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Nonlinear superposition formulas: some physically motivated examples

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***Abstract*— This paper gives an account of the nonlinear superposition principle for classes of nonlinear systems. The existence of a nonlinear superposition principle is ensured by a Lie algebraic analysis of the nonlinear system, whereas the nonlinear superposition formulas are obtained in closed form by computing some functionally independent first integrals of an auxiliary system that can be associated with the given one.**

I. Introduction

The purpose of this paper is to briefly review the nonlinear superposition principle for classes of nonlinear differential equations and to apply such a principle to some physically motivated examples, thus showing its applicability and emphasizing the related computational procedures.

In general, after the works of S. Lie [1], by nonlinear superposition principle, it is understood a pair of formulas (the explicit and implicit nonlinear superposition formulas) that allow one to express the general solution of a system of ordinary differential equations in terms of a finite number of particular solutions and a certain number of arbitrary constants. The system of linear time-invariant differential equations is a remarkable case, in which the explicit superposition formula allows one to expresses the general solution as a linear combination of n particular solutions, with n arbitrary constants, where n is the dimension of the state. Other remarkable classes of systems admitting a nonlinear superposition principle are given the bilinear ones [2] and by the switched systems [3]. Other related references are [4]-[8].

The existence of explicit and implicit nonlinear superposition formulas can be ensured on the basis of a Lie algebraic analysis, whereas the computation of such formulas in closed form has been done only for certain classes of functions [9], [10]. The knowledge of an explicit nonlinear superposition formula is important not only for the possibility of computing any solution of the considered system, but also for the possibility of deducing some properties of the general solution (such as stability and attractivity), on the basis of the properties of some particular solutions.

II. Preliminaries

Given an open and connected set $\mathcal{U} \subseteq \mathbb{R}^n$, the set $\mathcal{A}_n$ of all analytic functions $\alpha(x) : \mathcal{U} \to \mathbb{R}$, endowed with the usual operations of sum and product between functions, is a *ring*; denote by $\mathcal{K}_n$ the set of all functions $\alpha = \frac{a}{b}$, with $a, b \in \mathcal{A}_n$, with b that is not identically equal to 0; then, $\mathcal{K}_n$ is a *field* (the *quotient field* of the ring of analytic functions): $\alpha \in \mathcal{K}_n$ is called *meromorphic*. Actually, similarly to the field of rational functions, $\mathcal{K}_n$ is a field under the *equivalence relation* $\sim$ defined as follows: $\alpha_1, \alpha_2 \in \mathcal{K}_n$, $\alpha_i = \frac{a_i}{b_i}$, $a_i, b_i \in \mathcal{A}_n$, b_i not identically equal to 0, are *equivalent*, $\alpha_1 \sim \alpha_2$, if $a_1(x)b_2(x) = a_2(x)b_1(x), \forall x \in \mathcal{U}$; one can say that α_1 and α_2 *coincide* on $\mathcal{U}$. In the following, all the functions are assumed to be meromorphic.

Given two vector functions $f(x), g(x) \in \mathbb{R}^n$, $L_f g$ is the *directional derivative* of g by f, and $[f, g] = L_f g - L_g f$ is the *Lie bracket* of f and g. A scalar function $h(x) \in \mathbb{R}$ is a *first integral* associated with f if $L_f h = 0$. Given two matrices (not necessarily constant) $A, B \in \mathbb{R}^{n\times n}$, $[A, B] = BA - AB$ is the *Lie bracket* of A and B.

Some analytic functions $h_i(x) \in \mathbb{R}$, $i = 1, ..., m$, $m \leq n$, are *functionally independent* [11], [12] if and only if, letting $h = \begin{bmatrix} h_1 & \dots & h_m \end{bmatrix}^\top$, the Jacobian matrix $\frac{\partial h}{\partial x}$ of h has full rank over the field $\mathcal{K}_n$ of meromorphic functions, *i.e.*, $\frac{\partial h}{\partial x}$ has full rank for all x in some open and connected set.

