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Topological Equivalence of a Structure-Preserving Power Network Model and a Non-Uniform Kuramoto Model of Coupled Oscillators

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Abstract— **We study synchronization in the classic structure-preserving power network model proposed by Bergen and Hill. We find that, locally near the synchronization manifold, the phase and frequency dynamics of the power network model are topologically conjugate to the phase dynamics of a non-uniform Kuramoto model together with decoupled and stable frequency dynamics. This topological conjugacy implies the equivalence of local synchronization in power networks and in non-uniform Kuramoto oscillators. Hence, we can harness the results available for Kuramoto oscillators to analyze synchronization in power networks. We establish necessary and sufficient conditions for phase synchronization, sufficient conditions for frequency synchronization, and necessary and sufficient conditions for frequency synchronization with a uniform topology. These conditions also extend the results known for the classic first-order Kuramoto model and second-order consensus protocols. Our conditions all share a common physical interpretation: the non-uniformity between real power injections has be compensated by sufficiently strong coupling.**

I. INTRODUCTION

The vast North American power grid is often referred to as the largest and most complex machine engineered by humankind. Local instabilities can trigger cascading failures and ultimately result in wide-spread blackouts. In face of the rising complexity of the future power grid and the stochastic disturbances caused by renewables, the understanding of the system complexity becomes increasingly important.

Power system stability is broadly subdivided into rotor angle and voltage stability. Rotor angle stability is the ability of the power system to remain in synchronism when subjected to disturbances, and it is further classied as *transient stability* for severe disturbances. Generally, the complexity of a power network and the related (in)stability issues are not understood [1]. In particular, an open problem recognized by the power system community and not resolved yet by classical methods is the quest for explicit and concise conditions relating transient stability to the parameters and graph-theoretical properties of the underlying network [2]. In [3] we provided a solution to this problem for a network-reduced power system model with non-zero transfer conductances by means of a singular perturbation analysis and by using tools from coupled oscillators and consensus networks. These results on the reduced network can then be related to the original network via Kron reduction and algebraic graph theory [4].

In 1981 Bergen and Hill proposed a *structure-preserving* power network model [5] to represent the network components and topology explicitly and to overcome the difficulties in the analysis of network-reduced models with non-negligible transfer conductances. Since the structure-preserving power network model can be cast as mixed Hamiltonian and gradient-like system, the traditional transient stability analysis methods are based on Hamiltonian arguments together with computational tools [6]. Even though these Hamiltonian methods are very powerful, in particular to estimate the region of attraction of synchronous equilibria, they do not provide concise conditions for synchronization.

Synchronization recently attracted lots of interest in various scientific communities. Especially, the simple and rich coupled oscillator model proposed by Kuramoto [7] serves as a prototypical example for synchronization, and we refer to [8]–[10] for various applications and theoretic results. As observed in [3] and references therein, a power network model can be cast as a second-order Kuramoto model with inertia, viscous damping, non-complete coupling topology, and non-identical natural frequencies. The second-order consensus protocols studied in [11] can also be seen as a linearized version of the power network and Kuramoto models. For both the Kuramoto model and consensus protocol the relation between synchronization and the parameters and topology of the underlying network is well understood.

The contributions of this paper are three-fold.

First, we show that, locally near the synchronization manifold, the phase and frequency dynamics of the power network model are topologically conjugate to the phase dynamics of a non-uniform Kuramoto model and decoupled exponentially stable dynamics for the frequencies. The two decoupled dynamics correspond exactly to the fast and slow dynamics found via singular perturbation analysis in [3]. Compared to [3], this local topological conjugacy holds without any assumptions on the system parameters such as a sufficiently strong damping. Moreover, the following three statements are found to be equivalent: exponential synchronization in the structure-preserving power network model, exponential synchronization in the non-uniform Kuramoto model, and exponential stability of a topological Kuramoto model.

Second, we approach the outstanding problem proposed in [2] and provide novel and explicit conditions relating synchronization in the structure-preserving power model to the underlying network parameters and topology. Our conditions are necessary and sufficient for phase synchronization and sufficient for frequency synchronization. For a complete and uniform coupling graph, our conditions are also necessary and sufficient for frequency synchronization. In each case, the convergence rate of synchronizing solutions is exponential, we derive bounds on the ultimate phase cohesiveness,

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and our synchronization conditions can be interpreted as "the non-uniformity between real power injections has be compensated by a sufficiently strong coupling in the network", which confirms our results on the reduced model [3].

Third and finally, our synchronization conditions also extend the results known for second-order Kuramoto models and consensus protocols. Contrary to various results reviewed in [9], [10], we prove that the inertial terms do not affect the local synchronization conditions for second-order Kuramoto models. Furthermore, our approach extends the linear analysis in [11], where the eigenvalues of the first and second-order consensus protocols are related to another. We provide a nonlinear generalization of this result: the second-order power network model synchronizes if and only if the corresponding first-order Kuramoto model synchronizes.

