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CONSTRUCTING STABLE PATH in PARAMETER VARIATION SPACE

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Abstract. Although generating the *largest* or the *most-extended* set of points in the *"Parameter Variation Space" (PVS),* when its corresponding perturbed matrices are stable is the main topic of research in linear systems under multiple, large-parameter variations; a thorough analysis of this problem reveals that *bridging* two stable conditions forms its nucleus.

1. Introduction. Stability is an *intrinsic* property of a dynamical system, and as such, just as any law of physics, its degree of existence also known as the domain for parameter variations, is independent from any method of measurement. That means we *cannot change* these properties by using only **non-invasive measuring techniques.** Despite a large number of articles introducing various methods under which a given dynamical system with *multiple, large-parameter variations* remains stable, one important problem of connecting two stable conditions of such systems has escaped a proper scrutiny; and its direct solution when such a path exists, has not been addressed in the literature. Our current method aims at and always constructs the farthest bridge from the origin when possible. Now that we have introduced and solved this general problem, we welcome possible new solutions from others.

1.1. Genesis of Stability Robustness Measure. We only consider linear systems, which are modeled by a set of ordinary differential equations in a matrix form. When our models are described by polynomials, we transform them into matrix forms. Thus, our stability robustness analysis is directly related to that of an $n \times n$ stable matrix $S(\theta)$; where $\theta \in R^r$ is the vector of system physical parameters, in general $r \neq n$ and often $r \ll n$. We now have two ways of introducing Euclidean spaces in which θ changes. (*i*) We may choose the so-called *"Parameter Space,"* in which the origin is at $\theta \equiv 0$ and each parameter changes in $[0, \infty)$. This choice however, may not always be a proper one, because in every physical system there are constraints on the actual size and/or domain of parameter variations that would render such an infinite-horizon assumption fruitless. In fact, in most real-operating systems, the *physical range of parameter variations* is often finite, and each system has its own specific region for allowable range of variations with respect to its initial *operating condition,* shown by θ^o. Thus instead of (*i*), we choose (*ii*) the *"Parameter Variation Space" (PVS)* as depicted in Fig. 1.1 for $r = 3$ [6]. Here, the origin is at θ^o, and the variables are components of $\Delta\theta \in R^r$, where *theoretically* $\Delta\theta_i \in (-\infty, \infty)$, $i = 1, \cdots r$, when $\theta^o \to \theta^o + \Delta\theta$ [or $S(\theta^o) \to S(\theta^o) + \Delta S(\Delta\theta)$]. Using *PVS* also accounts for cases when different dynamical systems have the same mathematical model, but a different range of parameter variations. If we only had say $\Delta\theta_1$, then its maximum variation is found uniquely using a plethora of classical methodologies. Similarly, we may seek the maximum variations on all other axes of the *PVS.* In certain cases, variations on some axes may *theoretically* extend to infinity. Thus in any *PVS,* with its origin at θ^o (our *reference, stable-operating point*), we know unambiguously how far on each axis a parameter can change before violating the stability of θ^o. When our analysis is completed (i.e., variations on all axes are incorporated simultaneously), the resulting hyper-box is called the *"largest"* or the *"most-extended"* hyper-box for system stability that is often a non-convex set [6]. Herein the *"most"* refers to extending on *all axes* of *PVS.*

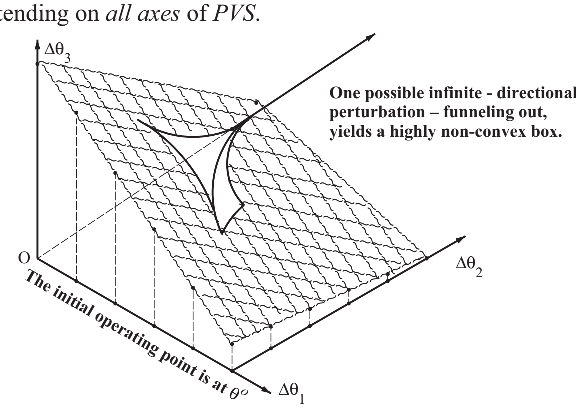

Fig. 1.1. The *largest* or the *most-extended* set of parameter variations, *E-Box* [6].

Constructing this *extended* hyper-box, also known as an *E-Box* [4], is a major task. Instead, its approximated version called *stability robustness measure (SRM)* is used that reflects the ability of a dynamical system to tolerate *"large"* $\Delta\theta$, while maintaining system stability, Fig. 1.2. Often we must also meet a pre-specified set of physical constraints in the system. For instance, when constructing a typical *SRM,* if $\Delta\theta_i \le \Delta\theta_i^{Max}$, *per se,* then any projected variation beyond $\Delta\theta_i^{Max}$ must be discarded, which leads us to a truncated *SRM,* cf., Fig. 1.3.