III. Nonlinear Superposition

Consider the class of time-varying nonlinear systems

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t)), \qquad (1)$$

where $x(t), f(t, x) \in \mathbb{R}^n$. A special subclass of systems belonging to class (1) is constituted by the linear ones:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)x(t), \qquad (2)$$

where $A(t) \in \mathbb{R}^{n\times n}$. Class (2) is very important because of the *linear superposition principle*; given n solutions $\xi^i(t) \in \mathbb{R}^n$, $i = 1, ..., n$, of (2),

$$\frac{\mathrm{d}\xi^i(t)}{\mathrm{d}t} = A(t)\xi^i(t), \quad i = 1, ..., n, \qquad (3)$$

such that $\det\left(\begin{bmatrix} \xi^1(t_0) & \dots & \xi^n(t_0) \end{bmatrix}\right) \neq 0$, for some initial time t_0, the linear superposition principle allows one to express any solution $x(t) \in \mathbb{R}^n$ of (2) as a linear combination of $\xi^1(t), ..., \xi^n(t)$,

$$x(t) = k_1\xi^1(t) + ... + k_n\xi^n(t), \qquad (4)$$

where $k := \begin{bmatrix} k_1 & \dots & k_n \end{bmatrix}^\top \in \mathbb{R}^n$ is given by

$$k = \begin{bmatrix} \xi^1(t) & \dots & \xi^n(t) \end{bmatrix}^{-1} x(t), \qquad (5)$$

and the inverse $\begin{bmatrix} \xi^1(t) & \dots & \xi^n(t) \end{bmatrix}^{-1}$ exists for all t in a sufficiently small open interval $\mathcal{T}_{t_0}$ containing the initial time t_0. Equation (4) is the *explicit linear superposition formula* and equation (5) is the *implicit linear superposition formula*

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for system (2). It is worth pointing out that each entry of the vector on the right-hand side of equation (5) is a first integral of the extended system constituted by system (2) and by its replicas (3), *i.e.*,

$$\frac{\partial k}{\partial x}A(t)x+\sum_{i=1}^{n}\frac{\partial k}{\partial \xi^i}A(t)\xi^i=0,\quad \forall t\in\mathcal{T}_{t_0}.$$

Example 1: Consider a linear oscillator with time-varying frequency,

$$\begin{cases}\frac{\mathrm{d}x_1}{\mathrm{d}t}=x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t}=-\omega(t)x_1,\end{cases}\tag{6}$$

where $\omega(t)$ is the time-varying oscillation frequency. Consider two replicas of the oscillator,

$$\begin{cases}\frac{\mathrm{d}\xi_1^1}{\mathrm{d}t}=\xi_2^1,\\ \frac{\mathrm{d}\xi_2^1}{\mathrm{d}t}=-\omega(t)\xi_1^1,\end{cases}\qquad \begin{cases}\frac{\mathrm{d}\xi_1^2}{\mathrm{d}t}=\xi_2^2,\\ \frac{\mathrm{d}\xi_2^2}{\mathrm{d}t}=-\omega(t)\xi_1^2.\end{cases}$$

The explicit and implicit superposition formulas are, respectively,

$$\begin{aligned}\begin{bmatrix}x_1\\x_2\end{bmatrix}&=k_1\begin{bmatrix}\xi_1^1\\ \xi_2^1\end{bmatrix}+k_2\begin{bmatrix}\xi_1^2\\ \xi_2^2\end{bmatrix},\\ \begin{bmatrix}k_1\\k_2\end{bmatrix}&=\begin{bmatrix}\frac{\xi_2^2x_1-\xi_1^2x_2}{\xi_1^1\xi_2^2-\xi_1^2\xi_2^1}\\ \frac{\xi_1^1x_2-\xi_2^1x_1}{\xi_1^1\xi_2^2-\xi_1^2\xi_2^1}\end{bmatrix}.\end{aligned}$$

■

In this paper, only time-varying nonlinear systems (1) that are sufficiently "close" to class (2) are considered: in particular, only those nonlinear systems that admit superposition formulas similar to (4) and (5).

Consider m particular solutions of (1), *i.e.*, m functions $\xi^i(t)\in\mathbb{R}^n$, $i=1,...,m$, such that

$$\frac{\mathrm{d}\xi^i(t)}{\mathrm{d}t}=f(t,\xi^i(t)),\quad i=1,...,m;\tag{7}$$

conditions on integer m and functions $\xi^i(t)$ will be given in the following.

