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The Use of Energy Constraints within Filtering and Smoothing

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Abstract—**This paper considers estimation problems where the energy of a linear system's output is constrained. A low-cost filtering procedure is proposed in which the input measurements are constrained by nonlinear censuring functions. A similar smoothing procedure for off-line applications is also described. It is shown that the filtered and smoothed output estimates satisfy a performance bound. It is also established that the resulting estimates are unbiased, and the estimation errors are bounded.**

I. INTRODUCTION

Constraints often appear within navigation problems. For example, vehicle trajectories are typically constrained by road, tunnel and bridge boundaries. Similarly, indoor pedestrian trajectories are constrained by walls and doors. However, as constraints are not easily described within state-space frameworks, many techniques for constrained filtering and smoothing are reported in the literature [1] – [18].

A common technique for constrained filtering involves augmenting the measurement vector with perfect observations – see the discussion [1, p. 248]. The application of the perfect-measurement approach to filtering and fixed-interval smoothing is described in [2]. Generalizations of this approach for time-varying problems are reported in [3] – [4]. Alternative proofs showing that the estimated states satisfy the constraints are presented in [5] - [6]. The use of perfect measurements can lead to singular state covariance matrices [1] which prompted the use of "nearly perfect" observations within an iterated extended Kalman filter [7] and nonzero measurement covariances within a linear filter [8], [9].

Constraints can be applied to state estimates, see [10], where a positivity constraint is used within a Kalman filter and a fixed lag smoother. Three different state equality constraint approaches, namely, maximum-probability, mean-square and projection methods are described in [11]. It can be shown that the constrained state estimates are unbiased [11] - [12]. It is established in [13] that if the states are known to be constrained in the null space of a constraint matrix then the Kalman predictor for a constrained system provides an error variance improvement. In [14], the projection method is used to apply nonlinear equality constraints. Under prescribed conditions, the perfect-measurement and projection approaches are equivalent [15] - [18].

In the state equality constrained methods [15] – [18], a constrained estimate can be calculated from a Kalman filter's unconstrained estimate at each time step. We seek simpler, low-computation-cost techniques suitable for implementation within real-time navigation systems. To this end, an on-line procedure is proposed that involves using nonlinear functions to constrain the measurements and subsequently applying the minimum-variance filter recursions [19]. An off-line procedure for retrospective analyses is also presented, where the minimum-variance fixed-interval smoother [20] – [23] is applied to the constrained measurements. In contrast to the afore-mentioned techniques [1] – [18], which employ constraint matrices and vectors, here constraint information is represented by an exogenous input process. This approach enables the nonlinearities to be designed so that the filtered and smoothed estimates satisfy a performance bound.

The remainder of this paper is organized as follows. The notation is introduced in Section IIA, suitable nonlinear censuring functions are discussed in Section IIB and the problem of interest is defined in Section IIC. Sections IID and IIE describe the constrained filtering and smoothing procedures, respectively. It is shown that the filtered and smoothed estimates are unbiased and the estimation errors are bounded. The conclusions follow in Section III.

II. CONSTRAINED FILTERING AND SMOOTHING

A. Notation

Let $w_k = \begin{bmatrix} w_{1,k} & \dots & w_{m,k} \end{bmatrix}^T \in \mathbb{R}^m$ represent a stochastic white input process, with

$$E\{w_k\} = 0, \quad E\{w_j w_k^T\} = Q_k \delta_{j,k}, \tag{1}$$

in which $\delta_{j,k}$ denotes the Kronecker delta function. It is convenient to abbreviate the entire record of w_k over the interval $k \in [0, N]$ by the stacked vector $w = [w_1^T \dots w_N^T]^T$.

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The two-norm of w is defined as $\|w\|_2 = \left(w^T w\right)^{1/2}$ and the set of processes having finite two-norms is known as the Lebesgue 2-space, which is denoted by ℓ_2. The energy of a stochastic process is the square of its two-norm, *i.e.*, $\|w\|_2^2 = \sum_{k=1}^{N}\sum_{j=1}^{p} w_{j,k}^2$. Thus, stipulating $w \in \ell_2$ means that w is required to have finite energy.

