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An Improved Algorithm for Frequency Weighted Balanced Truncation

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Abstract— **In this paper, a new frequency weighted balanced truncation based on Lin and Chiu's technique [1] is presented. The method proposed is a modification to Sreeram and Sahlan's technique [2] by using the relationship between the intermediate reduced order model and the final reduced order model. A numerical example and comparison with other well-known techniques shows that a significant approximation error reduction can be achieved using this improvement.**

I. Introduction

Frequency weighted balanced truncation was first introduced by Enns [3] based on a modification of balanced truncation [4]. The method may use input weighting, output weighting or both. The stability of the reduced order model is guaranteed only when one weighting is present. To overcome the potential drawback of instability when both weightings are present, the original Lin and Chiu's technique [1] and its generalization [5] present a simple modification to Enns' technique provided that there are no pole-zero cancellations between the original system and the weights [6]. Another modification to Enns' technique was proposed by Wang et al. [7] which not only guarantees stability in the case of double-sided weightings but also yields simple and elegant error bounds. Although the stability of reduced order models are guaranteed, but the reduction errors obtained from [1], [5]–[7] are at best slightly lower than Enns' method and hence may be considered still too large in most applications.

Sreeram and Sahlan [2] improved Lin and Chiu's technique by decomposing the transformed augmented system in [1] into a new augmented system and new weights. Their method does not only guarantee stability of the original system in case of double-sided weightings, but also simple, elegant and easily computable error bounds. However, the method can only reduce the approximation error slightly compared to Enns' technique by varying free parameters introduced in the technique. This is because, even though theoretically the new weights obtained from their method are supposed to be inner/co-inner functions [8], but they are not as one of the conditions for inner/co-inner functions is missing in their lemma.

However, the method of [2] is mathematically correct. A modification to the technique gives a significant approximation error reduction as presented in this paper. The proposed method is applicable to single-sided system and is illustrated by an example.

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II. Preliminaries

This section reviews some of the well-known frequency weighted balanced truncation techniques. Let $G(s)$, $V(s)$ and $W(s)$ be the stable original system and the stable input and output weights respectively. Let $\{A,B,C,D\}$, $\{A_v,B_v,C_v,D_v\}$ and $\{A_w,B_w,C_w,D_w\}$ be their corresponding minimal realizations respectively. Consider the augmented system $G(s)V(s)$ and $W(s)G(s)$ represented by the following realizations:

$$\begin{aligned} G(s)V(s) &= \left[\begin{array}{cc|c} A & BC_v & BD_v \\ 0 & A_v & B_v \\ \hline C & DC_v & DD_v \end{array}\right] \\ &= \begin{bmatrix} \tilde{A}_i & \tilde{B}_i \\ \tilde{C}_i & \tilde{D}_i \end{bmatrix} \\ W(s)G(s) &= \left[\begin{array}{cc|c} A_w & B_wC & B_wD \\ 0 & A & B \\ \hline C_w & D_wC & D_wD \end{array}\right] \\ &= \begin{bmatrix} \tilde{A}_o & \tilde{B}_o \\ \tilde{C}_o & \tilde{D}_o \end{bmatrix}. \end{aligned}$$

The controllability and observability Gramians of the augmented realization are given by:

$$\tilde{P}_i = \begin{bmatrix} P_E & P_{12} \\ P_{12}^T & P_v \end{bmatrix} \quad \tilde{Q}_o = \begin{bmatrix} Q_w & Q_{12} \\ Q_{12}^T & Q_E \end{bmatrix} \tag{1}$$

where $\tilde{P}_i$ and $\tilde{Q}_o$ satisfy the following Lyapunov equations:

$$\tilde{A}_i\tilde{P}_i + \tilde{P}_i\tilde{A}_i^T + \tilde{B}_i\tilde{B}_i^T = 0 \tag{2a}$$
$$\tilde{A}_o^T\tilde{Q}_o + \tilde{Q}_o\tilde{A}_o + \tilde{C}_o^T\tilde{C}_o = 0. \tag{2b}$$

Assuming that there are no pole-zero cancellations in $G(s)V(s)$ and $W(s)G(s)$, the Gramians, $\tilde{P}_i$ and $\tilde{Q}_o$ are positive definite.

A. Enns' Technique

Expanding $(1,1)$ block of (2a) and $(2,2)$ block of (2b) yield the following equations:

$$AP_E + P_EA^T + X_E = 0 \tag{3a}$$
$$A^TQ_E + Q_EA + Y_E = 0 \tag{3b}$$

where

$$\begin{aligned} X_E &= BC_vP_{12}^T + P_{12}C_v^TB^T + BD_vD_v^TB^T \\ Y_E &= C^TB_w^TQ_{12} + Q_{12}^TB_wC + C^TD_w^TD_wC. \end{aligned}$$

Diagonalizing the weighted Gramians $\{P_E, Q_E\}$ yielding

$$T_E^{-1}P_ET_E^{-T} = T_E^TQ_ET_E = diag(\sigma_1,\sigma_2,\ldots,\sigma_r,\sigma_{r+1},\ldots,\sigma_n)$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > \sigma_{r+1} \geq \ldots \geq \sigma_n > 0$.