Now, what if in a system with multiple-varying parameters, we were only interested to connect a typical $\Delta\theta_i^{Max}$ to another $\Delta\theta_j^{Max}$? Incidentally, in many linear systems, all we need for constructing its *SRM*, is this bridging algorithm, because of the specific structure of perturbation matrices as outlined in [6]. Indeed connecting any two stable matrices in the *PVS,* for which the entire *path* (a straight or segmented line) is stable, is the *nucleus for building any hyper-box.* Thus, we have a strong result that culminates all the preceding constructional methods once and for all. Here, we provide that missing link for stable matrices by choosing the farthest (the least conservative, yielding the most robust *SRM*) bridge from the origin (θ^o) in the *PVS* when connecting two stable points; and put in perspective all pertinent results, cf., Fig. 3.1. Extension of these results to discrete-time (or convergent) systems is forthcoming.

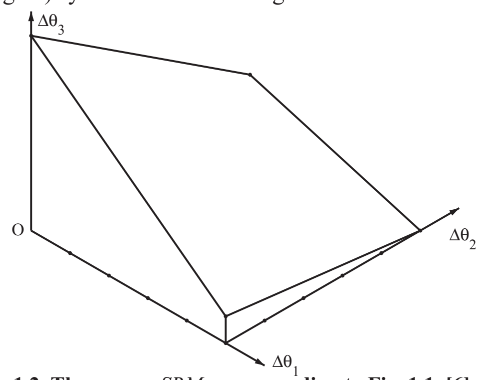

Fig. 1.2. The convex *SRM* corresponding to Fig. 1.1, [6].

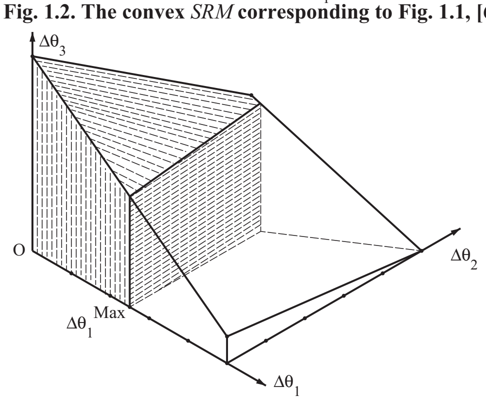

Fig. 1.3. The *SRM* of Fig. 1.2 with additional constraint(s) [6].

2. Earlier Results. Our results are stemming from the matrix equation $S^TP + PS + Q = 0$, where S is a stable matrix and $Q = Q^T > 0$ is chosen by us, yielding $P = P^T > 0$ [2] and [11]. When this so-called direct method of Lyapunov, was reintroduced in the West [8 to 10], many articles appeared for analyzing what if $S \to S + \Delta S$, then "how much" ΔS can the system tolerate before loosing its stability? Reza [12] was the first to use "common P", followed by Barnett *et al.* [1], [15 to 16] who published several pioneering articles for a class of "local" solutions.

2.1. Directional Perturbations. To prove a central piece of our work (*Lemma 4.1*), we use Theorem 2.1 that provides a sufficient condition for asymptotic stability of a perturbed matrix when $S \to S + \Delta S$. Below, by $N \geq M$, for $0 < N = N^T \in R^{n\times n}$ and $0 \leq M = M^T \in R^{n\times n}$, we mean $N - M \geq 0$. As elaborated in their original sources, utilizing these sufficient conditions repeatedly yields the maximum allowable range of variations for a single-parameter or a directional perturbation.

Theorem 2.1 ([3], [7], [13], [14]): Consider $\dot{x}(t) = S(\theta^o)x(t)$, where $S \in R^{n\times n}(R^r)$ is an asymptotically stable matrix satisfying the Lyapunov equation $S^TP + PS + Q = 0$, with $Q = Q^T > 0$ and $P = P^T > 0$. Let $S \to S + \Delta S$, then $S + \Delta S$ remains asymptotically stable if

$$(2.1)\qquad \Delta S\, Q^{-1}\, \Delta S^T < (1/4)P^{-1}\, Q\, P^{-1}, \quad \text{or}$$

$$(2.2)\qquad \Delta S^T\, P\, Q^{-1}\, P\, \Delta S < (1/4)Q.$$

When Lyapunov equation is set as $SP + PS^T + Q = 0$ (clearly this "P" is different from the preceding one), the above two inequalities respectively become:

$$(2.3)\qquad \Delta S^T\, Q^{-1}\, \Delta S < (1/4)P^{-1}\, Q\, P^{-1}, \quad \text{or}$$

$$(2.4)\qquad \Delta S\, P\, Q^{-1}\, P\, \Delta S^T < (1/4)Q.$$

2.2. Multiple Parameter Variations. When perturbing a multi-parameter system in all directions, we cannot use directional results alone to construct *SRM,* because that approach requires infinitely many directional perturbations. *Even so, we cannot connect the end points, because there is no theoretical foundation to support such a construction.* In search of expediting our method, we have discovered that when ΔS satisfies certain convexity property (Theorem 2.2), we can generate *set by set* of parameter variations in which its augmentation sweeps the *SRM.*