Paper organization: The remainder of this section introduces some notation. Section II introduces the structure-preserving power network model, the Kuramoto model, and different synchronization notions. These models are linked in Section III and synchronization conditions are derived in Section IV. Finally, Section V concludes the paper.

Geometry on n-torus: The *torus* is the set $\mathbb{T}^1 = [0, 2\pi]$, where $-\pi$ and $+\pi$ are associated with each other, an *angle* is a point $\theta \in \mathbb{T}^1$, and an *arc* is a connected subset of $\mathbb{T}^1$. The product set $\mathbb{T}^n$ is the n-dimensional torus. With slight abuse of notation, let $|\theta_1 - \theta_2|$ denote the *geodesic distance* between two angles $\theta_1 \in \mathbb{T}^1$ and $\theta_2 \in \mathbb{T}^1$. For $\gamma \in [0, \pi]$, let $\Delta(\gamma) \subset \mathbb{T}^n$ be the set of angle arrays $(\theta_1, \ldots, \theta_n)$ with the property that there exists an arc of length γ containing all $\theta_1, \ldots, \theta_n$ in its interior. Thus, an angle array $\theta \in \Delta(\gamma)$ satisfies $\max_{i,j\in\{1,\ldots,n\}} |\theta_i - \theta_j| < \gamma$. For $\gamma \in [0, \pi]$, we also define $\bar{\Delta}(\gamma)$ to be the union of the phase-synchronized set $\{\theta \in \mathbb{T}^n \mid \theta_i = \theta_j, i, j \in \{1, \ldots, n\}\}$ and the closure of the open set $\Delta(\gamma)$. Hence, $\theta \in \bar{\Delta}(\gamma)$ satisfies $\max_{i,j\in\{1,\ldots,n\}} |\theta_i - \theta_j| \leq \gamma$; the case $\theta \in \bar{\Delta}(0)$ corresponds simply to θ taking value in the phase-synchronized set.

Vectors and matrices: Given an n-tuple $(x_1, \ldots, x_n)$, let $x \in \mathbb{R}^n$ be the associated vector. The *inertia* of a matrix $A \in \mathbb{R}^{n\times n}$ are given by the triple $\{\nu_{\rm s}, \nu_{\rm c}, \nu_{\rm u}\}$, where $\nu_{\rm s}$ (respectively $\nu_{\rm u}$) denotes the number of stable (respectively unstable) eigenvalues of A in the open left (respectively right) complex half plane, and $\nu_{\rm c}$ denotes the number of center eigenvalues with zero real part. Finally, let $\mathbf{1}$ and $\mathbf{0}$ denote the matrices of unit and zero entries of appropriate dimension, and let I_n be the n-dimensional identity matrix.

Derivative operators: For a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, we adopt the shorthand $\nabla_i f(x) = \partial f(x)/\partial x_i$, $\nabla f(x) = (\partial f(x)/\partial x)^T \in \mathbb{R}^{n\times 1}$ is the gradient, and $\nabla^2 f(x) \in \mathbb{R}^{n\times n}$ is the Hessian matrix.

II. Mathematical Models and Synchronization

A. Structure-Preserving Power Network Model

In the following we briefly present the *structure-preserving power network model* introduced in [5] and refer to [12, Chapter 7] for detailed derivation from a higher order first principle model. Consider a connected power network with $m \geq 0$ generators and $n - m \geq 0$ load buses. The network is represented by the symmetric nodal admittance matrix $Y \in \mathbb{C}^{n\times n}$ (augmented with the generator transient reactances), where the indices $\{1, \ldots, m\}$ and $\{m+1, \ldots n\}$ correspond to the generators and loads, respectively. Define the *maximum real power transfer* between any two nodes i and j with *constant* voltage levels $|V_i|$ and $|V_j|$ as $a_{ij} = |V_i| \cdot |V_j| \cdot \Im(Y_{ij})$, which is positive if i and j are connected and zero otherwise.

With this notation the constant-voltage behind reactance swing dynamics of each generator i are given by

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)\,, i \in \{1, \ldots, m\}, \quad (1)$$

where $\theta_i \in \mathbb{T}^1$ and $\dot{\theta}_i \in \mathbb{R}^1$ are the generator rotor angle and frequency, and $P_i > 0$, $M_i > 0$, and $D_i > 0$ are the mechanical power input, inertia constant, and damping coefficient of generator i. The angles $\theta_j \in \mathbb{T}^1$, $j \in \{m + 1, \ldots, n\}$, are the voltage phase angles at the load buses.

