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Factorization of multipliers in passivity and IQC analysis

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Abstract— **Multipliers are often used to find conditions for the absolute stability of Lur'e systems. They can be used either in conjunction with passivity theory or within the more recent framework of integral quadratic constraints (IQCs). We compare the use of multipliers in both approaches. Passivity theory requires that the multipliers have a canonical factorization and it has been suggested in the literature that this represents an advantage of the IQC theory. We consider sufficient conditions on the nonlinearity class for the associated multipliers to have a canonical factorization.**

I. INTRODUCTION

The use of open–loop properties, such as applying the small gain theorem as well as the passivity theorem, in order to find absolute stability conditions for the Lur'e problem (see Fig. 1) is a common tool in nonlinear systems theory. In this problem the stability of a linear time-invariant (LTI) system, G, in a feedback interconnection with a nonlinear system, ϕ, is studied. Decoupling the linear and nonlinear parts reduces the complexity of the problem and allows a solution in terms of simple conditions on the linear part. An essential feature of this method is that stability is guaranteed for an entire class of nonlinearities.

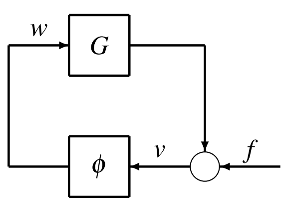

Fig. 1. Lur'e problem

Historically, a first general solution for a specific class of nonlinearities was given by Popov [1]; his result is generalized in [2] for multivariable systems (see [3] and references therein for different multivariable cases). The circle criterion was developed by several authors simultaneously, but a pair of papers can be highlighted [4], [5]. In the first [4], the definition of input–output stability using extended spaces, as proposed by [6], is used and the small gain and passivity theorems are established. In the second [5] the circle and Popov criteria are obtained as applications of these theorems. In the proof of the Popov criterion in [5], the abstract concept of multiplier is interpreted as a loop transformation, see Fig. 2.

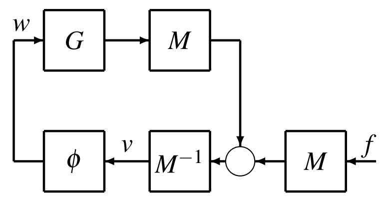

Fig. 2. Multiplier Transformation: stability of this systems implies stability of the original system in Fig. 1.

The multiplier is an artificial system that is introduced into the loop together with its inverse. Roughly speaking, an excess of positivity in the nonlinear part is exploited to redress a deficiency of positivity in the linear part. Passivity theory requires systems to be causal, but restricting the analysis to linear causal multipliers, i.e. systems without poles in the right half plane, leads to severe constraints on the choice of the phase. In [7] a factorization condition on non-causal multipliers is proposed to overcome this restriction and recover causality in the loop elements, (see Fig. 3 and Remark 2.7).

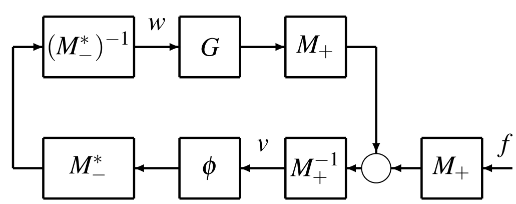

Fig. 3. Multiplier Transformation: recovering causality

The factorization condition on the multiplier is given by

$$M = M_- M_+, \tag{1}$$

where M_- and M_+ are invertible and M_+, M_+^{-1}, M_-^*, and M_-^{*-1} are causal and have finite gain. For the Lur'e problem where one part of the loop is LTI it is natural to restrict the multipliers themselves to be LTI. For a linear operator this is referred to as the canonical factorization (see Section II-B). Some special cases of this factorization, e.g. spectral factorization, inner–outer factorization and J-spectral factorization, have been used in $\mathbf{H}_\infty$ control theory [8]. The conditions for the existence of this factorization are summarized in the monograph [9] which takes an operator theoretical approach.

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In [10], an equivalent result was found from a control systems perspective. Only a few papers, for instance, [11], have used these results for control systems analysis.

In the multiplier approach the properties of a class Φ of positive nonlinearities ϕ are used to find the corresponding class $\mathscr{M}$ of multipliers M such that $M^*\phi$ is also positive. As an example, the original paper [7] was focused in preserving positivity for monotone and slope–restricted nonlinearities; this class of multipliers is known as the Zames–Falb multipliers. Then if there exists a multiplier M within this class such that MG is strictly positive, then the linear system G in a feedback interconnection with any of the nonlinearities within the class (Fig. 1) is stable.

