renders the goal set Γ in (2) asymptotically stable, and Γ_1 in (3) globally asymptotically stable for the closed-loop system.

Remark. If we replace the expression for u_1^i in (8) by that in (22) of part I, and we take the state space to be $\mathcal{X} = \mathbb{R}^{3n}$, then the set Γ becomes globally asymptotically stable relative to Γ_1, and the feedback above solves CFCP globally. As pointed out in part I to show this one cannot use the same method as that of Proposition II.2 because $x_3^i(t)$ is no longer a solution on the compact set S^1.

Simulations

We present simulation results for for 6 unicycles, for the two cases presented in part I of the paper:
A. The unicycles are uniformly distributed on the circle with $\alpha = \begin{bmatrix} 0 & \frac{2\pi}{6} & \frac{4\pi}{6} & \frac{6\pi}{6} & \frac{8\pi}{6} & \frac{10\pi}{6} \end{bmatrix}^\top$.
B. The unicycles are uniformly distributed on half the circle with $\alpha = \begin{bmatrix} 0 & \frac{2\pi}{10} & \frac{4\pi}{10} & \frac{6\pi}{10} & \frac{8\pi}{10} & \frac{10\pi}{10} \end{bmatrix}^\top$. This time the information flow structure corresponds to cyclic pursuit: unicycle i gets relative information with respect to unicycle $i+1$. The corresponding Laplacian is

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Figures 1 and 2 show the simulations results for cases A and B using feedback (8) with the following parameters: $r = 1$, $v = 1$, $v_1 = 0.2$, and $K = 0.7$. The function $\phi : \mathbb{R}^{2n} \to (0, +\infty)$ is chosen as

$$\phi(y) = \begin{cases} c & \|y\| \leq c \\ c^2/\|y\| & \|y\| > c \end{cases}$$

where $c = \sqrt{0.99v/Kr}$.

III. SOLUTION OF CFCP FOR GENERAL DIGRAPHS

The solution of CFCP in the case of circulant information flow digraph relies on the feedback transformation (5) and the design of two feedbacks $\bar{u}(\chi)$ and $\tilde{u}(\chi)$. The feedback $\bar{u}(\chi)$ asymptotically stabilizes Γ relative to Γ_1, while the feedback $\tilde{u}(\chi)$ asymptotically stabilizes Γ_1 and yields the LUB property. The stability analysis for the feedback $\tilde{u}(\chi)$ does not rely on the fact that the graph Laplacian L is circulant, and is therefore applicable to general information flow graphs that have a globally reachable node. On the other hand, the analysis for feedback $\bar{u}$ is based on Lemma II.1 and Lemma V.2 in part I, and cannot be used in the case when L is not circulant. In this section we develop a different analysis
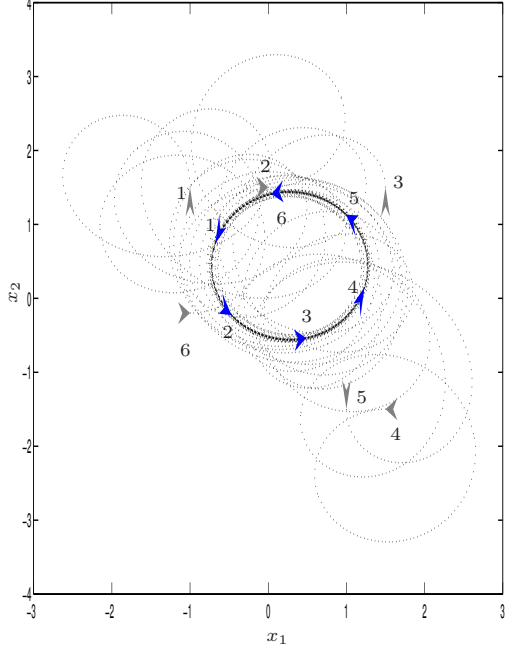
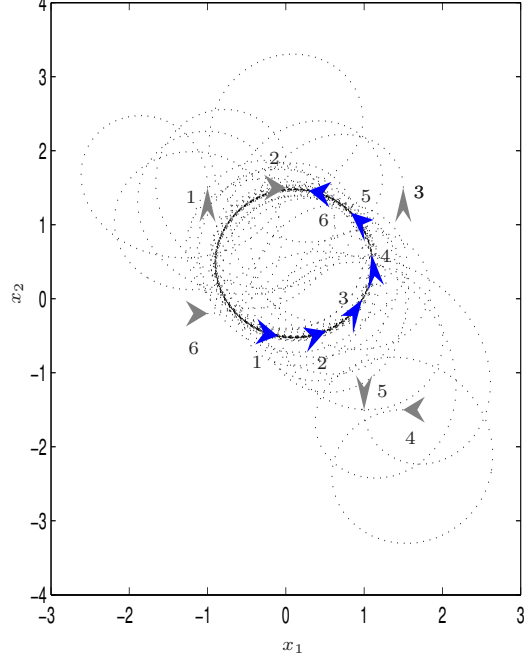Fig. 1.   CFCP Simulation - A: circulant digraph

Fig. 2.   CFCP Simulation - B: circulant digraph

proving that the feedback $\bar{u}(\chi)$ in (6) stabilizes Γ relative to Γ_1 even when L is not circulant, and thus the distributed feedback (8) solves CFCP in the general case of information flow graphs with a globally reachable node.