Theorem 2.2 (E-Box [4], [6]): Consider for instance the sufficient conditions (2.1) that is given in Theorem 2.1, with $Q = 2I$ ("I" means an *identity matrix*) repeated below.

$$(2.5)\qquad \Delta S(\Delta\theta)\, \Delta S^T(\Delta\theta) < P^{-2},$$

where $\theta \in R^r$ is the vector of system parameters. Suppose from this inequality we have extracted, in principle, all of its 2^r possible "solutions" $\Delta S_1(\Delta\theta_1, \ldots, \Delta\theta_r)$, $\Delta S_2(\Delta\theta_1, \ldots, \Delta\theta_r)$, ..., $\Delta S_{2^r}(\Delta\theta_1, \ldots, \Delta\theta_r)$, each corresponding to one point (or vertex) in the corresponding space of $(\Delta\theta_1, \ldots, \Delta\theta_r)$. Because we have assumed that the number of system parameters is finite, and these parameters change continuously, therefore it is obvious that these points or vertices in the *PVS* form a convex hull in that space. If for each and every point inside this hyper-box we can express the corresponding ΔS in terms of the above ΔS_i's as follows.

$$(2.6)\quad \Delta S = \textstyle\sum_{i=1}^{2^r} w_i \Delta S_i, \ \text{with} \ \sum_{i=1}^{2^r} w_i = 1, \quad 0 < w_i < 1, \ \text{for all i.}$$

Then each and every point inside and on the boundaries of this convex hull corresponds to an asymptotically stable $S + \Delta S$. Thus, when the system is stable at the above 2^r finite vertices, it is stable for each and every point inside the hyper-box that is constructed by connecting these vertices.

2.3. Extensions to Hurwitz Polynomials. Consider a monic-rational polynomial $\Delta(s, \theta) \triangleq \sum_{j=0}^{n} a_j(\theta)s^{n-j}$, $a_0 \equiv 1$, where again $\theta \in R^r$ is a vector of system physical parameters, and "s" is a complex variable. Determination of $\theta \in R^r$ and how this vector propagates into $a_j(\theta)$'s is a nontrivial task. For simplicity however, we often choose the coefficients $c \triangleq [a_1, \ldots, a_n]$, at any nominal operating point c^o, as the set of varying parameters; and seek conditions under which the multiple-large parameter variations Δc will not affect the stability of perturbed polynomial as

$c^o \to c^o + \Delta c$. Here, we have two equivalent sets of objects. *First* set corresponds to points $[\Delta a_1, \ldots, \Delta a_n]^T$ in the *PVS* (its origin is at c^o or θ^o). *Second* set corresponds to perturbed polynomials $\tilde{\Delta}(s, c^o + \Delta c)$, which each is computed at one new operating point.

Theorems 2.1 & 2 are applicable to the above polynomials as well. To that end, there corresponds *for instance* a matrix $S(\theta)$ in the phase-variable canonical form, $S(\theta) \triangleq \begin{pmatrix} \mathbf{0} & & \mathbf{I} \\ -a_n(\theta) & \cdots & -a_1(\theta) \end{pmatrix}$, such that $\det[sI - S(\theta)]$ becomes $\Delta(s, \theta)$. Thus the problem of studying the *largest* variations in coefficients of $\Delta(s, \theta)$, such that $\tilde{\Delta}(s, \theta + \Delta\theta)$ remains stable, becomes that of determining the "allowable range of variations" for this $S(\theta)$, when $S \to S + \Delta S$ and in the context of Theorems 2.1 & 2 [5].