Following Lie (see [13], [14]), equation (1) admits a *nonlinear superposition principle* if there exists a function $\Psi:\mathbb{R}^{n(m+1)}\to\mathbb{R}^n$ such that any solution $x(t)$ of (1) can be written for all t in a sufficiently small open interval $\mathcal{T}_{t_0}$ containing the initial time t_0 as

$$x(t)=\Psi\left(\xi^1(t),...,\xi^m(t),k\right),\tag{8}$$

where $k\in\mathbb{R}^n$ is constant; in particular, it is required that (8) computed at $t=t_0$ is locally invertible with respect to k, so that k can be expressed as a function of $x(t_0),\xi^1(t_0),...,\xi^m(t_0)$. It is worth pointing out that function Ψ does not depend explicitly on time t. Equation (8) is called an *explicit nonlinear superposition formula*. By the Implicit Function Theorem [15], the explicit superposition formula (8) can be locally inverted with respect to k, *i.e.*, there exists a function $\Theta:\mathbb{R}^{n(m+1)}\to\mathbb{R}^n$ such that the following equation holds for all $t\in\mathcal{T}_{t_0}$:

$$k=\Theta\left(x(t),\xi^1(t),...,\xi^m(t)\right);\tag{9}$$

equation (9) is called an *implicit nonlinear superposition formula*. In general, the implicit nonlinear superposition formula (9) holds on an open dense subset of $\mathbb{R}^{n(m+1)}$ rather than the whole $\mathbb{R}^{n(m+1)}$. It is worth pointing out that formula (9) is invariant with respect to any permutation of the $m+1$ vector arguments of Θ; for example, in case $m=1$, if $\Theta\left(x(t),\xi^1(t)\right)$ is a first integral, then $\Theta\left(\xi^1(t),x(t)\right)$ is a first integral too.

Example 2: Consider the single-input linear control system $\frac{\mathrm{d}x(t)}{\mathrm{d}t}=Ax(t)+Bu(t)$, $x(t)\in\mathbb{R}^n$, $A\in\mathbb{R}^{n\times n}$, $u(t)\in\mathbb{R}$, $B\in\mathbb{R}^n$; let $t_0=0$. Consider $m=n+1$ particular solutions $\xi^i(t)\in\mathbb{R}^n$, $i=0,...,n$, of such a control system, *i.e.*, such that $\frac{\mathrm{d}\xi^i(t)}{\mathrm{d}t}=A\xi^i(t)+Bu(t)$, $i=0,...,n$. Clearly, letting $\gamma^i(t)=\xi^i(t)-\xi^0(t)$, one has $\frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t}=A\gamma^i(t)$, $i=1,...,n$, and therefore letting $\Gamma:=\begin{bmatrix}\gamma^1 & ... & \gamma^n\end{bmatrix}$, one has $\frac{\mathrm{d}\Gamma(t)}{\mathrm{d}t}=A\Gamma(t)$, which yields $\Gamma(t)=\mathrm{e}^{At}\Gamma(0)$; this implies $\mathrm{e}^{At}=\Gamma(t)\Gamma^{-1}(0)$, under the assumption that $\det\left(\Gamma(0)\right)\neq 0$ (this is the condition to be satisfied in order that the particular solutions $\xi^0(t),...,\xi^n(t)$ can be used in the superposition formula). Finally, since $x(t)=\mathrm{e}^{At}c+\xi^0(t)$, for some constant $c\in\mathbb{R}^n$, the explicit and implicit nonlinear superposition formulas are, respectively, obtained:

$$\begin{aligned}x&=\xi^0+\begin{bmatrix}\xi^1-\xi^0 & ... & \xi^n-\xi^0\end{bmatrix}k\\ &=\xi^0+\sum_{i=1}^{n}\left(\xi^i-\xi^0\right)k_i,\end{aligned}$$

and

$$k=\begin{bmatrix}\xi^1-\xi^0 & ... & \xi^n-\xi^0\end{bmatrix}^{-1}\left(x-\xi^0\right).$$

■

The following theorem goes back to Lie [13].

Theorem 1: Equation (1) admits the superposition formulas (8), (9) if and only if

$$f(t,x)=\sum_{i=1}^{r}u_i(t)f_i(x),\tag{10}$$

where $u_i(t)\in\mathbb{R}$, $i=1,...,r$, are some functions of time and $f_1(x),...,f_r(x)\in\mathbb{R}^n$ are time-invariant vector functions such that the smallest Lie algebra over $\mathbb{R}$ that contains $f_1(x),...,f_r(x)$ is finite dimensional.

The interested reader is referred to [16] for a clear and modern proof of Theorem 1, whereas in [17] the ideas of the classical proof are sketched.