Proposition III.1. *Assume that the information flow graph has a globally reachable node. Let $v > v_1 > 0$ and $\phi : \mathbb{R}^{2n} \to (0, +\infty)$ be a locally Lipschitz function such that $\sup_{y \in \mathbb{R}^{2n}} \|\phi(y)y\| < \infty$. Then, there exists $K^\star > 0$ satisfying $\sup_{y \in \mathbb{R}^{2n}} \|\phi(y)y\| < v/(2K^\star r)$ such that for all $K \in (0, K^\star)$ the feedback (8) solves CFCP and renders the goal set Γ in (2) asymptotically stable, and Γ_1 in (3) globally asymptotically stable for the closed-loop system.*

The proof is omitted for space limitations.

A. Simulations

Figures 3 and 4 show the simulations results for cases A and B, given in Section II, using feedback (8) with the following parameters: $r = 1$, $v = 1$, $v_1 = 0.14$, $K = 1.9$ and

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The function $\phi : \mathbb{R}^{2n} \to (0, +\infty)$ is set as in Section II.

IV. SOLUTION OF CFCP FOR DYNAMIC UNICYCLES

Here we extend the solutions of the CFCP presented in Sections II and III, and Section V of part I to a system of dynamic unicycles (refer to the vertical rolling disc model in [5]),

$$\begin{aligned}
\dot{x}_1^i &= x_5^i \cos x_3^i \\
\dot{x}_2^i &= x_5^i \sin x_3^i \\
\dot{x}_3^i &= x_4^i \\
\dot{x}_4^i &= \frac{1}{J} w_2^i \\
\dot{x}_5^i &= \frac{R}{(I + mR^2)} w_1^i
\end{aligned} \tag{9}$$

for $i = 1, \ldots, n$, with state $x_d^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i) \in \mathbb{R}^2 \times S^1 \times \mathbb{R}^2$. We will denote the overall state by $\chi^d = \text{col}(x_d^1, \cdots, x_d^n)$. Moreover, as before, we will denote the kinematic states of each unicycle as $x^i = (x_1^i, x_2^i, x_3^i)$, and we will let χ denote the overall kinematic state of the unicycles, i.e., $\chi = \text{col}(x^1, \ldots, x^n)$. The scalars R and m in (9) are, respectively, the radius and mass of the unicycle; I and J are, respectively, the moments of inertia of the unicycle about axes perpendicular to and in the plane of the unicycle, passing through the centre, as shown in Figure 5. Finally, w_1^i and w_2^i are the torques about those axes. These are the new control inputs.

As before, the information flow among the n-unicycles is modeled by a digraph $\mathcal{G}$ with Laplacian L. An arc from node i to node j means that unicycle i has access to the relative displacement, relative heading, and relative linear and angular velocities with respect to unicycle j. Each unicycle is also assumed to have access to its own absolute orientation x_3^i, and its own velocities x_4^i, x_5^i.

In order to adapt the formulation of CFCP to system (9), we note the linear and angular velocities of the kinematic
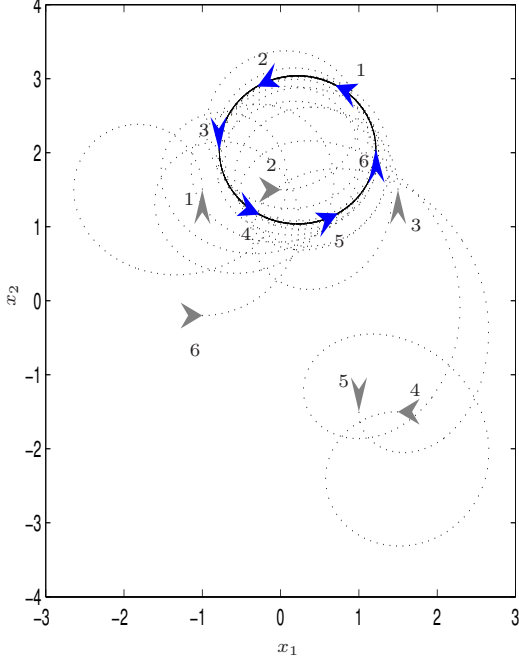
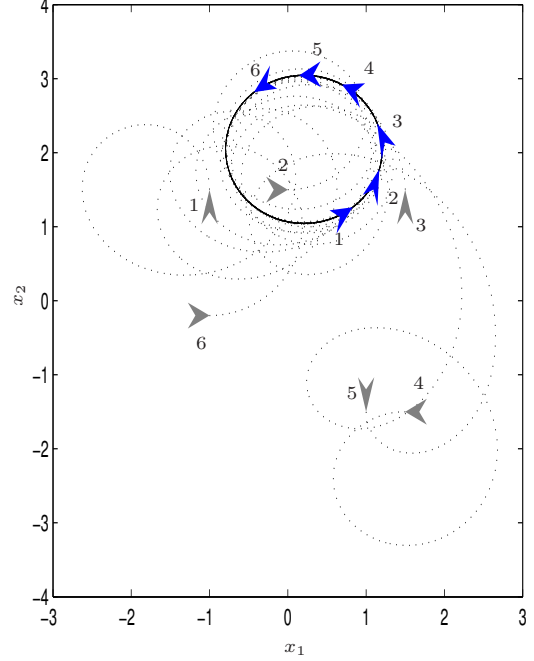Fig. 3.   CFCP Simulation - A: general digraph