Transforming and partitioning the original system realization, we have

$$\left[\begin{array}{c|c} T_E^{-1}AT_E & T_E^{-1}B \\ \hline CT_E & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Enns' reduced-order model is then given by $G_E(s) = \{A_{11}, B_1, C_1, D\}$.

Essentially, Enns' technique is based on diagonalizing simultaneously the solutions of Lyapunov equations as given in (3). However, Enns' technique cannot guarantee the stability of reduced order models as X_E and Y_E may not be positive semidefinite. Several modifications to Enns' technique are proposed in the literature to overcome the stability problem.

B. Generalization of Lin and Chiu's Technique

The generalization of Lin and Chiu's technique [5] modified Enns' technique [3] using the following transformation matrices

$$T_i = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \quad T_o = \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix}$$

where

$$X = P_{12}P_v^{-1} \quad \text{and} \quad Y = Q_w^{-1}Q_{12}, \tag{4}$$

to transform the original Gramians of augmented system $\{\tilde{P}_i, \tilde{Q}_o\}$ in (1) into block diagonal matrices as shown below:

$$\begin{aligned} \hat{P}_i &= T_i^{-1}\tilde{P}_iT_i^{-T} = \left[\begin{array}{c|c} P_{LC} & 0 \\ \hline 0 & P_v \end{array}\right] \\ \hat{Q}_o &= T_o^T\tilde{Q}_oT_o = \left[\begin{array}{c|c} Q_w & 0 \\ \hline 0 & Q_{LC} \end{array}\right] \end{aligned}$$

where $P_{LC} = P_E - P_{12}P_v^{-1}P_{12}^T$ and $Q_{LC} = Q_E - Q_{12}^TQ_w^{-1}Q_{12}$.

The corresponding state-space realizations have the following structures:

$$\begin{aligned} \left[\begin{array}{c|c} \hat{A}_i & \hat{B}_i \\ \hline \hat{C}_i & \hat{D}_i \end{array}\right] &= \left[\begin{array}{c|c} T_i^{-1}\tilde{A}_iT_i & T_i^{-1}\tilde{B}_i \\ \hline \tilde{C}_iT_i & \tilde{D}_i \end{array}\right] \\ &= \left[\begin{array}{cc|c} A & X_{23} & X_2 \\ 0 & A_v & B_v \\ \hline C & Y_2 & DD_v \end{array}\right] \end{aligned} \tag{5a}$$

$$\begin{aligned} \left[\begin{array}{c|c} \hat{A}_o & \hat{B}_o \\ \hline \hat{C}_o & \hat{D}_o \end{array}\right] &= \left[\begin{array}{c|c} T_o^{-1}\tilde{A}_oT_o & T_o^{-1}\tilde{B}_o \\ \hline \tilde{C}_oT_o & \tilde{D}_o \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_w & X_{12} & X_1 \\ 0 & A & B \\ \hline C_w & Y_1 & D_wD \end{array}\right] \end{aligned} \tag{5b}$$

where

$$X_{23} = AX - XA_v + BC_v \tag{6a}$$
$$X_2 = BD_v - XB_v \tag{6b}$$
$$Y_2 = CX + DC_v \tag{6c}$$
$$\hat{D}_i = DD_v \tag{6d}$$
$$X_{12} = YA - A_wY + B_wC \tag{6e}$$
$$Y_1 = D_wC - C_wY \tag{6f}$$
$$X_1 = B_wD + YB \tag{6g}$$
$$\hat{D}_o = D_wD \tag{6h}$$

The new realizations $\{\hat{A}_i, \hat{B}_i\}$ and $\{\hat{A}_o, \hat{C}_o\}$ now satisfy the following Lyapunov equations:

$$\begin{aligned} \hat{A}_i\hat{P}_i + \hat{P}_i\hat{A}_i^T + \hat{B}_i\hat{B}_i^T &= 0 \\ \hat{A}_o^T\hat{Q}_o + \hat{Q}_o\hat{A}_o + \hat{C}_o^T\hat{C}_o &= 0 \end{aligned}$$

Diagonalizing the weighted Gramians $\{P_{LC}, Q_{LC}\}$ of the new system $\{A, X_2, Y_1\}$ which satisfy

$$\begin{aligned} AP_{LC} + P_{LC}A^T + X_2X_2^T &= 0 \\ A^TQ_{LC} + Q_{LC}A + Y_1^TY_1 &= 0 \end{aligned}$$

yielding

$$T_{LC}^{-1}P_{LC}T_{LC}^{-T} = T_{LC}^TQ_{LC}T_{LC} = diag(\sigma_1, \sigma_2, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n)$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > \sigma_{r+1} \geq \ldots \geq \sigma_n > 0$. Reduced order model is then obtained by transforming, partitioning and truncating the original system realization.