Remark 1: According to Theorem 1, for any time-varying linear system (2), one can write

$$A(t)x(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}A_{i,j}(t)M_{i,j},$$

where the n^2 matrices $M_{i,j}:=e_ie_j^\top$ constitute a basis of the matrix Lie algebra $\mathbb{R}^{n\times n}$. ■

Since functions Ψ and Θ appearing in the nonlinear superposition formulas (8) and (9) are independent of time, the expressions of Ψ and Θ do not depend on scalar functions $u_i(t) \in \mathbb{R}$, $i = 1, ..., r$, but only on vector functions $f_1(x), ..., f_r(x) \in \mathbb{R}^n$; two systems $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{r} u_i(t) f_i(x)$ and $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{r} v_i(t) f_i(x)$ are described by the same superposition formulas, by the arbitrarines of functions u_i and v_i.

By (9), it is easy to see that the entries Θ_i, $i = 1, ..., n$, of Θ are functionally independent first integrals of the extended system constituted by equations (1), (7), whence, by the arbitrariness of the scalar functions $u_i(t)$, they are functionally independent joint first integrals associated with the extended vector functions

$$f_{1,e} = \begin{bmatrix} f_1(x) \\ f_1(\xi^1) \\ \vdots \\ f_1(\xi^m) \end{bmatrix}, ..., f_{r,e} = \begin{bmatrix} f_r(x) \\ f_r(\xi^1) \\ \vdots \\ f_r(\xi^m) \end{bmatrix},$$

which certainly exist when m is taken sufficiently high, because $f_{1,e}, ..., f_{r,e}$ generate a finite dimensional Lie algebra over $\mathbb{R}$. In particular, taking into account that $f_{i,e}\left(x_o, \xi_o^1, ..., \xi_o^m\right) \in \mathbb{R}^{n(m+1)}$, if there exists a point $(x_o, \xi_o^1, ..., \xi_o^m)$ such that $f_{1,e}\left(x_o, \xi_o^1, ..., \xi_o^m\right), ..., f_{r,e}\left(x_o, \xi_o^1, ..., \xi_o^m\right)$ are linearly independent, then the number of first integrals associated with $f_{1,e}, ..., f_{r,e}$, being functionally independent about point $(x_o, \xi_o^1, ..., \xi_o^m)$, is $n(m+1) - r$, which must be greater than or equal to n, thus yielding the inequality $nm \geq r$. Some techniques for the computation in closed-form of first integrals can be found in [18], [19], [20] starting from the knowledge of Lie symmetries and in [21], where a computationally valid procedure is given.

Remark 2: Two operations preserve the structure of Lie algebra, whence the existence of nonlinear superposition formulas, although their expression in closed-form may change: a nonlinear transformation on the state x, and a linear transformation on the time functions u_i.

(2.1) Given a finite dimensional Lie algebra over $\mathbb{R}$ $\mathrm{span}_{\mathbb{R}}\{f_1, ..., f_r\}$ and a diffeomorphism $y = \varphi(x)$, ones has that $\mathrm{span}_{\mathbb{R}}\{\varphi_* f_1, ..., \varphi_* f_r\}$ is a finite dimensional Lie algebra over $\mathbb{R}$, characterized by the same characteristic constants as $\mathrm{span}_{\mathbb{R}}\{f_1, ..., f_r\}$ (here $\varphi_* f(y) = \left(\frac{\partial \varphi}{\partial x} f\right) \circ \varphi^{-1}(y)$ denotes the push-forward of f by φ).

(2.2) Given an invertible matrix $Q \in \mathbb{R}^{r \times r}$, $u = QV$, where $u = \begin{bmatrix} u_1 & ... & u_r \end{bmatrix}^\top$ and $v = \begin{bmatrix} v_1 & ... & v_r \end{bmatrix}^\top$, (10) can be recast as follows:

$$\begin{aligned} f(t,x) &= \sum_{i=1}^{r} u_i(t) f_i(x) \\ &= \sum_{i=1}^{r} \sum_{j=1}^{r} Q_{i,j} v_j(t) f_i(x) \\ &= \sum_{j=1}^{r} \sum_{i=1}^{r} Q_{i,j} f_i(x) v_j(t) \\ &= \sum_{j=1}^{r} v_j(t) g_j(x), \end{aligned}$$

where $g_j(x) := \sum_{i=1}^{r} Q_{i,j} f_i(x)$, $j = 1, ..., r$. The two Lie algebras $\mathrm{span}_{\mathbb{R}}\{f_1, ..., f_r\}$ and $\mathrm{span}_{\mathbb{R}}\{g_1, ..., g_r\}$ over $\mathbb{R}$ are isomorphic, but in general they are described by different structure constants. ■