Fig. 4.   CFCP Simulation - B: general digraph

unicycles on the goal set Γ are $u_1^i = v$ and $u_2^i = v/r$ for $i = 1, \cdots, n$ so that, in steady-state, all unicycles follow a circle of radius r counter-clockwise with forward speed v. In light of this observation, we state CFCP for dynamic unicycles as follows.

CFCP for dynamic unicycles. Consider the n dynamic unicycles in (9). For a given information flow digraph $\mathcal{G}$ with a globally reachable node, and a desired formation specification expressed by a vector of angles $\alpha \in S^n$, design a distributed control law which asymptotically stabilizes the set

$$\Gamma_d = \{\chi^d : L(x_3 - \alpha) = 0, c^{i+1}(x^{i+1}) = c^i(x^i), \\ x_4^i = v/r, x_5^i = v, \ 1 \leq i \leq n\}, \quad (10)$$

where $c^i(x^i)$ is defined in (4). Additionally, as before, the unicycles should have a common asymptotic centre of rotation.

Note that the goal set Γ_d in (10) can be expressed as $\Gamma_d = \Gamma \cap \{\chi^d : x_4^i = v/r, x_5^i = v, \ 1 \leq i \leq n\}$, where Γ is the goal set for the kinematic unicycles, defined in (2). As we mentioned earlier, all feedbacks $u_1^i(\chi)$, $u_2^i(\chi)$ presented in part I and in this paper have the property that $u_1^i(\chi)|_\Gamma = v$, $u_2^i(\chi)|_\Gamma = v/r$. Therefore, letting

$$\mathcal{O} = \{\chi^d : x_4^i = u_2^i(\chi), \ x_5^i = u_1^i(\chi), i = 1 \cdots, n\},$$

we can express the goal set Γ_d as

$$\Gamma_d = \Gamma \cap \mathcal{O}.$$

All kinematic feedbacks presented earlier guarantee that the set Γ_d is asymptotically stable relative to $\mathcal{O}$ for (9). Therefore, in order to solve CFCP for dynamic unicycles, we may
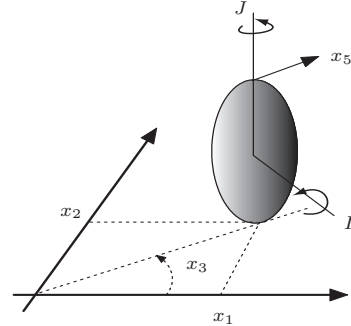
leverage once again the reduction principle in Theorem I.1 as follows:

Step 1. Given any of the kinematic feedbacks $u_1^i(\chi)$, $u_2^i(\chi)$ designed earlier, we design distributed feedbacks $w_1^i(\chi^d)$ and $w_2^i(\chi^d)$, $i = 1, \cdots, n$, rendering $\mathcal{O}$ globally asymptotically stable. We also show that the feedbacks $w_1^i(\chi^d)$, $w_2^i(\chi^d)$ guarantee that the closed-loop system is LUB near Γ_d, and that the unicycles have a common asymptotic centre of rotation.

Step 2. By using the fact that Γ_d is asymptotically stable relative to $\mathcal{O}$, and invoking the reduction theorem for asymptotic stability, Theorem I.1, we conclude that the feedback in question solves CFCP for dynamic unicycles.

The solution of CFCP for dynamic unicycles is given as follows.