C. Influence of Cross-terms

As pointed out in [9], frequency weighted balanced truncation technique has large frequency weighted error due to nonzero P_{12} and Q_{12} in (1).

Lemma 2.1: [9] The class of input weight $V(s) = \{A_v, B_v, C_v, D_v\}$ corresponding to $P_{12} = 0$ has to satisfy the following two equations:

$$\begin{aligned} B(C_vP_v + D_vB_v^T) &= 0 \\ A_vP_v + P_vA_v^T + B_vB_v^T &= 0 \end{aligned}$$

The class of output weight $W(s) = \{A_w, B_w, C_w, D_w\}$ corresponding to $Q_{12} = 0$ has to satisfy the following two equations:

$$\begin{aligned} C^T(B_w^TQ_w + D_w^TC_w) &= 0 \\ A_w^TQ_w + Q_wA_w + C_w^TC_w &= 0 \end{aligned}$$

Lemma 2.2: [8] $V(s) = \{A_v, B_v, C_v, D_v\}$ is a co-inner function ($V(s)V^*(s) = I$) if and only if

$$\begin{aligned} A_vP_v + P_vA_v^T + B_vB_v^T &= 0 \\ C_vP_v + D_vB_v^T &= 0 \\ D_vD_v^T &= I \end{aligned}$$

Similarly, $W(s) = \{A_w, B_w, C_w, D_w\}$ is an inner function ($W^*(s)W(s) = I$) if and only if

$$\begin{aligned} A_w^TQ_w + Q_wA_w + C_w^TC_w &= 0 \\ B_w^TQ_w + D_w^TC_w &= 0 \\ D_w^TD_w &= I \end{aligned}$$

Note that $V^*(s)$ and $W^*(s)$ are used to denote the complex conjugate transpose of $V(s)$ and $W(s)$ respectively.

Remark 1: The matrices functions $V(s)$ and $W(s)$ need not be square to be co-inner/inner function. If the co-inner and inner matrices functions ($V(s)$ and $W(s)$) are square then they satisfy the following:

$$V(s)V^*(s) = V^*(s)V(s) = W^*(s)W(s) = W(s)W^*(s) = I$$

which implies they are all-pass functions.

D. Sreeram and Sahlan's Technique

Sreeram and Sahlan's technique [2] improved Lin and Chiu's technique [5] using the properties of zero cross-terms (see *Lemma 2.1*) and inner/co-inner function (see *Lemma 2.2*). In [2], the third equations ($D_v D_v^T = I$ and $D_w^T D_w = I$) of *Lemma 2.2* are missing. Satisfying only the first two equations of the lemma yielding $V(s)V^*(s) = D_v D_v^T$ and $W^*(s)W(s) = D_w^T D_w$, but they are not equal to identity matrices implies they are not inner/co-inner functions.

Considering new conditions in *Lemma 2.2*, factorization becomes harder than before to satisfy all the equations in the lemma. So far, there is no way to decompose the original augmented system into inner/co-inner function. Using special weights (C_v and B_w are square and nonsingular, and $D_v = D_w = 0$), [10] showed that the original model reduction problem $W(s)(G(s) - G_r(s))$ and $(G(s) - G_r(s))V(s)$ can be decomposed into $\overline{W}(s)(\overline{G}_o(s) - \overline{G}_{r,o}(s))$ and $(\overline{G}(s) - \overline{G}_{r,i}(s))\overline{V}(s)$ respectively where $\overline{V}(s)$ and $\overline{W}(s)$ are all-pass functions (see *Remark 1*).

Considering the factors mentioned above, the mathematical derivation of Sreeram and Sahlan's technique [2] is modified here to single-sided case. In [2], they present an improved Lin and Chiu's technique by decomposing the transformed augmented systems $G(s)V(s)$ and $W(s)G(s)$ in (5) into new augmented systems as follows:

$$G(s)V(s) = \overline{G}_i(s)\overline{V}(s) \quad (7a)$$
$$W(s)G(s) = \overline{W}(s)\overline{G}_o(s) \quad (7b)$$

where $\overline{G}_i(s) = \{A, \overline{B}, C, \overline{D}_i\}$ and $\overline{G}_o(s) = \{A, B, \overline{C}, \overline{D}_o\}$ are the new original systems and the new weights $\overline{V}(s) = \{A_v, B_v, \overline{C}_v, \overline{D}_v\}$ and $\overline{W}(s) = \{A_w, \overline{B}_w, C_w, \overline{D}_w\}$. In the new weights, $\{A_v, B_v, \overline{C}_v, \overline{D}_v\}$ and $\{A_w, \overline{B}_w, C_w, \overline{D}_w\}$ the cross terms $P_{12} = 0$ and $Q_{12} = 0$ respectively.