Example 3: Consider the case $n = 1$. By [13], it is known that any Lie algebra over $\mathbb{R}$ spanned by scalar functions is at most three-dimensional. Hence, assume that $\mathfrak{X}$ is three-dimensional, *i.e.*, $\mathfrak{X} = \mathrm{span}_{\mathbb{R}}\{f_1, f_2, f_3\}$, with $\{f_1, f_2, f_3\}$ being a basis of $\mathfrak{X}$, and that $\mathrm{rank}_{\mathcal{K}_n}\{f_1, f_2, f_3\} = 1$. About a regular point of f_i, apart from a diffeomorphism, it can be assumed that $f_i = 1$; by $[1, g] = \frac{\partial g}{\partial x}$, one concludes that any g commuting with f_i satisfies $g = c f_i$, for some constant c. Therefore, it can be assumed that $[f_i, f_j]$ is not identically zero, because otherwise $\{f_1, f_2, f_3\}$ is not a basis of $\mathfrak{X}$. If f_1, f_2, f_3 are scalar functions of $x \in \mathbb{R}$, then it can be shown [13] (see also [22]) that, apart from a proper choice of the Lie algebra basis, the only Lie algebra satisfying the conditions $[f_i, f_j] \neq 0$, $i, j \in \{1, 2, 3\}$, $i \neq j$, is the split three-dimensional simple Lie algebra, described by the commutation relations

$$[f_1, f_2] = 2f_1, \quad [f_1, f_3] = f_2, \quad [f_2, f_3] = 2f_3.$$

Assume, apart from a diffeomorphism about any regular point, that $f_1(x) = 1$. Condition $[f_1, f_2] = 2f_1$ implies

$$\frac{\partial f_2(x)}{\partial x} = 2 \Longrightarrow f_2(x) = 2x + c_2;$$

condition $[f_1, f_3] = f_2$ implies

$$\frac{\partial f_3(x)}{\partial x} = 2x + c_2 \Rightarrow f_3(x) = x^2 + c_2 x + c_3;$$

condition $[f_2, f_3] = 2f_3$ implies

$$c_2^2 - 4c_3 = 0 \Rightarrow c_3 = \frac{1}{4} c_2^2;$$

therefore, $\{1, 2x + c_2, x^2 + c_2 x + \frac{1}{4} c_2^2\}$ is a basis of $\mathfrak{X}$; another basis of $\mathfrak{X}$ is $\{1, x, x^2\}$, which shows that any scalar differential equation, which admits nonlinear superposition formulas, is diffeomorphic to a scalar *Riccati differential equation* (see [10] and [23])

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = u_1(t) + u_2(t) x + u_3(t) x^2, \qquad (11)$$

with a proper choice of functions $u_i(t)$. Let $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$ and define the extended vector functions

$$f_{1,e} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad f_{2,e} = \begin{bmatrix} \xi^0 \\ \xi^1 \\ \xi^2 \\ \xi^3 \end{bmatrix}, f_{3,e} = \begin{bmatrix} \left(\xi^0\right)^2 \\ \left(\xi^1\right)^2 \\ \left(\xi^2\right)^2 \\ \left(\xi^3\right)^2 \end{bmatrix},$$

where $\xi^0 = x$. All first integrals associated with $f_{1,e}$ are given by arbitrary functions of $\xi^i - \xi^j$, for $i, j \in \{0, 1, 2, 3\}$; all first integrals associated with $f_{2,e}$ are given by arbitrary functions of $\frac{\xi^i - \xi^j}{\xi^h - \xi^k}$ for $i, j, h, k \in \{0, 1, 2, 3\}$, $(i, j) \neq (h, k)$; all first integrals associated with $f_{3,e}$ are given by arbitrary functions of $\frac{1}{\xi^i} - \frac{1}{\xi^j} = \frac{\xi^j - \xi^i}{\xi^i \xi^j}$, for $i, j \in \{0, 1, 2, 3\}$. Hence, these three vector functions admit as joint first integrals the arbitrary functions of the following quantity, which is often referred to as the *cross ratio*,

$$\Theta = \frac{(\xi^0 - \xi^1)(\xi^2 - \xi^3)}{(\xi^0 - \xi^2)(\xi^1 - \xi^3)}.$$