By contrast IQC theorem [12] is derived using a homotopy argument where causality is not required. As a result, in IQC theorem any multiplier preserving positivity for ϕ can be used and a canonical factorization is no longer required. This is sometimes stated as a distinguishing advantage of the IQC formulation [12], [13]. But to date no significantly wider class of multiplier or improved stability results have yet been found that exploit this feature. This suggests the question: is the existence of a canonical factorization a necessary feature of multipliers for standard nonlinearity classes? In addition some authors still use the classical multiplier approach [14]; are their results conservative because they must then impose the canonical factorization?

Recently, a few papers have examined the connection between dissipativity and IQC theory [15], [16]. In this paper we restrict our attention to the use of multipliers in the classical sense. In [17] a different factorization is analyzed, where M_+ and M_- are allowed to be "tall"; the use of this factorization does not demonstrate equivalency, since passivity theory requires invertible multipliers.

This paper focuses the two questions above. The main result is that if the class of nonlinearities includes the scaled identity (e.g. sector–bounded nonlinearities, slope–restricted nonlinearities, passive LTI systems) then both approaches lead to the same result. In particular, any LTI multiplier that preserves positivity must have a canonical factorization, except for certain pathological cases.

II. Problem definition

In this section some background concepts are summarized. The first subsection gives the notation and definitions that will be used throughout the paper. The second subsection introduces the canonical factorization and the condition for its existence. After that, the passivity theorem and its extension using multipliers are shown. Finally, the general IQC theorem is given. We assume the systems under consideration to be square. We make certain further restrictions on both the IQC framework and the passivity approach such that a straightforward comparison is possible.

A. Notation and definitions

$\mathscr{L}_2^m[0,\infty]$ is the Hilbert space of all square integrable and Lebesgue measurable functions $f:[0,\infty]\to\mathbb{R}^m$.A truncation of the function f at T is given by $f_T(t)=f(t),\ \forall t\leq T$ and $f_T(t)=0,\ \forall t>T$. In addition, f belongs to the extended space $\mathscr{L}_{2e}^m$ if $f_T\in\mathscr{L}_2^m$ for all $T>0$.

Let the system S be a map from $\mathscr{L}_{2e}^m[0,\infty)$ to $\mathscr{L}_{2e}^m[0,\infty)$, with input u and output Su. It is passive if $\langle u_T, Su_T\rangle\geq 0$ for all $T>0$ and $u\in\mathscr{L}_{2e}^m[0,\infty)$. It is positive if $\langle u, Su\rangle\geq 0$ for all $u\in\mathscr{L}_2^m[0,\infty)$. This system S is causal if $Su(t)=S(u_T)(t)$ for all $t<T$. Moreover, the system S is stable if for any $u\in\mathscr{L}_2^m[0,\infty)$, then $Su\in\mathscr{L}_2^m[0,\infty)$. The system S is bounded if there exists a constant γ such that $\|Su\|_2\leq\gamma\|u\|_2$.

This definition of a positive system is standard, but it is not equivalent to the standard definition of a positive real system [18], where causality is required. Although passivity and positivity definitions are often considered equivalent, the equivalence only holds for causal systems. Moreover, because passivity theorem requires a inner product between the input and output, the space of the input should be the dual space of the space of the output; therefore, this paper is restricted to square systems.

Lemma 2.1 (Section VI.9.1 in [19]): Let $S:\mathscr{L}_{2e}^m[0,\infty)\to\mathscr{L}_{2e}^m[0,\infty)$ be a causal system, then the system is passive if and only if it is positive.

Finally, this paper focused the stability of the feedback interconnection of a stable LTI system G and a bounded system ϕ, represented in Fig. 1 and given by

$$\begin{cases} v=f+Gw, \\ w=\phi v. \end{cases} \tag{2}$$

Since G is a stable LTI system, the exogenous input in this part of the loop can be taken as zero signal without loss of generality. It is well posed if the map $(v,w)\mapsto(0,f)$ has a causal inverse on $\mathscr{L}_2^{2m}[0,\infty)$, and this interconnection is stable if for any $f\in\mathscr{L}_2^m[0,\infty)$, then $Gw\in\mathscr{L}_2^m[0,\infty)$ and $\phi v\in\mathscr{L}_2^m[0,\infty)$. In addition, $G(s)$ means the matrix transfer function of the linear system G. G^* is the $\mathscr{L}_2$–adjoint of G, i.e., $G^*(s)=G(-s)^\top$. $\mathbf{RL}_\infty$ ($\mathbf{RH}_\infty$) is the space of all rational matrix transfer functions without poles in the imaginary axis (in the closed right-half plane).