The new parameters in the above equations are given by

$$\overline{B} = \begin{bmatrix} B & -X & AX \end{bmatrix} \quad (8a)$$
$$\overline{D}_i = \begin{bmatrix} D & 0 & CX \end{bmatrix} \quad (8b)$$
$$\overline{C}_v = \begin{bmatrix} C_v \\ A_v \\ I \end{bmatrix} \quad (8c)$$
$$\overline{D}_v = \begin{bmatrix} D_v \\ B_v \\ 0 \end{bmatrix} \quad (8d)$$
$$\overline{C} = \begin{bmatrix} C \\ -Y \\ YA \end{bmatrix} \quad (8e)$$
$$\overline{D}_o = \begin{bmatrix} D \\ 0 \\ YB \end{bmatrix} \quad (8f)$$
$$\overline{B}_w = \begin{bmatrix} B_w & A_w & I \end{bmatrix} \quad (8g)$$
$$\overline{D}_w = \begin{bmatrix} D_w & C_w & 0 \end{bmatrix} \quad (8h)$$

Using the matrices defined in (8), the equations in (6) can now be expressed as:

$$X_{23} = \overline{B}\,\overline{C}_v \quad (9a)$$
$$X_2 = \overline{B}\,\overline{D}_v \quad (9b)$$
$$Y_2 = \overline{D}_i\overline{C}_v \quad (9c)$$
$$\hat{D}_i = \overline{D}_i\overline{D}_v \quad (9d)$$
$$X_{12} = \overline{B}_w\overline{C} \quad (9e)$$
$$Y_1 = \overline{D}_w\overline{C} \quad (9f)$$
$$X_1 = \overline{B}_w\overline{D}_o \quad (9g)$$
$$\hat{D}_o = \overline{D}_w\overline{D}_o \quad (9h)$$

Diagonalizing the weighted Gramians $\{\overline{P}, \overline{Q}\}$ of the new system $\{A, \overline{B}, \overline{C}\}$ which satisfy

$$A\overline{P} + \overline{P}A^T + \overline{B}\,\overline{B}^T = 0 \quad (10a)$$
$$A^T\overline{Q} + \overline{Q}A + \overline{C}^T\overline{C} = 0 \quad (10b)$$

yielding

$$T_{SS}^{-1}\overline{P}T_{SS}^{-T} = T_{SS}^T\overline{Q}T_{SS} = diag(\sigma_1, \sigma_2, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n)$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > \sigma_{r+1} \geq \ldots \geq \sigma_n > 0$. Instead of reducing $G(s)$, the technique reduce the new original system $\overline{G}_x(s)$ by balanced truncation to obtain an r^{th} intermediate reduced-order model $\overline{G}_{r,x}(s)$ where $x = i, o$ depending on input or output weighting. The final reduced-order model $G_{r,x}(s)$ is obtained by simply deleting the extra rows in $\overline{C}_{r,o}$ and $\overline{D}_{r,o}$, and extra columns in $\overline{B}_{r,i}$ and $\overline{D}_{r,i}$. Although the method is simple and elegant, approximation error reduction obtained from this technique is very small and is often negligible.

As disscussed in the next section, the method is modified such that a significant approximation error reduction can be achieved.

III. Main Results

Instead of deleting extra rows or/and columns as in [2], the proposed method computes the final reduced order model $G_{r,x}(s)$, from the following equation:

$$G_{r,i}(s)V(s) = \overline{G}_{r,i}(s)\overline{V}(s) \quad (11a)$$
$$W(s)G_{r,o}(s) = \overline{W}(s)\overline{G}_{r,o}(s) \quad (11b)$$

Let $G_{r,x}(s) = C_{r,x}(sI - A_{r,x})^{-1}B_{r,x} + D_{r,x}$ and $D_{r,x} = D$, then the augmented systems $G_{r,i}(s)V(s)$ and $W(s)G_{r,o}(s)$ are given by:

$$G_{r,i}(s)V(s) = \left[\begin{array}{cc|c} A_{r,i} & B_{r,i}C_v & B_{r,i}D_v \\ 0 & A_v & B_v \\ \hline D_w C_{r,i} & DC_v & DD_v \end{array}\right] = \begin{bmatrix} \tilde{A}_{r,i} & \tilde{B}_{r,i} \\ \tilde{C}_{r,i} & \tilde{D}_{r,i} \end{bmatrix}$$

$$W(s)G_{r,o}(s) = \left[\begin{array}{cc|c} A_w & B_w C_{r,o} & B_w D \\ 0 & A_{r,o} & B_{r,o} \\ \hline C_w & D_w C_{r,o} & D_w D \end{array}\right] = \begin{bmatrix} \tilde{A}_{r,o} & \tilde{B}_{r,o} \\ \tilde{C}_{r,o} & \tilde{D}_{r,o} \end{bmatrix}$$

where the Gramians

$$\tilde{P}_{r,i} = \begin{bmatrix} P_{11,r} & P_{12,r} \\ P_{12,r}^T & P_v \end{bmatrix} \quad \tilde{Q}_{r,o} = \begin{bmatrix} Q_w & Q_{12,r} \\ Q_{12,r}^T & Q_{22,r} \end{bmatrix}$$

satisfy the following:

$$\begin{aligned} \tilde{A}_{r,i}\tilde{P}_{r,i} + \tilde{P}_{r,i}\tilde{A}_{r,i}^T + \tilde{B}_{r,i}\tilde{B}_{r,i}^T &= 0 \\ \tilde{A}_{r,o}^T\tilde{Q}_{r,o} + \tilde{Q}_{r,o}\tilde{A}_{r,o} + \tilde{C}_{r,o}^T\tilde{C}_{r,o} &= 0 \end{aligned}$$