This gives the implicit nonlinear superposition formula $k = \Theta$; by solving such an equation by $x = \xi^0$, one obtains the explicit nonlinear superposition formula $x = \Psi$, with

$$\Psi = \frac{k\xi^2 \left(\xi^1 - \xi^3\right) - \xi^1 \left(\xi^2 - \xi^3\right)}{k \left(\xi^1 - \xi^3\right) - \left(\xi^2 - \xi^3\right)}.$$

Similar explicit nonlinear superposition formulas can be easily determined by a permutation of the solutions ξ^1, ξ^2, ξ^3; for instance, if the triplet (ξ^1, ξ^2, ξ^3) is replaced with the triplet (ξ^2, ξ^3, ξ^1), one obtains the explicit nonlinear superposition formula

$$x = \frac{k\xi^3 \left(\xi^2 - \xi^1\right) - \xi^2 \left(\xi^3 - \xi^1\right)}{k \left(\xi^2 - \xi^1\right) - \left(\xi^3 - \xi^1\right)}.$$

Now, consider a planar linear system $\dot{y} = A(t)y$, where $y \in \mathbb{R}^2$ and

$$A(t) = \begin{bmatrix} A_{1,1}(t) & A_{1,2}(t) \\ A_{2,1}(t) & A_{2,2}(t) \end{bmatrix}.$$

Since $[A(t), E] = 0$ for any $t \in \mathbb{R}$, consider the projection $x = \frac{y_1}{y_2}$, which transform $\dot{y} = A(t)y$ into

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \begin{bmatrix} \frac{1}{y_2} & -\frac{y_1}{y_2^2} \end{bmatrix} \begin{bmatrix} A_{1,1}(t) & A_{1,2}(t) \\ A_{2,1}(t) & A_{2,2}(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= A_{1,2}(t) + (A_{1,1}(t) - A_{2,2}(t)) \frac{y_1}{y_2} - A_{2,1}(t) \frac{y_1^2}{y_2^2} \\ &= A_{1,2}(t) + (A_{1,1}(t) - A_{2,2}(t))\, x - A_{2,1}(t) x^2, \end{aligned}$$

i.e., the Riccati differential equation (11) with $u_1(t) = A_{1,2}(t)$, $u_2(t) = A_{1,1}(t) - A_{2,2}(t)$ and $u_3(t) = -A_{2,1}(t)$. Therefore, this shows that any scalar differential equation that admits nonlinear superposition formulas can be immersed into a planar linear system (see [13]), thus justifying the assertion that the scalar nonlinear systems that admit nonlinear superposition formulas are "close" to the linear ones. ∎

If one of the scalar functions $u_i(t)$ appearing in (10) is identically zero, the computation of the explicit and implicit nonlinear superposition formulas can be simplified. Note that the explicit and implicit nonlinear superposition formulas, also modulo permutation of the particular solutions, are not unique.

IV. Physically motivated examples

Example 4: Consider again the linear oscillator with time-varying frequency (6). Define the vector functions

$$f_1(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}.$$

Compute the Lie bracket

$$[f_1(x), f_2(x)] = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$$

and let $f_3(x) = [f_1(x), f_2(x)]$. Since $[f_1, f_2] = f_3$, $[f_1, f_3] = -2f_1$ and $[f_2, f_3] = 2f_2$, $\mathfrak{X} = \mathrm{span}_{\mathbb{R}}\{f_1, f_2, f_3\}$ is a three-dimensional Lie algebra. Compute the extended vector functions

$$f_{1,e}(x, \xi^1, \xi^2) = \begin{bmatrix} x_2 \\ 0 \\ \xi_2^1 \\ 0 \\ \xi_2^2 \\ 0 \end{bmatrix}, f_{2,e}(x, \xi^1, \xi^2) = \begin{bmatrix} 0 \\ x_1 \\ 0 \\ \xi_1^1 \\ 0 \\ \xi_1^2 \end{bmatrix}$$