3. Contribution – New Results. Again we use the direct method of Lyapunov for establishing our results. We acknowledge that [1], [15 to 16] (among many other articles by these authors) had influenced our earlier thought. We begin with $S^T P + SP + Q = 0$, where S is the stability matrix, $Q = Q^T > 0$ is chosen by us, and $P = P^T > 0$ is the solution of this equation. Now, what if, $S \to S + \Delta S$; then can we *"hold onto"* the same P (a *common* "P" for "S" and **a** "$S + \Delta S$"), by changing the Q? In other words, to maintain the stability of "$S + \Delta S$": can we adjust "Q", while holding onto the same common "P"? The answer is yes, however, it became clear that changing "Q" did not often correspond to a physically meaningful perturbation policy or a desirable ΔS [1]. Most importantly, all computations which were used to adjust "Q" for holding onto the same "P", were not utilized efficiently. There is also an element of "distance" between "S" and "$S + \Delta S$" when holding onto "P", which is why, we introduce below the *"radius of influence,"* in order to address this inherent limitation for having a common "P". Thus based on all lessons learned [6], and from the outset, we have chosen a fixed Q (often an **identity matrix** to eliminate computing Q^{-1}), and have solved for P; then we have iterated *outward* the procedure to expand the *computed stability region,* as was first introduced in [3]. Herein, the iterative procedure is *inward* rather than *outward,* and we set $\mathbf{Q} = \mathbf{I}$ as well.

N.B. *3.1:* In the next theorem where two directional perturbation matrices are constructed from a stable matrix $S(\theta^o)$ such that *the entire line from the origin of the PVS* (at θ^o) *to each of these two end points is asymptotically stable,* then we want to bridge these two end points appropriately. □

Definition 3.1 (Radius of Influence): Suppose we have two asymptotically stable matrices $S + \Delta S_\alpha$ and $S + \Delta S_\beta$ with their corresponding Lyapunov equations as follows.

$$(3.1) \qquad (S + \Delta S_\alpha)^T P_\alpha + P_\alpha(S + \Delta S_\alpha) + I = 0,$$

$$(3.2) \qquad (S + \Delta S_\beta)^T P_\beta + P_\beta(S + \Delta S_\beta) + I = 0.$$

Here, $P_{\alpha \& \beta} = P^T_{\alpha \& \beta} > 0$, and the generating matrix I for these equations is an **identity matrix.** Replacing P_α with P_β and *vice versa* in the above yield

$$(3.3) \qquad (S + \Delta S_\alpha)^T P_\beta + P_\beta(S + \Delta S_\alpha) = -Q^\alpha_\beta,$$

$$(3.4) \qquad (S + \Delta S_\beta)^T P_\alpha + P_\alpha(S + \Delta S_\beta) = -Q^\beta_\alpha.$$

If both $Q^\alpha_\beta = Q^{\alpha T}_\beta > 0$ and $Q^\beta_\alpha = Q^{\beta T}_\alpha > 0$, then we call the above two stable matrices exhibiting the *radius of influence property.*

Theorem 3.1: Let $S(\theta^o) \in R^{n \times n}(\theta^o)$ be an asymptotically stable matrix at the operating point $\theta^o \in R^r$, which is also the origin of the space of $(\Delta\theta_1, \ldots, \Delta\theta_r)$ (or *PVS*). Perturbing $S(\theta^o)$ in two directions of α and β yields perturbed matrices $S_\alpha = S + \Delta S_\alpha$ and $S_\beta = S + \Delta S_\beta$. If S_α and S_β exhibit the *radius of influence property,* then their convex combination is asymptotically stable.

Proof: Let $0 < w_\alpha < 1$ and $0 < w_\beta < 1$ such that $w_\alpha + w_\beta = 1$. Also, let $S_\gamma = S + \Delta S_\gamma$, such that $S_\gamma = w_\alpha S_\alpha + w_\beta S_\beta$ resulting in $S_\gamma = S + w_\alpha \Delta S_\alpha + w_\beta \Delta S_\beta$. Because, S_α and S_β are exhibiting a radius of influence property, by Definition 3.1, they satisfy (3.1) to (3.4). To prove that the convex combination of these two matrices (i.e., S_γ) is also asymptotically stable, we show that there exists a $P_\gamma = P^T_\gamma > 0$ such that the following symmetric matrix becomes positive definite.

$$(3.5) \quad \tilde{Q} \triangleq -(S + w_\alpha \Delta S_\alpha + w_\beta \Delta S_\beta)^T P_\gamma - P_\gamma(S + w_\alpha \Delta S_\alpha + w_\beta \Delta S_\beta).$$

Suppose we choose $P_\gamma = [(P_\alpha / w_\alpha) + (P_\beta / w_\beta)]$, where $P_{\alpha \& \beta}$ come from (3.1 & 2). Certainly, $P_\gamma = P^T_\gamma > 0$. Substituting this P_γ into (3.5) yields.