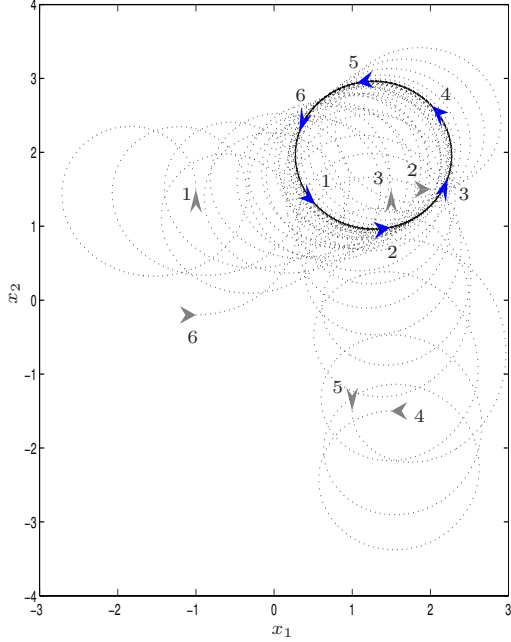**Proposition IV.1.**   *i. Assume the information flow graph is*

Fig. 5.   Dynamic Unicycle

Fig. 6.   Dynamic unicycles, Simulation - A: general digraph

Figures 6 and 7 show the simulations results for cases A and B, given in Section II, using feedback (11) with $K1 = 1$, $\phi(y) = lv/2rK(1 + \|y\|)$ with $l = 0.99$, $R = 1$, $J = 1$, $I = 1$, $m = 1$ and the rest of the parametrs as in Section III-A. Notice that K^* is unknown and so we used $K = K^*$ in $\phi(y)$. From this and the feedback (11) the parameter K is irrelevant in the controller. Empirically, we observed that increasing l beyond 1 gives better convergence of the centres of rotation up to a point, beyond which the performance degrages and solutions even become unbounded.
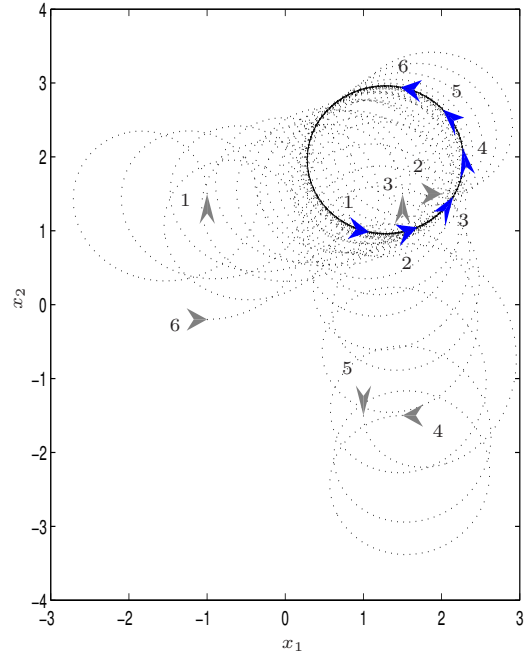

Fig. 7.   Dynamic unicycles, Simulation - B: general digraph

undirected and has a globally reachable node. If u_1^i and u_2^i, $i = 1, \cdots, n$ are chosen as in Proposition V.3 in part I, then the feedback

$$w_1^i = \frac{(I + mR^2)}{R} \left\{ \dot{u}_1^i(\chi) - K_1(x_5^i - u_1^i(\chi)) \right\}$$
$$w_1^i = J \left\{ \dot{u}_2^i(\chi) - K_1(x_4^i - u_2^i(\chi)) \right\} \tag{11}$$

where $K_1 > 0$ is a design constant, $\dot{u}_1^i(\chi)$ and $\dot{u}_2^i(\chi)$ are, respectively, the Lie derivatives of $u_1^i(\chi)$ and $u_2^i(\chi)$ along the dynamics (9), solves CFCP for dynamic unicycles and renders the goal set Γ_d in (10) asymptotically stable for the closed-loop system.

ii. *Assume the information flow graph is undirected and has a globally reachable node. If u_1^i and u_2^i, $i = 1, \cdots, n$ are chosen as in Proposition V.4 in part I, then the feedback (11) renders the goal set Γ_d in (10) globally asymptotically stable for the closed-loop system and solves CFCP for dynamic unicycles globally when the state space is taken to be $\mathcal{X} = \mathbb{R}^5$.*

iii. *For a general static information flow graph with a globally reachable node. If u_1^i and u_2^i, $i = 1, \cdots, n$ are chosen as in Proposition III.1 in this paper, then then feedback (11) solves CFCP for dynamic unicycles and renders the goal set Γ_d in (10) asymptotically stable for the closed-loop system. In addition, if u_1^i is chosen as in (22) of part I and the state space is taken to be $\mathcal{X} = \mathbb{R}^{5n}$, then the set Γ_d becomes globally asymptotically stable relative to $\mathcal{O}$, and the feedback above solves CFCP for dynamic unicycles globally.*

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