Similar to (7), $G_{r,i}(s)V(s)$ and $W(s)G_{r,o}(s)$ can also be decomposed into $\overline{G}_{r,i}(s)\overline{V}(s)$ and $\overline{W}(s)\overline{G}_{r,o}(s)$ respectively, where $\overline{G}_{r,x}(s)$ is an r^{th} order model of $\overline{G}_x(s)$.

Let $T_{r,i} = \begin{bmatrix} I & X_r \\ 0 & I \end{bmatrix}$ and $T_{r,o} = \begin{bmatrix} I & -Y_r \\ 0 & I \end{bmatrix}$ be the transformation matrices required to take the Gramians $\{\tilde{P}_{r,i}, \tilde{Q}_{r,o}\}$ into block diagonal matrices as follows:

$$\begin{aligned} \hat{P}_{r,i} &= T_{r,i}^{-1}\tilde{P}_{r,i}T_{r,i}^{-T} = \begin{bmatrix} \hat{P}_{11,r} & 0 \\ 0 & P_v \end{bmatrix} \\ \hat{Q}_{r,o} &= T_{r,o}^T\tilde{Q}_{r,o}T_{r,o} = \begin{bmatrix} Q_w & 0 \\ 0 & \hat{Q}_{22,r} \end{bmatrix} \end{aligned}$$

then the corresponding state-space realizations can be written as:

$$\begin{aligned} G_{r,i}(s)V(s) &= \left[\begin{array}{c|c} T_{r,i}^{-1}\tilde{A}_{r,i}T_{r,i} & T_{r,i}^{-1}\tilde{B}_{r,i} \\ \tilde{C}_{r,i}T_{r,i} & \tilde{D}_{r,i} \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_{r,i} & X_{23,r} & X_{2,r} \\ 0 & A_v & B_v \\ \hline C_{r,i} & Y_{2,r} & DD_v \end{array}\right] \end{aligned} \tag{12a}$$

$$\begin{aligned} W(s)G_{r,o}(s) &= \left[\begin{array}{c|c} T_{r,o}^{-1}\tilde{A}_{r,o}T_{r,o} & T_{r,o}^{-1}\tilde{B}_{r,o} \\ \tilde{C}_{r,o}T_{r,o} & \tilde{D}_{r,o} \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_w & X_{12,r} & X_{1,r} \\ 0 & A_{r,o} & B_{r,o} \\ \hline C_w & Y_{1,r} & D_wD \end{array}\right] \end{aligned} \tag{12b}$$

where

$$\begin{aligned} X_{23,r} &= A_{r,i}X_r - X_rA_v + B_{r,i}C_v \\ X_{2,r} &= B_{r,i}D_v - X_rB_v \\ Y_{2,r} &= C_{r,i}X_r + DC_v \\ X_{12,r} &= Y_rA_{r,o} - A_wY_r + B_wC_{r,o} \\ Y_{1,r} &= D_wC_{r,o} - C_wY_r \\ X_{1,r} &= B_wD + Y_rB_{r,o} \end{aligned}$$

From (7), we can obtain $\overline{W}(s), \overline{V}(s)$ and $\overline{G}_x(s)$. Similar to [2], an r^{th} intermediate reduced order model $\overline{G}_{r,x}(s)$ can then be computed directly using balanced truncation method [4]. Let $\overline{G}_{r,i}(s) = \{A_{r,i}, \overline{B}_{r,i}, C_{r,i}, \overline{D}_{r,i}\}$ and $\overline{G}_{r,o}(s) = \{A_{r,o}, B_{r,o}, \overline{C}_{r,o}, \overline{D}_{r,o}\}$ be the intermediate reduced order model obtained from $\overline{G}_x(s)$, then we can write the augmented systems as

$$\begin{aligned} \overline{G}_{r,i}(s)\overline{V}(s) &= \left[\begin{array}{c|c} A_{r,i} & \overline{B}_{r,i} \\ \hline C_{r,i} & \overline{D}_{r,i} \end{array}\right]\left[\begin{array}{c|c} A_v & B_v \\ \hline \overline{C}_v & \overline{D}_v \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_{r,i} & \overline{B}_{r,i}\overline{C}_v & \overline{B}_{r,i}\overline{D}_v \\ 0 & A_v & B_v \\ \hline C_{r,i} & \overline{D}_{r,i}\overline{C}_v & \overline{D}_{r,i}\overline{D}_v \end{array}\right] \end{aligned} \tag{13a}$$

$$\begin{aligned} \overline{W}(s)\overline{G}_{r,o}(s) &= \left[\begin{array}{c|c} A_w & \overline{B}_w \\ \hline C_w & \overline{D}_w \end{array}\right]\left[\begin{array}{c|c} A_{r,o} & B_{r,o} \\ \hline \overline{C}_{r,o} & \overline{D}_{r,o} \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_w & \overline{B}_w\overline{C}_{r,o} & \overline{B}_w\overline{D}_{r,o} \\ 0 & A_{r,o} & B_{r,o} \\ \hline C_w & \overline{D}_w\overline{C}_{r,o} & \overline{D}_w\overline{D}_{r,o} \end{array}\right] \end{aligned} \tag{13b}$$