The real power drawn by a load i consists of a constant term $P_i < 0$ and a frequency dependent term $D_i\dot{\theta}_i$ with $D_i > 0$. The resulting in the real power balance equation is

$$D_i\dot{\theta}_i = P_i - \sum_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)\,, \quad i \in \{m+1, \ldots, n\}\,. \quad (2)$$

The dynamics (1)-(2) evolve in $\mathbb{T}^n \times \mathbb{R}^m$ and feature an important symmetry, namely the rotational invariance of the angular variable θ on the unit circle $\mathbb{S}^1$. The power network model (1)-(2) can also be formulated as mixed gradient-like and dissipative Hamiltonian system with external forcing as

$$\begin{aligned} M\ddot{\theta}_i + D_i\dot{\theta}_i &= P_i - \nabla_i U(\theta)\,, \quad i \in \{1, \ldots, m\}\,, \\ D_i\dot{\theta}_i &= P_i - \nabla_i U(\theta)\,, \quad i \in \{m+1, \ldots, n\}\,, \end{aligned} \quad (3)$$

where $U : \mathbb{T}^n \to \mathbb{R}_{\geq 0}$ is the potential energy of the power flows given by $U(\theta) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(1 - \cos(\theta_i - \theta_j))$. Traditionally, the power injections P_i are associated with the artificial potential $W(\theta) = -\sum_{i=1}^{n} P_i\theta_i$ in the power systems literature. Besides the fact $W(\theta)$ is multi-valued on $\mathbb{T}^n$ and only locally a potential, our analysis will reveal that the power injections should rather be treated as external forcing.

B. Synchronization Notions

In power systems, synchronization and transient stability are usually defined as stability of an equilibrium of the dynamics (1)-(2) formulated in relative angle coordinates, which include the center of inertia and reference generator coordinates, as well as the infinite bus assumption [6], [12].

Alternatively, the controls, dynamical systems, and the physics community developed and studied various notions of synchronization and incremental stability without employing relative coordinates. The latter concepts are also amenable to relate synchronization to the underlying network topology and parameters. For the power network model (1)-(2) the following synchronization concepts are meaningful, where a synchronized solution has zero angular acceleration.

Definition II.1 (Synchronization) A solution $(\theta, \dot{\theta}) : \mathbb{R}_{\geq 0} \to (\mathbb{T}^n, \mathbb{R}^n)$ to the power network model (1)-(2) achieves

1) **phase synchronization** if there exist constants $\theta_{\text{sync}} \in \mathbb{T}^1$ and $\omega_{\text{sync}} \in \mathbb{R}^1$ such that all phases $\theta_i(t)$ converge exponentially fast to $\theta_{\text{sync}}+\omega_{\text{sync}}t \pmod{2\pi}$ as $t \to \infty$;
2) **phase-cohesiveness** if there exists a length $\gamma \in [0, \pi[$ such that $\theta(t) \in \bar{\Delta}(\gamma)$ for all $t \geq 0$;
3) **frequency synchronization** if there exists a constant $\omega_{\text{sync}} \in \mathbb{R}^1$ such that all frequencies $\dot{\theta}_i(t)$ converge exponentially fast to ω_{sync} as $t \to \infty$; and
4) **synchronization** if it is phase cohesive and it achieves frequency synchronization in the sense of 3) and 4).

Note that phase synchronization requires exponential stability in the incremental variables $|\theta_i(t) - \theta_j(t)|$ and an asymptotically linear phase. We will show that phase synchronization can occur if and only if all ratios P_i/D_i are identical. Otherwise, each distance $|\theta_i(t) - \theta_j(t)|$ converges to a constant value which is not necessarily zero. The concept of phase cohesiveness addresses exactly this point and means that at each time t there exists an arc of length γ containing all angles $\theta_i(t)$. Analogously, the concept of frequency synchronization combines incremental frequency stability together with an asymptotic property. Finally, synchronization combines phase cohesiveness with frequency synchronization. Of course, synchronization includes phase synchronization, and, in control-theoretic terms, it is best described by the terminology *practical phase synchronization*.

C. Kuramoto Model of Coupled Oscillators

A celebrated model for synchronization is due to Kuramoto [7]. The *Kuramoto model* considers $n \geq 2$ coupled phase oscillators, each represented by a natural frequency $\omega_i \in \mathbb{R}^1$ and phase variable $\theta_i \in \mathbb{T}^1$ obeying the dynamics

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j), \quad i \in \{1, \dots, n\}, \tag{4}$$

where $K > 0$ is the *coupling strength*. For the Kuramoto model we adopt the synchronization notions in Definition II.1. For identical natural frequencies ω_i, the Kuramoto model achieves phase synchronization from almost all initial conditions, see, e.g., [13, Corollary 6.11]. For non-identical natural frequencies, the Kuramoto model can achieve only frequency synchronization with a certain level of phase cohesiveness. In particular, the Kuramoto model achieves practical synchronization if and only if the coupling overcomes the worst non-uniformity among the natural frequencies, that is, $K > \max_{i,j\in\{1,\dots,n\}} |\omega_i - \omega_j|$, see [10, Theorem 4.1].