$$\begin{aligned} \tilde{Q} \triangleq & -\{[w_\alpha(S + \Delta S_\alpha) + w_\beta(S + \Delta S_\beta)]^T[(P_\alpha / w_\alpha) + (P_\beta / w_\beta)] \\ (3.6) \quad & + [(P_\alpha / w_\alpha) + (P_\beta / w_\beta)][w_\alpha(S + \Delta S_\alpha) + w_\beta(S + \Delta S_\beta)]\}. \end{aligned}$$

Rearranging (3.6) results in

$$\begin{aligned} \tilde{Q} \triangleq & -\{(S + \Delta S_\alpha)^T P_\alpha + \frac{w_\alpha}{w_\beta}(S + \Delta S_\alpha)^T P_\beta + \frac{w_\beta}{w_\alpha}(S + \Delta S_\beta)^T P_\alpha \\ & + (S + \Delta S_\beta)^T P_\beta + P_\alpha(S + \Delta S_\alpha) + \frac{w_\beta}{w_\alpha} P_\alpha(S + \Delta S_\beta) \\ (3.7) \quad & + \frac{w_\alpha}{w_\beta} P_\beta(S + \Delta S_\alpha) + P_\beta(S + \Delta S_\beta)\}. \end{aligned}$$

Substituting (3.1) and (3.2) in (3.7) yields

$$\begin{aligned} \tilde{Q} \triangleq & -\{-I + \frac{w_\alpha}{w_\beta}[(S + \Delta S_\alpha)^T P_\beta + P_\beta(S + \Delta S_\alpha)] \\ (3.8) \quad & + \frac{w_\beta}{w_\alpha}[(S + \Delta S_\beta)^T P_\alpha + P_\alpha(S + \Delta S_\beta)] - I\}. \end{aligned}$$

Because $S + \Delta S_\alpha$ and $S + \Delta S_\beta$ by Definition 3.1 exhibit the radius of influence property; thus (3.3 & 4) are satisfied, yielding $Q^\beta_\alpha = Q^{\beta T}_\alpha > 0$ and $Q^\alpha_\beta = Q^{\alpha T}_\beta > 0$. Now, incorporating (3.3 & 4) in (3.8) yields the following that completes our proof.

$$(3.9) \quad \tilde{Q} \triangleq 2I + \frac{w_\alpha}{w_\beta} Q^\alpha_\beta + \frac{w_\beta}{w_\alpha} Q^\beta_\alpha > 0.$$

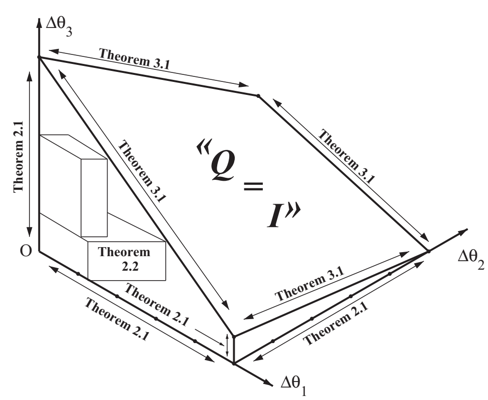

Fig. 3.1. The construction of a typical *SRM.*

Comment 3.1: When *both tests* (3.3 & 4) pass, P_α and P_β that are already in our disposal (3.1 & 2), can be interchanged, and thus $P_\alpha + P_\beta$ becomes common at both ends, and of course each end has a different "Q" relative to this common "P". Here, Definition 3.1 is different from seeking at the *outset* a common "P" corresponding to two end points. *Because seeking such a common* "P" *at the outset requires changing* "Q", *which we do not undertake in our entire approach,* as depicted by "$\mathbf{Q}=\mathbf{I}$" in Fig. 3.1. Thus, it is imperative to appreciate the fact that knowing $P_\alpha + P_\beta$ becomes a common "P" at both ends *does not add any value to our constructional procedure* that follows.

4. Algorithm. We begin with two stable end points α and β in the *PVS* with stable origin "O" (corresponding to θ^o), and search a possible stable path in the plane of $\alpha O \beta$ that is the farthest from "O" (i.e., aside from $\alpha O \beta$ itself). We solve for the corresponding P_α and P_β from (3.1 & 2), where each Lyapunov equation uses $Q = I$. Then we test whether we can substitute P_α for P_β and *vice versa,* and maintain (3.3 & 4), i.e., both $Q_\alpha^\beta > 0$ and $Q_\beta^\alpha > 0$. When either of these *two* tests fails, we bisect the straight line between α and β and study the stability of its mid-point. If this mid-point is not stable, then we proceed to shrink its distance from the origin in the *PVS* (cf., Example 5.1). But, if the mid-point is stable, then we use this point as a new *anchor* to proceed with either end point individually. (In certain cases, the Algorithm extends the line from the origin to the new stable mid-point that is further away from the line connecting end points directly, and uses that new point as an anchor.) This bisection process continues until (3.3 & 4) hold between one original end point and one generated mid-point. Then the process is repeated reversely until the path connecting two original end points is constructed completely. Of course, all of these are contingent upon the existence of a path other than $\alpha O \beta$, because by the method of our construction the initial path $\alpha O \beta$ is already stable (cf., **N.B.** *3.1*). However, in general we want to enhance this path in the sense of distancing it from the origin "O" rather than going through it, and there is a trade off for that construction.