Equating equations (12) and (13) gives

$$X_{23,r} = \overline{B}_{r,i}\overline{C}_v \tag{14a}$$
$$X_{2,r} = \overline{B}_{r,i}\overline{D}_v \tag{14b}$$
$$X_{12,r} = \overline{B}_w\overline{C}_{r,o} \tag{14c}$$
$$Y_{1,r} = \overline{D}_w\overline{C}_{r,o} \tag{14d}$$
$$Y_{2,r} = \overline{D}_{r,i}\overline{C}_v \tag{14e}$$
$$DD_v = \overline{D}_{r,i}\overline{D}_v \tag{14f}$$
$$X_{1,r} = \overline{B}_w\overline{D}_{r,o} \tag{14g}$$
$$D_wD = \overline{D}_w\overline{D}_{r,o} \tag{14h}$$

Rewriting the first four of (14) we get

$$X_{23,r} = A_{r,i}X_r - X_rA_v + B_{r,i}C_v = \overline{B}_{r,i}\overline{C}_v \tag{15a}$$
$$X_{2,r} = B_{r,i}D_v - X_rB_v = \overline{B}_{r,i}\overline{D}_v \tag{15b}$$
$$X_{12,r} = Y_rA_{r,o} - A_wY_r + B_wC_{r,o} = \overline{B}_w\overline{C}_{r,o} \tag{15c}$$
$$Y_{1,r} = D_wC_{r,o} - C_wY_r = \overline{D}_w\overline{C}_{r,o} \tag{15d}$$

In (15), the matrices $\{A_{r,i}, \overline{B}_{r,i}, A_{r,o}, \overline{C}_{r,o}\}$ are obtained from the intermediate reduced order model $\overline{G}_{r,x}(s)$. Solving (15) for $X_r, B_{r,i}, Y_r, C_{r,o}$ one can obtain $G_{r,i}(s) = \{A_{r,i}, B_{r,i}, C_{r,i}, D\}$ or $G_{r,o}(s) = \{A_{r,o}, B_{r,o}, C_{r,o}, D\}$ depending on input or output weighting.

Note that, since $D_{r,x} = D$, and to ensure the last four of (14) are satisfied, the matrices $\overline{D}_{r,i}$ and $\overline{D}_{r,o}$ are defined as:

$$\begin{aligned} \overline{D}_{r,i} &= \begin{bmatrix} D & 0 & C_{r,i}X_r \end{bmatrix} \\ \overline{D}_{r,o} &= \begin{bmatrix} D \\ 0 \\ Y_rB_{r,o} \end{bmatrix} \end{aligned}$$

using the matrices obtained from (15).

To solve the equations (15c) and (15d), we can rewrite them as

$$\begin{bmatrix} -I\otimes A_w + A_{r,o}^T \otimes I & I\otimes B_w \\ -I\otimes C_w & I\otimes D_w \end{bmatrix}\begin{bmatrix} Vec(Y_r) \\ Vec(C_{r,o}) \end{bmatrix} = \begin{bmatrix} Vec(\overline{B}_w\overline{C}_{r,o}) \\ Vec(\overline{D}_w\overline{C}_{r,o}) \end{bmatrix} \tag{16}$$

where $Vec(X)$ denotes the vector formed by stacking the columns of X into one long vector. The coefficient matrix on the left of the above equation has full rank, guaranteeing solvability of the equation when

$$\begin{bmatrix} -A_w + \lambda I & B_w \\ -C_w & D_w \end{bmatrix}$$

has full rank for all $\lambda = \lambda_i(A_{r,o}), i = 1, \ldots, r$ [11], where $\lambda(X)$ denotes the eigenvalues of X. However, there is a unique solution if and only if the matrix on the left of (16) is square. Similarly X_r and $B_{r,i}$, provided they exist, are uniquely determined if and only if $V(s)$ is square.

Remark 2: The condition that

$$\begin{bmatrix} -A_w + \lambda I & B_w \\ -C_w & D_w \end{bmatrix}$$

has full rank at some λ_i is effectively a condition that $W(\lambda_i)$ has full rank there. This observation follows immediately from the identity:

$$\begin{aligned} &\begin{bmatrix} -A_w + \lambda I & B_w \\ -C_w & D_w \end{bmatrix} \\ = &\begin{bmatrix} I & 0 \\ C_w(A_w - \lambda_i I)^{-1} & I \end{bmatrix} \begin{bmatrix} -A_w + \lambda I & B_w \\ 0 & W(\lambda_i) \end{bmatrix}. \end{aligned}$$

We say effectively, since there remains open the possibility that $W(s)$ could have a pole at λ_i. A similar remark applies to the input weight $V(\lambda_i)$.