(there is no need to compute $f_{3,e}$). Two joint functionally independent first integrals associated with $f_{1,e}$ and $f_{2,e}$ are given by $x_1\xi_2^1 - \xi_1^1 x_2$ and $x_1\xi_2^2 - \xi_1^2 x_2$. The implicit superposition formula is

$$\begin{aligned} k_1 &= x_1\xi_2^1 - \xi_1^1 x_2, \\ k_2 &= x_1\xi_2^2 - \xi_1^2 x_2; \end{aligned}$$

by the inverse with respect to x, it is obtained the explicit superposition formula:

$$\begin{aligned} x_1 &= \frac{k_1\xi_1^2 - k_2\xi_1^1}{\xi_2^1\xi_1^2 - \xi_1^1\xi_2^2}, \\ x_2 &= \frac{k_1\xi_2^2 - k_2\xi_2^1}{\xi_2^1\xi_1^2 - \xi_1^1\xi_2^2}, \end{aligned}$$

under the assumption that $\det\left(\begin{bmatrix} \xi^1 & \xi^2 \end{bmatrix}\right) = \xi_1^1\xi_2^2 - \xi_2^1\xi_1^2$ is not identically zero. ∎

Example 5: (A knife edge) Consider the kinematic equations of motion of a *knife edge* [24]

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= \cos(x_3)u_1(t), \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= \sin(x_3)u_1(t), \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} &= u_2(t). \end{aligned}$$

Define

$$f_1(x) = \begin{bmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since

$$[f_1(x), f_2(x)] = \begin{bmatrix} \sin(x_3) \\ -\cos(x_3) \\ 0 \end{bmatrix},$$

define $f_3(x) = [f_1(x), f_2(x)]$. Since $[f_1, f_2] = f_3$, $[f_1, f_3] = 0$ and $[f_2, f_3] = -f_1$, one concludes that $\mathfrak{X} = \mathrm{span}_{\mathbb{R}}\{f_1, f_2, f_3\}$ is a three-dimensional Lie algebra over $\mathbb{R}$. Compute the extended vector functions

$$f_{1,e}(x, \xi^1) = \begin{bmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \\ \cos(\xi_3^1) \\ \sin(\xi_3^1) \\ 0 \end{bmatrix}, \quad f_{2,e}(x, \xi^1) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(there is no need to compute $f_{3,e}$). Three joint functionally independent first integrals associated with $f_{1,e}$ and $f_{2,e}$ are $x_1 - \xi_1^1 \cos\left(\xi_3^1 - x_3\right) - \xi_2^1 \sin\left(\xi_3^1 - x_3\right)$, $x_2 + \xi_1^1 \sin\left(\xi_3^1 - x_3\right) - \xi_2^1 \cos\left(\xi_3^1 - x_3\right)$ and $\left(x_3 - \xi_3^1\right)$, thus obtaining the implicit nonlinear superposition formula:

$$\begin{aligned} k_1 &= x_1 - \xi_1^1 \cos\left(\xi_3^1 - x_3\right) - \xi_2^1 \sin\left(\xi_3^1 - x_3\right), \\ k_2 &= x_2 + \xi_1^1 \sin\left(\xi_3^1 - x_3\right) - \xi_2^1 \cos\left(\xi_3^1 - x_3\right), \\ k_3 &= x_3 - \xi_3^1; \end{aligned}$$

by the inverse, it is obtained the explicit nonlinear superposition formula:

$$\begin{aligned} x_1 &= \xi_1^1 \cos(k_3) - \xi_2^1 \sin(k_3) + k_1, \\ x_2 &= \xi_1^1 \sin(k_3) + \xi_2^1 \cos(k_3) + k_2, \\ x_3 &= \xi_3^1 + k_3. \end{aligned}$$

■

Example 6: (Chained system) Consider a three-dimensional *chained system*: [24]

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= u_1(t), \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= u_2(t), \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} &= x_2 u_1(t). \end{aligned}$$

Define

$$f_1(x) = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Letting

$$f_3(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

it is easy to see that $[f_1, f_2] = -f_3$, $[f_1, f_3] = 0$ and $[f_2, f_3] = 0$, whence that $\mathfrak{X} = \mathrm{span}_{\mathbb{R}}\{f_1, f_2, f_3\}$ is a three-dimensional Lie algebra over $\mathbb{R}$. Compute the extended vector functions