Instead of the complete and uniform coupling K/n, the Kuramoto model (4) is also studied with more general symmetric coupling topologies with weights $a_{ij} = a_{ji}$. In this case, the dynamics are simply $\dot{\theta}_i = \omega_i - \nabla_i U(\theta)$ similar to the power network model (3). The potential function $U(\theta)$ depends on the weighted coupling graph, and for a connected and undirected graph the phase-synchronized state is a local minimum of $U(\theta)$. Of particular interest are so-called $\mathbb{S}^1$-*synchronizing graphs* for which all critical points are hyperbolic, the phase-synchronized state is the only local minimum of $U(\theta)$, and all other critical points are local maxima or saddle points. For identical natural frequencies ω_i the class of $\mathbb{S}^1$-synchronizing graphs (including the complete graph and trees) yields almost global phase synchronization [13], [14]. For non-identical ω_i no exact synchronization conditions are known in the case of non-uniform coupling a_{ij}.

III. A One-Parameter Family of Dynamical Systems and its Properties

We link the structure-preserving power network model (1)-(2) and the Kuramoto model (4) through a parametrized system. The proofs of the results presented in this section can be found in [10]. Consider for $n_1, n_2 \geq 0$ and $\lambda \in [0, 1]$ the one-parameter family $\mathcal{H}_\lambda$ of dynamical systems combining forced gradient-like and dissipative Hamiltonian dynamics as

$$\begin{aligned} D_1\dot{x}_1 &= F_1 - \nabla_1 H(x), \\ \begin{bmatrix} I_{n_2} & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} \lambda D_2^{-1} F_2 \\ (1-\lambda) F_2 \end{bmatrix} + \left((1-\lambda) \begin{bmatrix} \mathbf{0} & I_{n_2} \\ -I_{n_2} & \mathbf{0} \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} \lambda D_2^{-1} & \mathbf{0} \\ \mathbf{0} & D_2 \end{bmatrix} \right) \begin{bmatrix} \nabla_2 H(x) \\ \nabla_3 H(x) \end{bmatrix}, \end{aligned} \tag{5}$$

where $x = (x_1, x_2, x_3) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathbb{R}^{n_2} = \mathcal{X}$ is the state, and the sets $\mathcal{X}_1$ and $\mathcal{X}_2$ are smooth manifolds of dimensions n_1 and n_2, respectively. The matrices $D_1 \in \mathbb{R}^{n_1 \times n_1}$, $D_2 \in \mathbb{R}^{n_2 \times n_2}$ and $M \in \mathbb{R}^{n_2 \times n_2}$ are positive definite, $F_1 \in \mathbb{R}^{n_1}$ and $F_2 \in \mathbb{R}^{n_2}$ are constant forcing terms, and $H : \mathcal{X} \to \mathbb{R}$ is a twice continuously differentiable potential function.

The parameterized system (5) continuously interpolates, as a function of $\lambda \in [0, 1]$, between gradient-like and mixed dissipative Hamiltonian/gradient-like dynamics. For $\lambda = 1$, the system (5) reduces to the forced gradient-like dynamics

$$\mathbf{D}\dot{x} = \mathbf{F} - \nabla H(x), \tag{6}$$

where $\mathbf{F} = [F_1^T, F_2^T, \mathbf{0}]^T$ is the external forcing term and $\mathbf{D} = \text{blkdiag}(D_1, D_2, D_2^{-1}M)$ is a block-diagonal *time constant* matrix (or system metric). For $\lambda = 0$, the dynamics (5) reduce to gradient-like dynamics for x_1 and dissipative Hamiltonian (or Newtonian) dynamics for (x_2, x_3) written as

$$\begin{aligned} D_1\dot{x}_1 &= F_1 - \nabla_1 H(x), \\ \begin{bmatrix} I_{n_2} & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} \mathbf{0} \\ F_2 \end{bmatrix} + \left(\begin{bmatrix} \mathbf{0} & I_{n_2} \\ -I_{n_2} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_2 \end{bmatrix} \right) \begin{bmatrix} \nabla_2 H(x) \\ \nabla_3 H(x) \end{bmatrix}. \end{aligned} \tag{7}$$

It turns out that, independently of $\lambda \in [0, 1]$, all parameterized systems (5) have the same equilibria with the same local stability properties determined by the potential $H(x)$.

Theorem III.1 (Properties of the $\mathcal{H}_\lambda$ family) *Consider the one-parameter family $\mathcal{H}_\lambda$, $\lambda \in [0, 1]$, of dynamical systems* (5) *with arbitrary positive definite matrices D_1, D_2, and M. The following statements hold:*

1) **Equilibria:** *For all $\lambda \in [0, 1]$ the equilibria of $\mathcal{H}_\lambda$ are given by the set $\mathcal{E} \triangleq \{x \in \mathcal{X} : \nabla H(x) = \mathbf{F}\}$; and*
2) **Local stability:** *For any equilibrium $x^* \in \mathcal{E}$ and for all $\lambda \in [0, 1]$, the inertia of the Jacobian of $\mathcal{H}_\lambda$ is given by the inertia of $-\nabla^2 H(x^*)$ and the corresponding center-eigenspace is given by* $\ker\nabla^2 H(x^*)$.