B. Canonical factorization

The condition for the existence of a canonical factorization is given in [9]. Since this result is given in a different framework, this section shows the definition of canonical factorization that will be used. The canonical factorization has a general definition using a Cauchy contour for linear operator [9]. However, we are going to use the definition given in [8] when this Cauchy contour is the imaginary axis.

Definition 2.2 (Canonical factorization): Let $M(s)$ be a square matrix transfer function such that $M(s)\in\mathbf{RL}_\infty$ and $M^{-1}(s)\in\mathbf{RL}_\infty$. Then, $M(s)=M_-(s)M_+(s)$ is a canonical factorization of $M(s)$ if $M_+(s)\in\mathbf{RH}_\infty$, $M_+^{-1}(s)\in\mathbf{RH}_\infty$, $M_-^*(s)\in\mathbf{RH}_\infty$, and $(M^*(s)_-)^{-1}\in\mathbf{RH}_\infty$.

The next corollary is a simplified version of the Theorem 15.3 in [9], using the above definition.

Corollary 2.3: [1] Let $M(s)\in\mathbf{RL}_\infty$ be an $n\times n$ rational matrix function such that $M^{-1}(s)\in\mathbf{RL}_\infty$. Assume that

[1]Proofs are available in a longer version of this paper, available from the authors on request.

$\mathrm{herm}(M(j\omega)) = \frac{1}{2}(M(j\omega)+M^*(j\omega)) > 0$, $\forall\omega \in \mathbb{R}$. Then, $M(s)$ admits a canonical factorization.

C. Passivity theorem

Following [19], the passivity theorem, and its version with multipliers, can be written as follows. The following theorems are simplified versions applied for the case where G is LTI stable and ϕ is bounded. More general versions of the passivity theorem can be found in [20], [21].

Theorem 2.4 (Passivity theorem): Let G be a stable LTI system and let ϕ be a bounded system from $\mathscr{L}_{2e}^m[0,\infty)$ to $\mathscr{L}_{2e}^m[0,\infty)$. Assume that the feedback interconnection of G and ϕ is well posed and there exists a constant $\varepsilon > 0$ such that the following conditions hold

$$\begin{aligned} \langle u, Gu\rangle &\leq -\varepsilon\|u\|^2, && (3)\\ \langle u, \phi u\rangle &\geq 0 && (4)\end{aligned}$$

for all $T > 0$ and $u \in \mathscr{L}_{2e}^m[0,\infty)$. Then, the feedback interconnection (2) is stable.

Remark 2.5: The classical theorem has been modified for positive feedback interconnection.

The conservatism applying the passive theorem can be decreased using the multiplier approach. The following theorem establishes the use of LTI multiplier with a canonical factorization.

Theorem 2.6: Let G be a stable LTI system and let ϕ be a bounded system from $\mathscr{L}_{2e}^m[0,\infty)$ to $\mathscr{L}_{2e}^m[0,\infty)$. Assume that the feedback interconnection of G and ϕ is well posed and there exist a constant $\varepsilon > 0$ and LTI multiplier M, such that $M(s)$ has a canonical factorization and the following conditions hold

$$\begin{aligned} \langle u, MGu\rangle &\leq -\varepsilon\|u\|^2, && (5)\\ \langle u, M^*\phi u\rangle &\geq 0 && (6)\end{aligned}$$

for all $u \in \mathscr{L}_2^m[0,\infty)$. Then, the feedback interconnection (2) is stable.

Remark 2.7: The passivity theorem cannot be applied directly, since the systems are not passive, they are positive (note that there are no truncations in the equations). Nevertheless, the factorization allows a causal equivalent representation given by $-M_+G(M_-^*)^{-1}$ and $M_-^*\phi M_+^{-1}$, see Fig. 3, and equations (5) show these causal systems are (strictly) passive. For example, following Lemma 15 in section VI.9.2 in [19], let x and u belong to $\mathscr{L}_2[0,\infty)$ and be related by $x = M_-^* u$. Taking into account the conditions on the canonical factorization, i.e. M_-^* and $(M_-^*)^{-1}$ to be bounded, for all $x \in \mathscr{L}_2[0,\infty)$, then $u \in \mathscr{L}_2[0,\infty)$, and vice versa. As a consequence, left-hand side of (5) can be rewritten as follows

$$\langle u, M_-M_+Gu\rangle = \langle M_-^*u, M_+Gu\rangle = \langle x, M_+G(M_-^*)^{-1}x\rangle.$$

Since M_+, G, and $(M_-^*)^{-1}$ are causal, using Lemma 2.1, $-M_+G(M_-^*)^{-1}$ is strictly passive.