Remark 3: Note that if the weights $W(s)$ and $V(s)$ have full row and column rank respectively, the requirement for them to have this property for the particular values of $\lambda = \lambda_i(A_{r,o})$ will be generally satisfied.

Theorem 3.1: If $G(s) = \{A,B,C,D\}$ is stable and minimal then the final reduced order model $G_{r,x}(s)$ obtained from the proposed method is also stable and minimal.

Proof: It has been proven in [2] that for a stable and minimal original system $G(s) = \{A,B,C,D\}$, the new realization $\overline{G}_x(s)$ is also stable and minimal. Since $\overline{G}_{r,x}(s)$ is obtained by balanced truncation of $\overline{G}_x(s)$, stability of $\overline{G}_{r,x}(s)$ follows immediately. As a result, $G_{r,x}(s)$ which is the reduced order model obtained using the proposed technique is also guaranteed to be stable for stable original systems as it has the same $A_{r,x}$ as $\overline{G}_{r,x}(s)$.

Theorem 3.2: If $G_{r,x}(s)$ is an r^{th} order model of the given original system $G(s)$ and $\overline{G}_{r,x}(s)$ is an r^{th} order model of the new system $\overline{G}_x(s)$, then

$$\begin{aligned} \|(G(s) - G_{r,i}(s))V(s)\|_\infty &= \left\|(\overline{G}_i(s) - \overline{G}_{r,i}(s))\overline{V}(s)\right\|_\infty \\ \|W(s)(G(s) - G_{r,o}(s))\|_\infty &= \left\|\overline{W}(s)(\overline{G}_o(s) - \overline{G}_{r,o}(s))\right\|_\infty \end{aligned}$$

Proof: From (7), we have

$$\begin{aligned} G(s)V(s) &= \overline{G}_i(s)\overline{V}(s) && (17a) \\ W(s)G(s) &= \overline{W}(s)\overline{G}_o(s) && (17b) \end{aligned}$$

From (11) we also have

$$\begin{aligned} G_{r,i}(s)V(s) &= \overline{G}_{r,i}(s)\overline{V}(s) && (18a) \\ W(s)G_{r,o}(s) &= \overline{W}(s)\overline{G}_{r,o}(s) && (18b) \end{aligned}$$

Substracting (18) from (17) we have

$$\begin{aligned} (G(s) - G_{r,i}(s))V(s) &= (\overline{G}_i(s) - \overline{G}_{r,i}(s))\overline{V}(s) \\ W(s)(G(s) - G_{r,o}(s)) &= \overline{W}(s)(\overline{G}_o(s) - \overline{G}_{r,o}(s)) \end{aligned}$$

Corollary 1:

$$\begin{aligned} \|(G(s) - G_{r,i}(s))V(s)\|_\infty &= \left\|(\overline{G}_i(s) - \overline{G}_{r,i}(s))\overline{V}(s)\right\|_\infty \\ &\leq 2\left\|\overline{V}(s)\right\|_\infty \sum_{i=r+1}^{n} \sigma_i \end{aligned}$$

$$\begin{aligned} \|W(s)(G(s) - G_{r,o}(s))\|_\infty &= \left\|\overline{W}(s)(\overline{G}_o(s) - \overline{G}_{r,o}(s))\right\|_\infty \\ &\leq 2\left\|\overline{W}(s)\right\|_\infty \sum_{i=r+1}^{n} \sigma_i \end{aligned}$$

where σ_i are the singular values of $\overline{G}_x(s)$.

Remark 4: If the reduced order model $G_{r,x}(s)$ is obtained without frequency weighting, then $V(s) = W(s) = I$. The following result of [3], [12] can be obtained easily:

$$\|(G(s) - G_r(s))\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_i.$$

Algorithm A step-by-step algorithm for the proposed method can be obtained as follows:

i) Given a stable and minimal $G(s)$ and $V(s)$ solve (2a) for the Gramians $\tilde{P}$.
ii) Compute X from (4).
iii) Compute the fictitious input and output matrices $\overline{B}$ from (8a).
iv) Calculate the transformation matrix, T_i which balance $\{A,\overline{B},C\}$ to diagonalize the Gramians:
$$T^{-1}\overline{P}T^{-T} = T^T QT = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$
v) Compute the frequency weighted balanced realization
$$\left[\begin{array}{c|c} T^{-1}AT & T^{-1}\overline{B} \\ \hline CT & \overline{D}_i \end{array}\right] = \left[\begin{array}{cc|c} A_{r,i} & A_{12} & \overline{B}_{r,i} \\ A_{21} & A_{22} & B_2 \\ \hline C_{r,i} & C_2 & \overline{D}_{r,i} \end{array}\right].$$
vi) Solve (15a) to (15b) for $B_{r,i}$.
vii) An r^{th} order model is given by $G_{r,i}(s) = \{A_{r,i}, B_{r,i}, C_{r,i}, D\}$.
viii) Calculate the weighted error =
$$\|(G(s) - G_{r,i}(s))V(s)\|_\infty$$