Statements 1) and 2) assert that normal hyperbolicity of the critical points of $H(x)$ can be directly related to local exponential (set) stability for any $\lambda \in [0, 1]$. This implies that all vector fields $\mathcal{H}_\lambda$, $\lambda \in [0, 1]$, are *locally* *topologically conjugate* [15] near a hyperbolic equilibrium point $x^* \in \mathcal{E}$. In particular, near $x^* \in \mathcal{E}$, trajectories of the gradient vector field (6) can be continuously deformed to match trajectories of the Hamiltonian vector field (7) while preserving parameterization of time. This topological conjugacy holds also for hyperbolic one-dimensional equilibrium trajectories [16, Theorem 6] considered in synchronization.

The similarity between second-order Hamiltonian systems and the corresponding first-order gradient flows is well-known in mechanical control systems [17], [18], in dynamic optimization [19], [20], and in transient stability studies for power networks [21], [22], but we are not aware of any result as general as Theorem III.1. In [21], [22], statements 1) and 2) are proved under the more stringent assumptions that $\mathcal{H}_\lambda$ has a finite number of isolated and hyperbolic equilibria. If the dynamical system $\mathcal{H}_\lambda$ is analyzed on an Euclidean state space, then various convergence statements and other minimizing properties can be deduced, see [19]–[22].

As a consequence of Theorem III.1, we can link synchronization in the power network model (1)-(2) and in a variant of the Kuramoto model (4). Since Theorem III.1 is valid only for equilibria, we convert synchronization to stability of an equilibrium manifold by changing coordinates to a rotating frame. The explicit synchronization frequency $\omega_{\text{sync}} \in \mathbb{R}^1$ of the power network model (1)-(2) is obtained by summing over all equations power network model (1)-(2) as

$$\sum\nolimits_{i=1}^{m} M_i \ddot{\theta}_i + \sum\nolimits_{i=1}^{n} D_i \dot{\theta}_i = \sum\nolimits_{i=1}^{n} P_i \,. \tag{8}$$

In the frequency-synchronized case when all $\ddot{\theta}_i = 0$ and $\dot{\theta}_i = \omega_{\text{sync}}$, equation (8) simplifies to $\sum_{i=1}^{n} D_i \omega_{\text{sync}} = \sum_{i=1}^{n} P_i$. We conclude that the synchronization frequency of the power network model is given by $\omega_{\text{sync}} \triangleq \sum_{i=1}^{n} P_i / \sum_{i=1}^{n} D_i$.

Accordingly, define the *non-uniform Kuramoto model* (featuring the same ω_{sync}) by dropping the inertial terms as

$$D_i \dot{\theta}_i = P_i - \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \,, \quad i \in \{1, \dots, n\} \,, \tag{9}$$

and the globally exponentially stable *frequency dynamics* as

$$\frac{d}{dt} \dot{\theta}_i = -M_i^{-1} D_i \left(\dot{\theta}_i - \omega_{\text{sync}} \right) , \quad i \in \{1, \dots, m\} \,, \tag{10}$$

where M_i, D_i, P_i, and a_{ij} take the same values as the corresponding parameters of the power network (1)-(2).

Finally, let $\tilde{P}_i \triangleq P_i - D_i \cdot \omega_{\text{sync}}$ be the frequency deviation and define the *topological Kuramoto model* by

$$\dot{\theta}_i = \tilde{P}_i - \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \,, \quad i \in \{1, \dots, n\} \,, \tag{11}$$

and its associated *scaled frequency dynamics* by

$$\frac{d}{dt} \dot{\theta}_i = -M_i^{-1} D_i \, \dot{\theta}_i \,, \quad i \in \{1, \dots, m\} \,. \tag{12}$$

The scaled model (11)-(12) corresponds to the dynamics (9)-(10) formulated in a rotating frame with frequency ω_{sync} and after normalizing all time constants D_i in (9).

Notice that the power network model (1)-(2), the non-uniform Kuramoto model (9) together with frequency dynamics (10) (formulated in a rotating frame with frequency ω_{sync}), and the topological Kuramoto model (11) together with the scaled frequency dynamics (12) are instances of the parameterized system (5) with the forcing terms $\tilde{P}_i$ and the overall potential function $H(\theta, \dot{\theta}) = \frac{1}{2} \dot{\theta}^T \dot{\theta} + U(\theta)$. In the sequel, we seek to apply Theorem III.1 to these three models.

For a rigorous reasoning, we define a two-parameter family of functions $\phi_{r,s} : \mathbb{R}_{\geq 0} \to \mathbb{T}^1$ of the form $\phi_{r,s}(t) \triangleq r + s \cdot t \pmod{2\pi}$, where $r \in \mathbb{T}^1$ and $s \in \mathbb{R}^1$. Consider for $(r_1, \dots, r_n) \in \bar{\Delta}(\gamma)$, $\gamma \in [0, \pi[$ the composite function

$$\Phi_{\gamma,s} : \mathbb{R}_{\geq 0} \to \mathbb{T}^n \,, \; \Phi_{\gamma,s}(t) \triangleq \big(\phi_{r_1,s}(t), \dots, \phi_{r_n,s}(t) \big) \tag{13}$$

parameterizing synchronized trajectories of the three models (1)-(2), (9)-(10), and (11)-(12). In the sequel, we will also refer to $\Phi_{\gamma,\omega_{\text{sync}}}(t)$ as the *synchronization manifold*.