B. Constraining functions

We consider nonlinear censuring functions whose outputs are constrained to lie within prescribed bounds. In particular, for α_k, $\beta \in \mathbb{R}$, $k \in [1, N]$, let $f(\alpha_k, \beta)$ denote an odd function of α_k about $E\{\alpha\}$, *i.e.*, $f(-\alpha_k - E\{\alpha\}, \beta) = -f(\alpha_k - E\{\alpha\}, \beta)$, so that

$$E\{f(\alpha,\beta)\} = E\{\alpha\}. \qquad (2)$$

Consider the hard-limiting function

$$f(\alpha_k,\beta) = \begin{cases} E\{\alpha\}+\beta & \text{if } E\{\alpha\}+\beta \le \alpha_k \\ \alpha_k & \text{if } E\{\alpha\}-\beta < \alpha_k < E\{\alpha\}+\beta \\ E\{\alpha\}-\beta & \text{if } \alpha_k \le E\{\alpha\}-\beta \end{cases}. \qquad (3)$$

By inspection, $f(\alpha,\beta)$ is constrained within $E\{\alpha\} \pm \beta$ and since it is an odd function, (2) is satisfied. In this paper, the attention is confined to zero-mean processes, namely $E\{\alpha\} = 0$ and

$$f(\alpha_k,\beta) = \begin{cases} \beta & \text{if } \beta \le \alpha_k \\ \alpha_k & \text{if } -\beta < \alpha_k < +\beta \\ -\beta & \text{if } \alpha_k \le -\beta \end{cases}, \qquad (4)$$

for which $\frac{df(\alpha,\beta)}{d\alpha} = 1$ when $-\beta < \alpha_k < \beta$.

An example will be presented subsequently that uses the saturating nonlinearity

$$g(\alpha,\beta) = 2\beta\pi^{-1}\arctan\left(\pi\alpha(2\beta)^{-1}\right), \qquad (5)$$

which is a continuous approximation of (4) and satisfies $|g(\alpha,\beta)| \le |\beta|$ and $\frac{dg(\alpha,\beta)}{d\alpha} = \left(1 + (\pi\alpha)^2(2\beta)^{-2}\right)^{-1} \approx 1$ when $(\pi\alpha)^2(2\beta)^{-2} \ll 0$.

C. Problem statement

Let $\theta_k \in \mathbb{R}^p$ denote an exogenous input process in ℓ_2. Suppose that a system's internal states, $x_k \in \mathbb{R}^n$, and outputs, $y_k \in \mathbb{R}^p$, are generated by

$$x_{k+1} = A_k x_k + B_k w_k, \qquad (6)$$

$$y_k = \begin{bmatrix} y_{1,k} \\ \vdots \\ y_{p,k} \end{bmatrix} = \begin{bmatrix} f(C_{1,k}x_k, \theta_{1,k}) \\ \vdots \\ f(C_{p,k}x_k, \theta_{p,k}) \end{bmatrix}, \qquad (7)$$

where $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$ and $C_{j,k}$ is the j^{th} row of $C_k \in \mathbb{R}^{p\times n}$. It can be seen from (7) that the $y_{j,k}$ lie within $\pm\theta_{j,k}$, *i.e.*, $-\theta_{j,k} \le y_{j,k} \le \theta_{j,k}$, $j = 1, \ldots p$. Consequently, the input θ_k is called a constraint process. For example, if the outputs represent the trajectories of pedestrians within a building then the constraint process could include knowledge about wall, floor and ceiling positions. Similarly, a vehicle trajectory constraint process could include information about building boundaries and road edges.

Assume that observations

$$z_k = y_k + v_k \qquad (8)$$

are available, where $v_k \in \mathbb{R}^p$ is a stochastic, white measurement noise process with

$$E\{v_k\} = 0, \quad E\{v_j v_k^T\} = R_k\delta_{jk}, \quad E\{w_j v_k^T\} = 0. \qquad (9)$$

Denote $y = [y_1^T \ \ldots \ y_N^T]^T$ and $\theta = [\theta_1^T \ \ldots \ \theta_N^T]^T$, where $\theta_k = [\theta_{1,k} \ \ldots \ \theta_{p,k}]^T$. Equation (7) implies

$$\|y\|_2^2 \le \|\theta\|_2^2, \qquad (10)$$

namely, the energy of the system's output is bounded from above by the energy of the constraint process. In view of (10), it is desired to exploit knowledge of the model (6), (7) together with the assumptions (1), (9), to produce estimates $\hat{y} = [\hat{y}_1^T \ \ldots \ \hat{y}_N^T]^T$ of y from the measurements $z = [z_1^T \ \ldots \ z_N^T]^T$ such that

$$\|\hat{y}\|_2^2 \le \|\theta\|_2^2 \qquad (11)$$

and the output estimation error, $\tilde{y}_k = y_k - \hat{y}_k$, is in ℓ_2.