Lemma 4.1: The iteration number for bisection procedure when a path exists is finite.

Proof: Consider two stable, starting end-points matrices $S + \Delta S_\alpha$ and $S + \Delta S_\beta$, with its mid-point (averaging point, also called the "45^o line" [6]) as follows.

$$(4.1)\quad S + \Delta S_{45^o} \triangleq (½)[(S + \Delta S_\alpha) + (S + \Delta S_\beta)] = S + (½)\Delta S_\alpha + (½)\Delta S_\beta.$$

If this mid-point is not stable, then we shrink the line connecting the origin of *PVS* to this point, and use it as the new end point. That means we replace $S + \Delta S$ with $S + q_J \Delta S$, for $q_J < 1$, and J is the iteration number that is the same as bisection number described below. We calculate the correct length of a typical "q" (by iteratively using Theorem 2.1 [3 & 7]) and the result is

$$(4.2)\quad S + (½) q_1 \Delta S_\alpha + (½) q_1 \Delta S_\beta. \quad q_1 < 1,$$

We now use this new end point or anchor and continue on the α-side, *per se,* to reach the end point corresponding to ΔS_α. The averaging procedure yields a new "45^o line" as follows.

$$(½)[S + (½) q_1 \Delta S_\alpha + (½) q_1 \Delta S_\beta] + (½)(S + \Delta S_\alpha) =$$

$$(4.3)\quad S + [(½)^2 q_1 + (½)]\Delta S_\alpha + (½)^2 q_1 \Delta S_\beta.$$

If this mid-point is not stable, we use the next q_J to shrink it again

$$(4.4)\quad S + [(½)^2 q_1 q_2 + (½) q_2]\Delta S_\alpha + (½)^2 q_1 q_2 \Delta S_\beta, \quad q_1, q_2 < 1.$$

After N_α trials, we have:

$$S + [(½)^{N_\alpha} q_1 q_2 \cdots q_{N_\alpha} + (½)^{N_\alpha - 1} q_2 \cdots q_{N_\alpha} + \cdots + (½)]\Delta S_\alpha$$

$$(4.5)\quad + (½)^{N_\alpha} q_1 q_2 \cdots q_{N_\alpha} \Delta S_\beta, \quad q_J < 1, \text{ for } J = 1, \ldots, N_\alpha.$$

Clearly, and because each $q_J < 1$ for all J (and thus $q_1 q_2 \cdots q_{N_\alpha} \approx 0$), the *influence* of ΔS_β in choosing the first segment on the α-side decreases substantially, confirming the inherent distance limitation between two end points, which prohibits having "one P" in general for the entire bridge. Now, approximating the above bracket with:

$$[(½)^{N_\alpha} q_1 q_2 \cdots q_{N_\alpha} + (½)^{N_\alpha - 1} q_2 \cdots q_{N_\alpha} + \cdots + (½)] <$$

$$(4.6)\quad [(½)^{N_\alpha} + (½)^{N_\alpha - 1} + \cdots + (½)] = 1 - (½)^{N_\alpha}.$$

Also, because of $q_J < 1$ for all J (therefore $q_1 q_2 \cdots q_{N_\alpha} \approx 0$), the (4.5) on using (4.6) becomes

$$S + [1 - (½)^{N_\alpha}]\Delta S_\alpha + (½)^{N_\alpha} q_1 q_2 \cdots q_{N_\alpha} \Delta S_\beta$$

$$(4.7)\quad \approx S + [1 - (½)^{N_\alpha}]\Delta S_\alpha = S + \Delta S_\alpha - (½)^{N_\alpha} \Delta S_\alpha.$$

Again, recalling that $S + \Delta S_\alpha$ satisfies (3.1), thus $-(½)^{N_\alpha} \Delta S_\alpha$ plays the role of its perturbation and according to Theorem 2.1, also with $Q = I$, we have the following estimate for meeting stability

$$(4.8)\quad (½)^{2N_\alpha} \Delta S_\alpha \Delta S_\alpha^T < (¼) P_\alpha^{-2}.$$

A similar analysis on the β-side yields an estimate for the number of trials toward its end point, N_β, as follows.

$$(4.9)\quad (½)^{2N_\beta} \Delta S_\beta \Delta S_\beta^T < (¼) P_\beta^{-2}.$$

An estimate of the total number of bisections connecting α to β is

$$(4.10)\quad N = N_\alpha + N_\beta,$$

which is finite and upon generating (4.8 & 9), it can be estimated by inspection (cf., Example 5.1).