We now have all ingredients to relate synchronization in the three (1)-(2), (9)-(10), and (11)-(12).

Theorem III.2 (Synchronization Equivalence) *Consider the power network model* (1)-(2)*, the non-uniform Kuramoto model* (9)*, and the topological Kuramoto model* (11) *with* $\tilde{P}_i = P_i - D_i \, \omega_{\text{sync}}$*, where* $\omega_{\text{sync}} = \sum_{k=1}^{n} P_k / \sum_{k=1}^{n} D_k$*. The following statements are equivalent for any* $\gamma \in [0, \pi[$*,* $t \geq 0$*, and any function* $\Phi_{\gamma,\omega_{\text{sync}}}(t)$ *defined in* (13)*:*

(i) $(\Phi_{\gamma,\omega_{\text{sync}}}(t), \omega_{\text{sync}} \mathbf{1})$ *is a locally exponentially stable synchronized trajectory* $(\theta(t), \dot{\theta}(t))$ *of the power network model* (1)-(2)*;*

(ii) $\Phi_{\gamma,\omega_{\text{sync}}}(t)$ *is a locally exponentially stable synchronized trajectory* $\theta(t)$ *of the first-order non-uniform Kuramoto model* (9)*; and*

(iii) $\Phi_{\gamma,0}(t)$ *is a locally exponentially stable synchronized equilibrium trajectory* $\theta(t)$ *of the topological Kuramoto model* (11)*.*

If the equivalent statements (i), (ii), and (iii) are true, then, locally near their respective synchronization manifolds $(\Phi_{\gamma,\omega_{\text{sync}}}(t), \omega_{\text{sync}} \mathbf{1})$ *and* $(\Phi_{\gamma,0}(t), \mathbf{0})$*, the three models* (1)-(2)*,* (9)-(10)*, and* (11)-(12) *are topologically conjugate.*

For purely second-order power network dynamics (1)-(2) (with $n = m$), e.g., in a network-reduced power system model with constant-current loads, Theorem III.1 and Theorem III.2 essentially state that the locations and stability properties of the *foci* of the second-order (damped oscillatory) power system dynamics (1)-(2) are equivalent to those of the *nodes* of the (overdamped) topological Kuramoto dynamics (11) and its scaled frequency dynamics (12). Furthermore, the trajectories near those equilibria are topologically conjugate. Figure 1 illustrates these conclusions.

Theorems III.1 and III.2 show that the location and local stability properties of the equilibria of the power network power model (1)-(2) are *independent* of the inertial terms M_i. Hence, the inertial terms *do not affect* any local bifurcations or synchronization conditions. This fact is perfectly aligned with the results known for second-order consensus protocols [11] and in apparent contradiction to various results reported for second-order Kuramoto models, see [10] for a comprehensive discussion. However, notice also that all results presented in this section are local, and the inertial terms still affect the transient synchronization behavior as well as the the separatrices and the region of attraction.

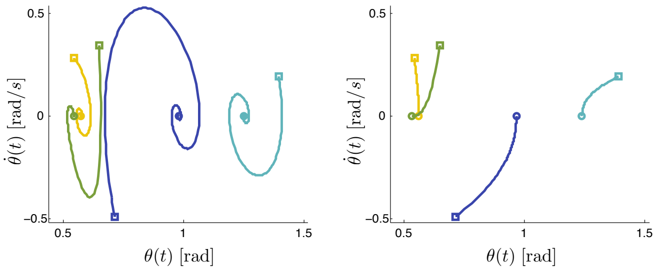

Fig. 1. Phase space plot of second-order power network dynamics (1)-(2) with $m = 4$ generators (left plot) and the corresponding first-order topological Kuramoto oscillators (11) together with the scaled frequency dynamics (12) (right plot). From the same initial configuration $\theta(0)$ (■) both models converge to the same nearby equilibria (●) and their trajectories are topologically conjugate, i.e., they can be "bent" to match each other.