D. IQC theorem

In [12], in a similar argument as in the multiplier approach, the properties of an artificial system, whose inputs are the input and the output of the original one, is used to obtain the stability of the Lur'e system using a homotopy argument. As an advantage, causality is not needed.

Definition 2.8: Two signals, $u \in \mathscr{L}_2^m[0,\infty]$ and $w \in \mathscr{L}_2^m[0,\infty]$ are said to satisfy the IQC defined by a measurable Hermitian–valued $\Pi : j\mathbb{R} \to \mathbb{C}^{(2m)\times(2m)}$, if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega)\\ \widehat{w}(j\omega)\end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{u}(j\omega)\\ \widehat{w}(j\omega)\end{bmatrix} d\omega \geq 0, \tag{7}$$

where $\widehat{u}$ and $\widehat{w}$ are the Fourier transform of the signals u and w, resp.

In this framework, Π is also referred to as a multiplier. Here, in order to avoid a confusion between Π and M, Π will be referred to as generalized multiplier.

Definition 2.9: A bounded system $\phi : \mathscr{L}_2^m[0,\infty] \mapsto \mathscr{L}_2^m[0,\infty]$ is said to satisfy the IQC defined by Π if the signals u and ϕu satisfy the IQC defined by Π for all $u \in \mathscr{L}_2^m[0,\infty]$.

Theorem 2.10 (IQC theorem): Assume that

(i) the feedback interconnection of G and $\tau\phi$ is well posed for all $\tau \in [0,1]$,

(ii) there exists a generalized multiplier Π such that $\tau\phi$ satisfies the IQC defined by Π for all $\tau \in [0,1]$,

(iii) there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega)\\ I\end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega)\\ I\end{bmatrix} \leq -\varepsilon I \qquad \forall\omega \in \mathbb{R} \tag{8}$$

Then, the feedback interconnection (2) is stable.

III. Conservatism Analysis

It has been suggested that IQC analysis is less conservative that passivity theory on the selection of the multiplier for absolute stability [12], [13]. In order to establish a comparison, we first write versions of both the IQC theorem and the passivity theorem in a common notation, following [22]. After that, the main results of this paper can be presented: we establish conditions under which the two approaches are equivalent.

A. Common notation

For the passivity theorem, condition (6) can be written in the frequency domain as

$$\int_{-\infty}^{\infty} \left(\widehat{u}^* M^*(j\omega)\widehat{\phi u} + \widehat{\phi u}^* M(j\omega)\widehat{u}\right) d\omega \geq 0 \tag{9}$$

since the Fourier transform preserves the inner product [8]. The linearity of the multiplier M has been used. Equation (9) means that the signals u and ϕu satisfy the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & M^*(j\omega)\\ M(j\omega) & 0\end{bmatrix} \tag{10}$$

A version of the passivity theorem can be written as follows:

Corollary 3.1: Assume that

(i) the feedback interconnection of G and ϕ is well posed, (*ii-a*) there exists a multiplier M such that ϕ satisfies the IQC defined by (10).

(*ii-b*) M has a canonical factorization,

(*iii*) there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} 0 & M^*(j\omega) \\ M(j\omega) & 0 \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I \quad (11)$$

for all $\omega \in \mathbb{R}$.

Then, the feedback interconnection (2) is stable.

On the other hand, applying the IQC theorem to the specific Π given by (10), we obtain:

Corollary 3.2: Assume that

(*I*) for all $\tau \in [0,1]$, the feedback interconnection of G and $\tau\phi$ is well posed,

(*II*) there exists a multiplier M such that ϕ satisfies the IQC defined by (10).

(*III*) there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} 0 & M^*(j\omega) \\ M(j\omega) & 0 \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I \quad (12)$$

for all $\omega \in \mathbb{R}$.

Then, the feedback interconnection (2) is stable.