D. Constrained filter procedure

A procedure is proposed in which the minimum-variance filter recursions [19] are used to calculate estimates $\hat{y}$ from measurements which are constrained using the nonlinear function (4). Suppose that the constrained measurements are obtained as

$$\underline{z}_k = \begin{bmatrix} \underline{z}_{1,k} \\ \vdots \\ \underline{z}_{p,k} \end{bmatrix} = \begin{bmatrix} f(z_{1,k}, \gamma^{-1}\theta_{1,k}) \\ \vdots \\ f(z_{p,k}, \gamma^{-1}\theta_{p,k}) \end{bmatrix} \qquad (12)$$

for a positive $\gamma \in \mathbb{R}$ to be designed.

The minimum-variance filter recursions are given by [19]

$$\hat{x}_{k+1/k} = (A_k - K_k C_k)\hat{x}_{k/k-1} + A_k \underline{z}_k \qquad (13)$$

$$\hat{x}_{k/k} = (I - L_k C_k)\hat{x}_{k/k-1} + L_k \underline{z}_k \qquad (14)$$

$$\hat{y}_{k/k} = C_k \hat{x}_{k/k} \qquad (15)$$

where $L_k = P_k C_k^T (C_k P_k C_k^T + R_k)^{-1}$ is the filter gain, $K_k = A_k L_k$ is the predictor gain and $P_k = P_k^T \in \mathbb{R}^{n\,x\,n}$ is the solution of the Riccati difference equation (RDE)

$$P_{k+1} = A_k P_k A_k^T - K_k (C_k P_k C_k^T + R_k) K_K^T + B_k Q_k B_k^T . \quad (16)$$

For linear problems, namely if the measurements are given by $C_k x_k + v_k$, the state prediction, (13), is a conditional mean estimate, for which the prediction error, $\tilde{x}_{k/k-1} = x_k - \hat{x}_{k/k-1}$, is unbiased, *i.e.*,

$$E\{\tilde{x}_{k/k-1}\} = 0, \quad (17)$$

and the filter minimizes the covariance $E\{\tilde{x}_{k/k-1}\tilde{x}_{k/k-1}^T\}$ [18]. Although applying the filter (13) – (15) to the constrained measurements (12) does not minimize $E\{\tilde{x}_{k/k-1}\tilde{x}_{k/k-1}^T\}$, the predicted and filtered estimates are unbiased.

Lemma 2.1: In respect of the filter (13) – (15) operating on the constrained measurements (12), the following applies.

(i) the predicted state estimates, $\hat{x}_{k+1/k}$, are unbiased;

(ii) the corrected state estimates, $\hat{x}_{k/k}$, are unbiased provided that $\hat{x}_{0/0} = E\{x_0\}$; and

(iii) the output estimates, $\hat{y}_{k/k}$, are unbiased provided that $\hat{x}_{0/0} = E\{x_0\}$.

Proof: see [26].

The nonlinearity design relies on following property of linear systems.

Lemma 2.2 [25]: Suppose that $y_k \in \mathbb{R}^p$ is generated from $w_k \in \mathbb{R}^m$ by the system

$$x_{k+1} = \bar{A}_k x_k + \bar{B}_k w_k \quad (18)$$

$$y_k = \bar{C}_k x_k + \bar{D}_k w_k , \quad (19)$$

where $\bar{A}_k$, $\bar{B}_k$, $\bar{C}_k$ and $\bar{D}_k$ are real-valued matrices of appropriate dimensions. Then with a $\gamma \in \mathbb{R}$, the existence of solutions $M_k = M_k^T > 0$ to the RDE

$$M_{k+1} = \bar{A}_k M_k \bar{A}_k^T + \bar{B}_k \bar{B}_k^T + (\bar{A}_k M_k \bar{C}_k^T + \bar{B}_k \bar{D}_k^T) \times (\gamma^2 I - \bar{D}_k \bar{D}_k^T - \bar{C}_k M_k \bar{C}_k^T)^{-1} \times (\bar{A}_k M_k \bar{C}_k^T + \bar{B}_k \bar{D}_k^T)^T , \quad (20)$$

satisfying

$$(\gamma^2 I - \bar{D}_k \bar{D}_k^T - \bar{C}_k M_k \bar{C}_k^T) > 0, \quad (21)$$

for all $k \in [1, N]$, is necessary and sufficient to ensure that $\|y\|_2^2 + x_0^T M_0 x_0 \le \gamma^2 \|\underline{z}\|_2^2$ for all $\underline{z} \in \ell_2$.