Remark III.3 **(Alternatives)** Alternative methods to relate stability properties from the first-order Kuramoto model (4) to the power network model (1)-(2) include second-order Gronwall's inequalities [23], strict Lyapunov functions for mechanical systems [17], and singular perturbation analysis [3]. It should be noted that the approaches [17], [23] are limited to purely second-order systems, the second-order Gronwall inequality approach [23] has been carried out only for uniform inertial terms and unit damping, and the Lyapunov approach [17] is limited to potential-based Lyapunov functions. Finally, the singular perturbation approach [3] requires a sufficiently small inertia over damping ratio $\epsilon = \max_{i\in\{1,\dots,m\}}\{M_i/D_i\}$. As compared with these alternative methods, Theorem III.2 applies to the power network model (1)-(2) with mixed first and second-order dynamics, for all values of $M_i > 0$ and $D_i > 0$, and without additional assumptions. Finally, it is instructive to note that the first-order non-uniform Kuramoto dynamics (9) and the frequency dynamics (10) (in the time-scale t/ϵ) correspond to the reduced slow system and the fast boundary layer model in the singular perturbation approach [3, Theorem IV.2]. □

IV. Synchronization in Power Networks

In this section we will establish synchronization conditions for the power network model (1)-(2) by means of the topological Kuramoto model (11). We begin by studying phase synchronization, which though not relevant in power network applications still yields important insights for the synchronization problem. For zero power injections $P_i = 0$, the Hamiltonian formulation (3) (no external forcing for $P_i = 0$) of the power network model and Theorem III.1 imply that phase synchronization depends exclusively on the potential $U(\theta)$. The following result confirms this intuition.

Theorem IV.1 **(Phase synchronization)** *Consider the power network* (1)-(2). *The following statements are equivalent:*

(i) **Phase synchronization:** *there exists a exponentially stable phase-synchronized solution with constant synchronization frequency $\bar{\omega}_{\text{sync}} \in \mathbb{R}$; and*

(ii) **Uniformity:** *there exists a constant $\bar{P} \in \mathbb{R}$ such that $P_i = D_i \cdot \bar{P}$ for all $i \in \{1,\dots,n\}$.*

Moreover, in either of the two equivalent cases (i) or (ii), $\bar{P} \equiv \bar{\omega}_{\text{sync}}$ and the following three statements hold:

1) **Explicit phase:** *The asymptotic synchronization phase is given by $\sum_{i=1}^n D_i\theta_i(0)/\sum_{i=1}^n D_i + \bar{\omega}_{\text{sync}}t \pmod{2\pi}$.*
2) **Global convergence:** *For all initial conditions $(\theta(0),\dot{\theta}(0)) \in \mathbb{T}^n \times \mathbb{R}^m$ all frequencies $\dot{\theta}_i(t)$ converge to $\bar{P}$ and all phases $\theta_i(t) - \bar{\omega}_{\text{sync}}t \pmod{2\pi}$ converge to the critical points of the potential function $U(\theta)$.*
3) **Almost global stability:** *If the graph induced by a_{ij} is $\mathbb{S}^1$-synchronizing, then for almost all initial conditions $(\theta(0),\dot{\theta}(0)) \in \mathbb{T}^n \times \mathbb{R}^m$, the phases of the power network model synchronize exponentially.*

Proof: By Theorem III.2, there exists an exponentially stable phase-synchronized solution of the power network model (1)-(2) if and only if the topological Kuramoto model (11) features an exponentially stable phase-synchronized equilibrium. The condition (ii) for the latter statement to be true as well as the properties 1), 2), and 3) follow are known results for Kuramoto oscillators, see [10], [13], [14]. ■

According to Theorem IV.1, phase synchronization occurs if and only if all power injections P_i are identical to $D_i\bar{\omega}_{\text{sync}}$. This intuition carries over to a necessary condition for frequency synchronization. In particular, synchronization of a node i with the rest of the network cannot occur if its coupling to the network via the power transfers $\sum_{j=1}^n a_{ij}$ is too weak and the power injection P_i diverges too much from $D_i \cdot \omega_{\text{sync}}$, where $\omega_{\text{sync}} = \sum_{k=1}^n P_k/\sum_{k=1}^n D_k$ is the network synchronization frequency derived in Section III.

Lemma IV.2 **(Necessary synchronization condition)** *Consider the power network model* (1)-(2) *and let $\omega_{\text{sync}} = \sum_{k=1}^n P_k/\sum_{k=1}^n D_k$. If the model synchronizes, then*

$$\sum\nolimits_{j=1}^n a_{ij} \geq \left|P_i - D_i\cdot\omega_{\text{sync}}\right|, \quad i \in \{1,\dots,n\}. \quad (14)$$

Physically speaking, the necessary condition (14) is based on the simple fact that the power network (1)-(2) cannot synchronize if there are no angles satisfying the real power balance equations in a rotating frame with frequency ω_{sync}.

Proof of Lemma IV.2: Consider the power network model (1)-(2) written in a rotating frame with frequency ω_{sync} such that a synchronized solution with synchronization frequency ω_{sync} is an equilibrium solution determined by $\dot{\theta} = \mathbf{0}$ and

$$0 = P_i - D_i\cdot\omega_{\text{sync}} - \sum\nolimits_{j=1}^n a_{ij}\sin(\theta_i^* - \theta_j^*), \quad i \in \{1,\dots,n\},$$

where $\theta^* \in \mathbb{T}^n$. Since $\sin(x) \in [-1,1]$ is bounded, the previous equation has no solution if (14) does not hold. ■

Before stating a sufficient condition for synchronization of the power network model (1)-(2), we first consider the simpler case of a *uniform* power network, where all coupling coefficients take the same value. For such a uniform power network model we can state *exact* synchronization conditions which generalize the results known for the classic Kuramoto model (4) to second-order dissipative Kuramoto models.