Remark 5: To reduce the approximation error, the matrices $\overline{B}$ and $\overline{C}$ used in the proposed algorithm can be made to be functions of free parameter α as follows:

$$\overline{B} = \begin{bmatrix} B & -\alpha X & AX \end{bmatrix}$$

To ensure that equations in (9) are valid, we need to have

$$\overline{C}_v = \begin{bmatrix} C_v \\ \frac{A_v}{\alpha} \\ I \end{bmatrix} \qquad \overline{D}_v = \begin{bmatrix} D_v \\ \frac{B_v}{\alpha} \\ 0 \end{bmatrix}$$

Note that, α can be any scalar values other than zeros. By varying the scalar α, one can easily reduce the weighted approximation errors.

Remark 6: Similar to [2], [7] the proposed method is realization dependent. For different realization of input and output weights, different reduced order models and weighted approximation errors are obtained.

IV. Example

For comparison purposes, we consider the fourth-order system used in [1], [2], [5], [7] with the following input weight [1]:

$$V(s) = \frac{4.5}{s+4.5} I_2$$

TABLE I

WEIGHTED ERRORS FOR SINGLE-SIDED

	Enns' [3]	LC's [5]	VA's [6]	Wang's [7]	SS's [2]		Proposed	
Order	Error	Error	Error	Error	α_{SS}	Error	α	Error
1	0.5568	0.5605	0.5545	0.5556	4	0.5545	200	0.5368
2	0.0620	0.0619	0.0616	0.0663	54	0.0619	150	0.0602
3	0.0322	0.0314	0.0319	0.0326	39	0.0314	3	0.0285

where I_2 denotes a 2^{nd} order of identity matrix. The maximum singular value of input weight $V(s)$ is given in Fig. 1. From the figure, we can see that the considered weighting function is a low pass filter with the passband frequency is all frequencies lower than 4.5 rad/s.

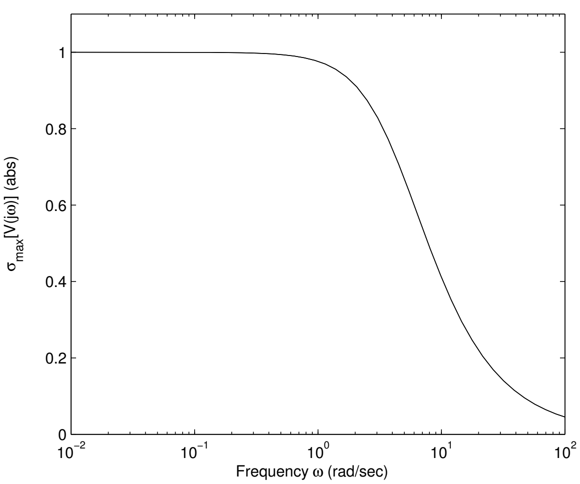

Fig. 1. Maximum singular value of input weight V(s)

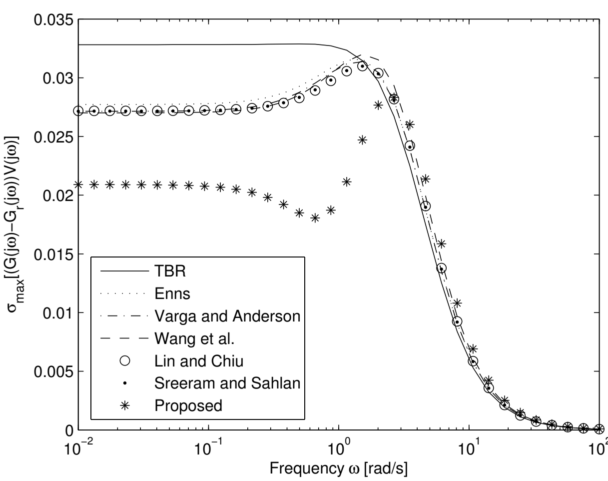

Fig. 2. Maximum singular value of reduction error for various methods

Table I shows the reduction errors $\|(G(s)-G_r(s))V(s)\|_\infty$ obtained from the proposed method and other existing techniques. It is clear from the table that the proposed technique gives the lowest errors compared to other well-known techniques. If we plot the same figure as in Fig. 5 of [1], we can have a Fig. 2. The figure shows clearly that the proposed method gives a significant improvement to existing techniques in the selected band of frequencies.

V. CONCLUSIONS

An improved frequency weighted balanced truncation based on Lin and Chiu's technique is presented. The method modified the technique in [2] using the relationship between the intermediate and the final reduced order model. By varying user chosen free parameter, the example indicates a significant improvement over the existing techniques [1]–[3], [5]–[7].

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