Remark 3.3: Using Remark 2 in [12], the homotopy condition on (*II*) is not needed, since the inequality in the IQC is satisfied for any positive constant $\tau \in [0,1]$ if it is satisfied for $\tau = 1$.

Remark 3.4: There are more general versions of both theorems. The IQC theorem can be applied with a more general form of Π [12]. Similarly the passivity theorem can be applied with other supply rates using the dissipativity theory [23], [20]. Moreover, an extension of dissipativity theory has been proposed in [24], [25].

Remark 3.5: A similar comparison is given in [12], but using another version of the IQC theorem and the passivity theorem. Despite that (12) is the standard condition in the IQC Theorem, Remark 3 [12] allows that the right-hand side of (12) is replaced by $-\varepsilon G^*(j\omega)G(j\omega)$ since the nonlinearity is bounded. Similarly, passivity theorem remains true is the right-hand side of (5) is replaced by $-\varepsilon\|Gu_T\|^2$.

If we confine our attention to Corollaries 3.1 and 3.2, the two approaches differ in that Corollary 3.1 requires only the well posed condition for the nonlinearity itself, whereas Corollary 3.2 requires the condition for all nonlinearities given by $\tau\phi$, for all $\tau \in [0,1]$. In this paper we assume the well posed condition holds for all $\tau \in [0,1]$. On the other hand, Corollary 3.2 requires no counterpart to condition (*ii-b*) in Corollary 3.1. This is the difference between the two theories that we analyse.

B. Main results

In this section, the main results of this paper are given. The results seem simple, but are novel to the best of our knowledge. It turns out there is an equivalence between both approaches when applied to standard classes of nonlinearities. Certain reasonable assumptions on both the class of nonlinearities and the multiplier are sufficient to ensure the multiplier has a canonical factorization.

Assumption 3.6: The multipliers are rational matrix transfer functions that satisfy $M(s) \in \mathbf{RL}_\infty$ and $M^{-1}(s) \in \mathbf{RL}_\infty$.

We also require the hermitian part of the multiplier to be positive definite. The following two lemmas show that a small scaled identity can be added to any multiplier without loss of generality, and therefore we need not exclude multipliers whose hermitian part is positive semi-definite.

Lemma 3.7: Let M be a multiplier such that $\text{herm}(M) \geq 0$ and it satisfies (12) for some $\varepsilon > 0$. Then, there exist a constant $\zeta > 0$ such that the multiplier $\overline{M}(j\omega) = M(j\omega) + \zeta I$ with $\text{herm}(\overline{M}) > 0$ satisfies (12) for $\frac{\varepsilon}{2} > 0$.

Lemma 3.8: If M^* preserves the positivity of class of nonlinearities, then $M^* + \zeta I$ for all $\zeta > 0$ also preserves the positivity of the class.

We also require a mild condition on the class of nonlinearities.

Assumption 3.9: There exists $k > 0$ such that $kI \in \Phi$ where Φ is the class of nonlinearities.

The following proposition establishes that when the class of nonlinearities includes a scaled identity, the canonical factorization is not a limitation on the class of multipliers.

Proposition 3.10: Let Φ be a class of nonlinearities satisfying Assumption 3.9, let G be a stable LTI system and let M be a multiplier satisfying Assumption 3.6. Under these conditions, if M satisfies (*II*) for all $\phi \in \Phi$ and M and G satisfy (*III*), then:

- either M satisfies (*ii-a*), (*ii-b*), and M and G satisfy (*iii*)
- or there exists some small $\zeta > 0$ such that $\overline{M} = M + \zeta I$ satisfies (*ii-a*), (*ii-b*) and $\overline{M}$ and G satisfy (*iii*).

Roughly speaking, we have shown that under Assumption 3.9 on the class of nonlinearities and under Assumption 3.6 on the multiplier, the existence of a canonical factorization is no restriction on the class of nonlinearities and hence Corollary 3.2 offers no advantage over Corollary 3.1.

C. Discussion on the Assumptions

If we want to find an example where IQC theory offers a direct advantage over passivity theory, we must either find a multiplier that breaks Assumption 3.6 or a class of nonlinearities that breaks Assumption 3.9. We show in the following section that Assumption 3.9 is satisfied for several standard classes of nonlinearities. Even if Assumption 3.9 is not satisfied, the positivity of the multiplier is sufficient for a canonical factorization to exist.