Theorem IV.3 **(Synchronization in a uniform power network)** *Consider the power network model* (1)-(2) *with uniform and complete coupling $a_{ij} = K/n$ for $K > 0$ and $i,j \in \{1,\dots,n\}$. Define the scaled power inputs $\tilde{P}_i = P_i - D_i\cdot\omega_{\text{sync}}$, where $\omega_{\text{sync}} = \sum_{k=1}^n P_k/\sum_{k=1}^n D_k$.*

The following two statements are equivalent:

(i) *The coupling strength K is larger than the maximum non-uniformity among the scaled power inputs, i.e., $K > K_{\text{critical}} \triangleq \max_{i,j \in \{1,\dots,n\}}\{\tilde{P}_i - \tilde{P}_j\}$.*

(ii) *There exists a locally exponentially stable synchronized solution of the power network with uniform coupling.*

Moreover, in either of the two equivalent cases (i) and (ii), the synchronized solution has the frequency ω_{sync} and it is phase cohesive in $\bar{\Delta}(\gamma_{\min})$, where $\gamma_{\min} \in [0, \pi/2[$ is the unique solution to the equation $\sin(\gamma_{\min}) = K_{\text{critical}}/K$.

Proof: By Theorem III.2, there exists a locally exponentially stable synchronized solution of the power network model (1)-(2) with uniform coupling if and only if the classic Kuramoto model (4) with $\omega_i = \tilde{P}_i$ features a locally exponentially stable synchronized equilibrium. According to [10, Theorem 4.1], the latter statement is true if and only if condition (i) holds, and the phase cohesiveness in $\bar{\Delta}(\gamma_{\min})$ also follows from [10, Theorem 4.1]. ■

Lemma IV.2 and Theorem IV.3 can be interpreted as "the coupling in the network has to dominate the non-uniformity in the scaled power injections $\tilde{P}_i$" such that the power network synchronizes. If the coupling is quantified in terms of the algebraic connectivity $\lambda_2(L(a_{ij}))$, that is, the second smallest eigenvalue of the Laplacian matrix $L(a_{ij})$ of the weighted power network graph, then a sufficient condition with an analogous interpretation can be established.

Theorem IV.4 (Sufficient synchronization condition) *Consider the power network model* (1)-(2)*. Define the scaled power inputs $\tilde{P}_i = P_i - D_i \cdot \omega_{\text{sync}}$, where $\omega_{\text{sync}} = \sum_{k=1}^n P_k / \sum_{k=1}^n D_k$. Assume that the algebraic connectivity dominates the non-uniformity in scaled power inputs, i.e.,*

$$\lambda_2(L(a_{ij})) > \lambda_{\text{critical}} \triangleq \left(\sum\nolimits_{i,j=1,i<j}^{n} \left|\tilde{P}_i - \tilde{P}_j\right|^2\right)^{1/2}. \quad (15)$$

Then there exists a locally exponentially stable synchronized trajectory with synchronization frequency ω_{sync} and phase cohesiveness in $\bar{\Delta}(\gamma_{\min})$, where $\gamma_{\min} \in [0, \pi/2[$ is the unique solution to $\sin(\gamma_{\min}) = \lambda_{\text{critical}}/\lambda_2(L(a_{ij}))$.

Proof: By Theorem III.2, there exists a locally exponentially stable synchronized solution of the power network model (1)-(2) if and only if the topological Kuramoto model (11) features a locally exponentially stable synchronized equilibrium. A sufficient condition for the latter to be true is given by condition (15), as proved in [3, Theorem V.5] (where all phase shifts φ_{ij} in [3] are set to zero). Moreover, [3, Theorem V.5] asserts that such a synchronized solution is phase cohesive in $\bar{\Delta}(\gamma_{\min})$ with $\gamma_{\min}$ as stated above. ■

The sufficient synchronization condition (15) is conservative for more than $n = 2$ nodes and further research is necessary to derive a tighter condition. However, the results in this section highlight the crucial role of the dissipation terms D_i, the scaled power injections $\tilde{P}_i$, and the network topology and weights a_{ij} for the synchronization problem. The insights gained by these results may be beneficial develop new strategies for coordinated (damping) controller design and controlled islanding and load shedding in a power grid.

V. Conclusions

We showed the equivalence of local synchronization in a structure-preserving power network model, local synchronization in a non-uniform Kuramoto model, and local stability of a topological Kuramoto model. Based on these equivalences, we presented various synchronization conditions for power networks. Our results are only a first step and should be extended towards tighter and non-local synchronization conditions, for more detailed system models, and for the design of coordinated controllers and remedial action schemes.

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