A candidate for γ within (12) can be tested by applying Lemma 2.2 to the system (13) – (15) as follows.

Lemma 2.3: Suppose that the minimum-variance filter recursions (13) – (15) operate on the constrained measurements (12). Let $\bar{A}_k = A_k - K_k C_k$, $\bar{B}_k = K_k$, $\bar{C}_k = (I - C_k^T L_k^T) C_k^T$ and $\bar{D}_k = L_k C_k$. For a given $\gamma > 0$, if a solution $M_k = M_k^T > 0$ for the RDE (20) satisfying (21) exists for all $k \in [0, N]$, then the performance (11) is achieved.

Proof: see [26].

A search is required for a minimum γ^2 so that positive definite solutions for the RDE specified within Lemma 2.3 exist. This search is tractable because M_{k+1} is a convex function of γ^2, since $\frac{d^2 M_{k+1}}{d^2 \gamma^2} > 0$, provided that (26) holds.

It is shown below that the filter's output estimation error is stable.

Lemma 2.4: Define the filter output estimation error as $\tilde{y} = y - \hat{y}$. Under the conditions of Lemma 2.3, $\theta \in \ell_2$ and $v \in \ell_2$ imply $\tilde{y} \in \ell_2$.

Proof: see [26].

E. Constrained smoother procedure

In this sequel, it is described how the censuring function within (12) can be designed for use by a fixed-interval smoother. The minimum-variance smoother for output estimation is derived in [20] - [22] and a version that is robust to problem uncertainties appears in [23]. The smoother recursions are given by

$$x_{k+1} = \bar{A}_k x_k + \bar{B}_k \underline{z}_k \quad (22)$$

$$\alpha_k = \bar{C}_k x_k + \bar{D}_k \underline{z}_k \quad (23)$$

$$\lambda_{k-1} = \bar{A}_k^T \lambda_k - \bar{C}_k^T \alpha_k , \ \lambda_N = 0 \quad (24)$$

$$\beta_k = -\bar{B}_k^T \lambda_k + \bar{D}_k^T \alpha_k \quad (25)$$

$$\hat{y}_{k/N} = \underline{z}_k - R_k \beta_k , \quad (26)$$

where $\bar{A}_k$, $\bar{B}_k$ are defined above, $\bar{D}_k = (C_k P_k C_k^T + R_k)^{-1/2}$ and $\bar{C}_k = -\bar{D}_k C_k$. It is shown below that the smoothed estimates are unbiased.

Lemma 2.5: In respect of the smoother (22) – (26) operating on the constrained measurements (12), the smoothed estimates, $\hat{y}_{k/N}$, are unbiased.

Proof: see [26].

A method for testing candidates for the scalar γ within (12) is described below.

Lemma 2.6: In respect of the smoother (22) – (26) operating on the constrained measurements (12), suppose the following.

(i) γ_2, $\gamma_3 > 0$ are given.

(ii) With $\bar{A}_k$, $\bar{B}_k$ defined above, $\bar{D}_k = (C_k P_k C_k^T + R_k)^{-1/2}$, $\bar{C}_k = -\bar{D}_k C_k$ and $\gamma = \gamma_2$, a solution $M_k = M_k^T > 0$ for the RDE (20) satisfying (21) exists for all $k \in [0, N]$.

(iii) With $\bar{A}_k$, $\bar{B}_k$ defined above, $\bar{D}_k$ =

$R_k(C_kP_kC_k^T + R_k)^{-1/2}$, $\bar{C}_k = -R_k\bar{D}_kC_k$ and $\gamma = \gamma_3$, a solution $M_k = M_k^T > 0$ for the RDE (20) satisfying (21) exists for all $k \in [0, N]$.

Then the design

$$\gamma^2 = 1 + \gamma_2^2\gamma_3^2 \quad (27)$$

within (12) is sufficient to achieve (11).

Proof: see [26].

It is argued below that the smoother's output estimation error is stable.

Lemma 2.7: Define the smoother output estimation error as $\tilde{y} = y - \hat{y}$. Under the conditions of Lemma 2.6, $\theta \in \ell_2$ and $v \in \ell_2$ imply $\tilde{y} \in \ell_2$.

The proof follows *mutatis mutandis* from that of Lemma 2.4.