From a loop transformation point of view, as originally proposed in [7], [27], [19], Assumption 3.6 is mandatory in order to recover the $\mathscr{L}_2$-stability of the original system from the stability of the transformed system. However, there is an important class of multipliers in the literature which do not satisfy Assumption 3.6: the Popov multipliers.

The relation between our work and Popov multipliers is beyond the scope of this paper. Note that when using passivity theory, the $\mathscr{L}_2$-stability is degraded when a Popov multiplier is used as the derivative of the input in the nonlinearity must also belong to $\mathscr{L}_2$ (Section 6.6 in [21])[2].

[2] A similar observation was also made when studying the stability of interconnections of negative-imaginary systems [28], [29] via passivity.

Similarly, when a Popov multiplier is used within the IQC theory some special considerations must be taken into account because the generalized multiplier is not measurable. For example, in [13], the generalized multiplier is split up into two terms: a bounded part and the Popov multiplier. As in the passivity analysis, the $\mathscr{L}_2$-stability appears to be degraded (see Definition 3 in [13]).

IV. Applications

Two of the classical nonlinearities (saturations and perturbations), where the multiplier approach has been used, hold the conditions for Prop. 3.10. In the case of saturation, the nonlinearities are usually characterized with the following properties: memoryless, monotone, slope–restrict and odd. A scaled identity is within this description. In the same way, standard classes of systems considered as perturbations also include a scaled identity.

A. Monotone, slope–restricted and odd nonlinearities

The class of monotone, slope–restricted and odd nonlinearities has received considerable attention since the celebrated paper [7] by Zames and Falb introduced the multiplier factorization. Even though they were proposed more than 40 years ago, novel work on Zames-Falb multipliers has appeared recently. For example, their application to repeated nonlinearities are established in [30] and [31]; in [32], [33], the multipliers are proposed for robust stability analysis of input-constrained Model Predictive Control; in [34], a subclass of the Zames–Falb multiplier is proposed in order to restrict the constraints on the nonlinear system.

Multipliers M^* that preserve the positivity of this class of nonlinearities are referred to as Zames–Falb multipliers. There are two definitions in the literature: we will distinguish them with the terminology Open and Closed Zames–Falb multipliers. Originally, they were designed to satisfy two properties:

1) To preserve the positivity of the nonlinearity.
2) To ensure the canonical factorization.

Under these conditions, the original class of Zames–Falb multiplier was defined as follows.

Definition 4.1 (Open Zames–Falb Multiplier): A rational transfer function, M, is said to be an Open Zames-Falb multiplier, $\mathscr{M}_{OZF}$, if it is given by $M(s) = M_0 - Z(s)$, where the unit impulse response of $Z(s)$, $z(t)$, satisfies $\|z\|_1 = \int_{-\infty}^{\infty} |z(t)|dt < M_0$.

An appeal to the properties of Banach Algebras guarantees the canonical factorization for this class of multiplier (see Section VI.9.5 in [19]). Proposition 3.10 provides an alternative guarantee, because a scaled identity belongs to such a class of nonlinearities. In addition:

Lemma 4.2: If $M \in \mathscr{M}_{OZF}$, then M satisfies Assumption 3.6.

Hence, for this class of multiplier, IQC analysis offers no advantage over passivity theory. However, for IQC theory the second condition can be removed. This means a wider class of multiplier can be used [12].

Definition 4.3 (Closed Zames–Falb Multiplier): A rational transfer function, M, is said to be a Closed Zames-Falb multiplier, $\mathscr{M}_{CZF}$, if it is given by $M(s) = M_0 - Z(s)$, where the unit impulse response of $Z(s)$, $z(t)$, satisfies that $\|z\|_1 = \int_{-\infty}^{\infty} |z(t)|dt \le M_0$.

This definition includes multipliers which cannot be factorized, because they do not satisfy Assumption 3.6. For example, the multiplier given by $M(s) = 1 - \frac{1}{s+1}$ belongs to $\mathscr{M}_{CZF}$, but it cannot be factorized since it has a zero at $s = 0$, so $M^{-1} \notin \mathbf{RL}_\infty$.

Nevertheless, we may still conclude from Proposition 3.10 that IQC analysis holds no advantage over passivity theory for this case. The key insight is that the class of nonlinearity still satisfies Assumption 3.9, irrespective of the choice of multiplier.