$$f_{1,e}(x, \xi^1) = \begin{bmatrix} 1 \\ 0 \\ x_2 \\ 1 \\ 0 \\ \xi_2^1 \end{bmatrix}, \quad f_{2,e}(x, \xi^1) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(there is no need to compute $f_{3,e}$). Three joint functionally independent first integrals associated with $f_{1,e}$ and $f_{2,e}$ are given by $x_1 - \xi_1^1$, $x_2 - \xi_2^1$ and $x_3 - \xi_3^1 - \xi_1^1 x_2 + \xi_1^1 \xi_2^1$, thus obtaining the implicit nonlinear superposition formula:

$$\begin{aligned} k_1 &= x_1 - \xi_1^1, \\ k_2 &= x_2 - \xi_2^1, \\ k_3 &= x_3 - \xi_3^1 - \xi_1^1 x_2 + \xi_1^1 \xi_2^1; \end{aligned}$$

by the inverse, it is obtained the explicit nonlinear superposition formula:

$$\begin{aligned} x_1 &= \xi_1^1 + k_1, \\ x_2 &= \xi_2^1 + k_2, \\ x_3 &= \xi_3^1 + \xi_1^1 k_2 + k_3. \end{aligned}$$

■

Example 7: (DC-to-DC electric power conversion systems) Consider a DC-to-DC electric power conversion systems described by [2]

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= \frac{u(t) - 1}{L} x_2 + \frac{E}{L}, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -\frac{u(t) - 1}{L} x_1 - \frac{1}{RC} x_2, \end{aligned}$$

where the DC supply is E and the load resistance is R. The state variables are the current x_1 through the inductor L and the output voltage x_2 on the capacitor C; $u(t)$ is a piecewise constant function of time, $u(t) \in \{0, 1\}$. Since the system parameters E, L, R and C my be subject to time-varying uncertainties, it would be nice to obtain a superposition formula independent of them. Define

$$\begin{aligned} f_1(x) &= \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, f_2(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, f_3(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}, \\ f_4(x) &= \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, f_5(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, f_6(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \end{aligned}$$

which span a six-dimensional Lie algebra over $\mathbb{R}$, described by the commutation relations $[f_1, f_2] = -f_2$, $[f_1, f_3] = -f_3$, $[f_1, f_4] = 0$, $[f_1, f_5] = -f_5$, $[f_1, f_6] = 0$, $[f_2, f_3] = f_4 - f_1$, $[f_2, f_4] = -f_2$, $[f_2, f_5] = 0$, $[f_2, f_6] = -f_5$, $[f_3, f_4] = f_3$, $[f_3, f_5] = -f_6$, $[f_3, f_6] = 0$, $[f_4, f_5] = 0$, $[f_4, f_6] = -f_6$, $[f_5, f_6] = 0$. Proceeding as in the previous examples, explicit and implicit nonlinear superposition formulas are, respectively, obtained:

$$\begin{aligned} x_1 &= \xi_1^1 + \left(\xi_1^2 - \xi_1^1\right) k_1 + \left(\xi_1^3 - \xi_1^1\right) k_2, \\ x_2 &= \xi_2^1 + \left(\xi_2^2 - \xi_2^1\right) k_1 + \left(\xi_2^3 - \xi_2^1\right) k_2, \end{aligned}$$

and

$$
\begin{aligned}
k_1 &= \frac{-\xi_2^3 x_1 + \xi_1^1\xi_2^3 + \xi_2^1 x_1 + \xi_1^3 x_2 - \xi_1^3\xi_2^1 - \xi_1^1 x_2}{-\xi_1^2\xi_2^3 + \xi_1^2\xi_2^1 + \xi_1^1\xi_2^3 + \xi_1^3\xi_2^2 - \xi_1^3\xi_2^1 - \xi_1^1\xi_2^2},\\
k_2 &= \frac{\xi_2^2 x_1 - \xi_1^1\xi_2^2 - \xi_2^1 x_1 - \xi_1^2 x_2 + \xi_1^2\xi_2^1 + \xi_1^1 x_2}{-\xi_1^2\xi_2^3 + \xi_1^2\xi_2^1 + \xi_1^1\xi_2^3 + \xi_1^3\xi_2^2 - \xi_1^3\xi_2^1 - \xi_1^1\xi_2^2}.
\end{aligned}
$$

■

V. Conclusions

In this paper we have reviewed the nonlinear superposition principle for classes of nonlinear systems: the existence of a nonlinear superposition principle can be ensured by a Lie algebraic analysis of the nonlinear system, whereas the nonlinear superposition formulas can be obtained in closed form by computing some functionally independent first integrals of an auxiliary system that can be associated with the given one.

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