Comment 4.1 (On the inward iterative procedure): Since these kinds of computations (averaging two matrices with finite-value elements, and solving Lyapunov equations) are well-posed and straightforward, we have experienced no difficulty and anticipate none in repeatedly using our results. The Algorithm is completely automated, and uses subroutines: to solve two versions of Lyapunov equations with $Q = I$ or $2I$; to calculate eigenvalues of matrices and to perform standard matrix algebra operations.

Algorithm 4.1:

0. Given two directional stable end points α [or $(S + \Delta S_\alpha)$] and β [or $(S + \Delta S_\beta)$] in the *PVS*, an Euclidean space with stable origin "O" [or (S)]; we search a possible path in the plane of $\alpha O \beta$. Let $J = 1, \ldots, N$, where N is the total number of possible iterations or bisections, cf., (4.10). Let the mid-point between two points be shown by M^J.

1. For stable matrices $S + \Delta S_\alpha$ and $S + \Delta S_\beta$, test the *radius of influence property* (3.3 & 4).

1.1. If true, then connect the two end points. END.

1.2. If untrue, Go to Step 2.

2. Select α side. Bisect the line connecting two end points, and test the stability of its mid-point M^J. [*Comment:* In certain cases a line connecting "O" to M^J must be extended from the line connecting end points directly, if one wishes to connect the path for constructing the *largest* stable set, which is a separate issue.]

2.1. If true, Go to Step 3.

2.2. If untrue, Go to Step 6.

3. Test the corresponding (3.3 & 4), between α and the stable mid-point (anchor) M^J.

3.1. If true, connect α to M^J. Go to Step 7.

3.2. If untrue, Go to Step 4.

4. Continue bisecting between α and M^J to find *stable* anchor M^{J+1}.

4.1. If (3.3 & 4) are true, then connect α to M^{J+1}. Go to Step 7.

4.2. If (3.3 & 4) is untrue, Go to Step 5.

5. Repeat bisecting between α and the mid-point M^{J+1}, seeking new *stable* anchor, Go to Step 3.

6. If M^J is not stable, *SRM* is non-convex. Now on a line connecting the origin "O" to M^J, find the farthest extreme point from "O" for which the system is stable (iteratively using Theorem 2.1). That point becomes the new anchor between the underlying two end points, Go to Step 3.

7. Finally, when α is connected to M^J, reverse the construction by replacing M^J for α in Step 2, while tracing the previous mid-points as anchor points until reaching M^1.

8. Go to Step 2, and replace α with β and continue repeating the construction. END.

5. Numerical Example. We have selected one example, in order to explore the contribution of our results. Our method works for matrices with multiple-varying parameters, and can be extended to polynomials with linear coefficient variations.

Example 5.1: Consider the following stable matrix S that is perturbed in two symmetric directions of ΔS_α and ΔS_β, only for the ease of demonstration, as follows.

$$(5.1)\quad S=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},\ \Delta S_\alpha=\begin{pmatrix}0 & 1.3\\ 0.75 & 0\end{pmatrix},\ \Delta S_\beta=\begin{pmatrix}0 & 0.75\\ 1.3 & 0\end{pmatrix}.$$

Here, $\det[\lambda I-(S+\Delta S_{\alpha\text{ or }\beta})]=\lambda^2+2\lambda+1-0.975$, which shows both directional perturbed matrices are stable. However, the mid-point (M_α^β in Fig. 5.1) between two end points corresponding to

$$(5.2)\quad S+(½)(\Delta S_\alpha+\Delta S_\beta)\triangleq S+\Delta S_{45^o}=\begin{pmatrix}-1 & 1.025\\ 1.025 & -1\end{pmatrix},$$

yields $\det[\lambda I-(S+\Delta S_{45^o})]=\lambda^2+2\lambda+1-1.025^2$, which is unstable. What is the implication of this development and how does that fit in the context of our results?

Solution: Clearly, in this case the convex combination of matrices at two end points does not hold on the straight line connecting both ends, because the *SRM* of the original system is non-convex. In other words, we must work within the constraint imposed by the system dynamics to construct the stable path, and that is the beginning of correctly applying our new procedure.