Lemma 4.4: Let M be a multiplier such that $M \in \mathscr{M}_{CZF}$ and $M \notin \mathscr{M}_{OZF}$, i.e. $\|z\|_1 = M_0$. If M satisfies (*II*) and (*III*) for some plant $G(s)$, then there exists $\zeta > 0$ such that $\overline{M}(s) = (M_0 + \zeta) - Z(s)$ satisfies (*ii-a*), (*ii-b*), and (*iii*) for some plant $G(s)$.

As a conclusion, the class of Zames–Falb multipliers can be taken as Definition 4.1 without loss of generality and both theories are equivalent for the absolute stability of this class of nonlinearities.

B. Passive uncertainties

In 1994, two papers were submitted to journals using multipliers for the same class of nonlinearities. They were addressing different problems: in [35], the problem of $\mathbf{H}_2$ performance is addressed using an embryonic version of the IQC theorem; in [36], robustness analysis is carried out using the passivity theorem.

In each case, the nonlinear class is a diagonal LTI perturbation where each diagonal term is passive, i.e. $\Delta = \mathrm{diag}(\Delta_1, \Delta_2, ..., \Delta_{n_p})$, where Δ_i is passive for $i = 1, 2, \ldots, n_p$. It is clear that the identity is within this class of nonlinearities and hence the results in Section III can be applied. Both papers define the multiplier as a mapping from $\mathscr{L}_2[-\infty, \infty]$ into $\mathscr{L}_2[-\infty, \infty]$, i.e. $M(s) \in \mathbf{RL}_\infty$. However, the definitions are slightly different.

Definition 4.5 ([35]): Given a multiplier $M(s) \in \mathbf{RL}_\infty$, if this is a diagonal transfer function and

$$M(j\omega) = M^*(j\omega) \ge 0 \qquad \forall \omega \in \mathbb{R}, \tag{13}$$

then it is said that $M \in \mathscr{M}_F$.

Definition 4.6 ([36]): Given a multiplier $M(s) \in \mathbf{RL}_\infty$, if this is a diagonal transfer function and there exists $\varepsilon > 0$ such that

$$M(j\omega) = M^*(j\omega) \ge \varepsilon I \qquad \forall \omega \in \mathbb{R}, \tag{14}$$

then it is said that $M \in \mathscr{M}_B$.

In [36], the canonical factorization is required, and it is suggested that by following [19] this factorization is ensured. Our analysis confirms that the multipliers within Definition 4.6 can be factorized. In addition the conditions on the multiplier imposed in [19] are no longer required.

Corollary 4.7: If $M \in \mathscr{M}_B$ then M satisfies Assumption 3.6.

Lemma 4.8: Given a rational transfer function $M \in \mathbf{RL}_\infty$ such that $M(j\omega) = M^*(j\omega) \geq \varepsilon I$, $\forall \omega \in \mathbb{R}$, satisfying (*II*) and (*III*) for some plant $G(s)$, then $M(s)$ satisfies (*ii-a*), (*ii-b*), and (*iii*) for some plant $G(s)$.

As in the previous application, the difference between both classes of multipliers is reduced to the limiting case $\varepsilon = 0$, where $M^{-1} \notin \mathbf{RL}_\infty$. But we can argue as before:

Lemma 4.9: Given a rational transfer function $M \in \mathscr{M}_F$, satisfying (*II*), and (*III*) for some plant $G(s)$, then there exists $\zeta > 0$ such that $\overline{M}(s) = \zeta I + M(s)$ satisfies (*ii-a*), (*ii-b*), and (*iii*) for some plant $G(s)$. In addition, $\overline{M}(s) \in \mathscr{M}_B$.

As an conclusion, the equivalence between both classes of multiplier, $\mathscr{M}_B$ and $\mathscr{M}_F$, has been shown.

V. Conclusion

Following the comparison proposed in [22], [12], an analysis of the conservatism imposed by the requirement for a canonical factorization of the multiplier in passivity theory has been carried out. It has been shown that if the class of nonlinearities includes a scaled identity, then the class of multipliers satisfying Assumption 3.6 is equivalent using both theories. The canonical factorization does not introduce conservatism into the stability analysis for this case.

The results have been applied to two widely used classes of nonlinearities, which both include a scaled identity. The results in this paper allow an easy method to analyze the conservatism, because only the cases where the multiplier is not within Assumption 3.6 must be considered. For these two applications, the equivalence between IQC theory and passivity theory is established. The further implications of violating Assumption 3.6 are subject to current investigation.

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