Once again, a γ for (12) can be found iteratively by searching for minimum $\gamma_2, \gamma_3 > 0$, so that positive definite RDE solutions exist in accordance with Lemma 2.6. Although the conditions of Lemma 2.6 serve to ensure (11) and $\tilde{y} \in \ell_2$, the choice (27) arises out of a sufficient condition. Consequently, the design $\gamma^{-1}\theta_{j,k}$, $j = 1 \ldots p$, within (12) may be too conservative and yield poor mean-square-error performance. That is, nonlinearity designs with $\gamma^2 < 1 + \gamma_2^2\gamma_3^2$ may provide improved performance.

F. Example

Measurements were generated using (6), the nonlinearity (5) within (7), $A = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}$, $B = C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q = R = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$ and Gaussian processes. The constraint process within (7) was chosen to be fixed, namely, $\theta_k = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, $k \in [1, 10^6]$, for which $\theta \in \ell_2$. A minimum value of $\gamma = 0.738$ was found for the solutions of the RDE specified within Lemma 2.3 to be positive definite. The limits of the observed distribution of the filtered estimates, $\hat{y}_{k/k} = \begin{bmatrix} \hat{y}_{1,k/k} \\ \hat{y}_{2,k/k} \end{bmatrix}$, are indicated by the outer black region of Fig. 1. It can be seen from the figure that -0.5 ≤ $\hat{y}_{1,k/k}$ ≤ 0.5 and -0.5 ≤ $\hat{y}_{2,k/k}$ ≤ 0.5, which illustrates Lemma 2.3.

Minimum values of $\gamma_2 = 2.79$ and $\gamma_3 = 2.06$ were found for the solutions of the RDEs specified within Lemma 2.6 to be positive definite. However, constraining the smoother measurements using $\gamma = \sqrt{1+\gamma_2^2\gamma_3^2} = 5.83$ was found to be too conservative and yielded poor error performance. Instead, the ratio of the unconstrained smoother and measurement energies, namely, $\gamma = \|\hat{y}\|_2 \|\underline{z}\|_2^{-1} = 0.99$, was used within (12) to constrain the smoother measurements. The observed distribution of the smoothed estimates, $\hat{y}_{k/N} = \begin{bmatrix} \hat{y}_{1,k/N} \\ \hat{y}_{2,k/N} \end{bmatrix}$, are indicated by the inner white region of Fig. 1. Mean square errors of 0.0478, 0.0339, 0.0478 and 0.0327 were observed for the unconstrained filter, constrained filter, unconstrained smoother and constrained smoother, respectively, which demonstrate that constrained filtering and smoothing can be advantageous.

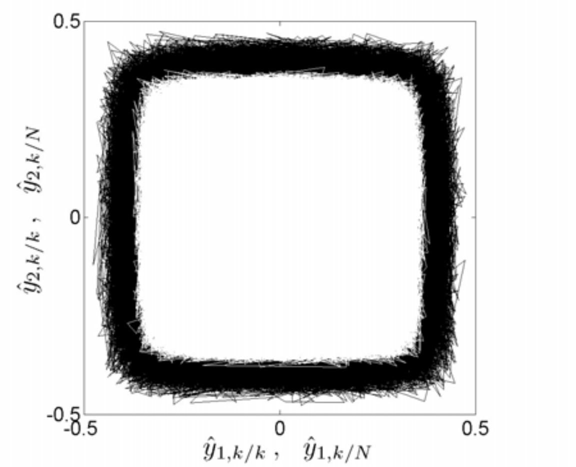

Fig. 1. Superimposed distributions of $\hat{y}_{k/k}$ (outer black) and $\hat{y}_{k/N}$ (inner white) for

III. Conclusion

This paper develops constrained filtering and smoothing procedures that differ from the approaches of [1] – [18] and are novel in the following respects.

- Problems are considered where the output of a linear system, *y*, satisfies $\|y\|_2^2 \le \|\theta\|_2^2$, in which $\theta \in \ell_2$ is an exogenous constraint process.
- Nonlinear functions to censure the measurements are designed so that the estimates, $\hat{y}$, satisfy $\|\hat{y}\|_2^2 \le \|\theta\|_2^2$.
- Although the filter and fixed interval smoother that operate on censured measurements do not minimize the error variance, it is shown that the estimates are unbiased and the estimation errors, $\tilde{y} = y - \hat{y}$, are in ℓ_2.

Constrained filtering is illustrated by a loosely-coupled GPS/INS integration application. In particular, *a priori* knowledge about altitude and velocity constraints is used in the calculation of navigation solutions. It is demonstrated that constraining the filter's inputs can provide a performance benefit when GPS outages occur.

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