Looking for a path connecting two stable end points, we first study the corresponding (4.8 & 9) (or (4.10)) for estimating the total number of bisections needed to complete the task. For ΔS_α and ΔS_β in (5.1), we solve for $P_{\alpha\,\&\,\beta}$ as follows.

$$(5.3)\quad P_\alpha=\begin{pmatrix}15.875 & 20.5\\ 20.5 & 27.15\end{pmatrix},\ P_\beta=\begin{pmatrix}27.15 & 20.5\\ 20.5 & 15.875\end{pmatrix}.$$

Constructing (4.8 & 9) yields $N_\alpha\geq 4$ and $N_\beta\geq 4$. Thus we expect the total number of bisections to be $N=N_\alpha+N_\beta\geq 8$. Certainly the trade off is: higher the number of iterations; the less conservative the path becomes, yielding a more robust *SRM.*

If we did not know the situation in this example (i.e., (5.2)) and had applied Theorem 3.1, then both Q_α^β and Q_β^α [cf., (3.3 & 4)] would have failed. Under these conditions, and according to Algorithm 4.1, we seek directional perturbations along the "45^o line". Here, we have already established that M_α^β in Fig. 5.1 is unstable. Along the "45^o line" the farthest stable point from origin is M_{45}, Fig. 5.1, which is closer to the origin than M_α^β. Our numerical computation shows that M_{45} corresponds to $\Delta a_{12}=\Delta a_{21}=1<1.025$ (of that for M_α^β). To enhance numerical computations, we choose M_{45} below this value, say at $\Delta a_{12}=\Delta a_{21}=0.9$. Then with the preceding Δa_{12} and Δa_{21}, we get

$$(5.4)\qquad P_{M_{45}}=\begin{pmatrix}2.631 & 2.368\\ 2.368 & 2.631\end{pmatrix}.$$

We evaluate (3.3 & 4) between M_{45} and each end point. Starting with the α-side, we have

$$(5.5)\qquad (S+\Delta S_\alpha)^T P_{M_{45}}+P_{M_{45}}(S+\Delta S_\alpha)=-Q_{M_{45}}^\alpha.$$

Using the corresponding ΔS_α, we have

$$(5.6)\qquad Q_{M_{45}}^\alpha=\begin{pmatrix}1.71 & -0.66\\ -0.66 & -0.89\end{pmatrix}<0.$$

This shows there might be another segmentation between M_{45} and α; thus we seek another directional perturbation to bisect the line connecting M_{45} and α, shown by M_{45}^α, and as follows.

$$(5.7)\qquad \Delta S_{M_{45}^\alpha}=(½)(\Delta S_\alpha+\Delta S_{M_{45}})=\begin{pmatrix}0 & 1.1\\ 0.825 & 0\end{pmatrix}.$$

This point is stable, and to follow the pictorial description in Fig. 5.1, we choose point $M^{\#1}$, which is calculated on the line between the origin and M_{45}^α and is proportionally below M_{45}^α, Fig. 5.1.

N.B. *5.1:* To be consistent with Algorithm 4.1, we should have called M_{45} as M^1 and $M^{\#1}$ as M^2, *etc.,* however we thought this selection may read better. □

For the following $\Delta S_{M^{\#1}}$

$$(5.8)\qquad \Delta S_{M^{\#1}}\triangleq\begin{pmatrix}0 & 1\\ 0.75 & 0\end{pmatrix},$$

the corresponding $P_{M^{\#1}}$ and $Q_{M^{\#1}}^\alpha$ are

$$(5.9)\qquad P_{M^{\#1}}=\begin{pmatrix}1.8125 & 1.75\\ 1.75 & 2.25\end{pmatrix},\ Q_{M^{\#1}}^\alpha=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}>0.$$

However,

$$(5.10)\qquad Q_\alpha^{M_{\#1}}=\begin{pmatrix}1 & 4.7625\\ 4.7625 & 13.3\end{pmatrix}<0.$$

That means we must bisect again. We now proceed to select another mid-point between $M^{\#1}$ and α, as shown by $M^{\#2}$ in Fig.

5.1. The corresponding $\Delta S_{M^{\#2}}$, $P_{M^{\#2}}$, $Q^{\alpha}_{M^{\#2}}$ and $Q^{M^{\#2}}_{\alpha}$ are as follows.

$$(5.11)\qquad \Delta S_{M^{\#2}} = \begin{pmatrix} 0 & 1.15 \\ 0.75 & 0 \end{pmatrix}, \quad P_{M^{\#2}} = \begin{pmatrix} 1.676 & 1.568 \\ 1.568 & 2.303 \end{pmatrix}.$$

$$(5.12)\qquad Q^{\alpha}_{M^{\#2}} = \begin{pmatrix} 1 & -0.51865 \\ -0.51865 & 0.9996 \end{pmatrix} > 0,$$

$$(5.13)\qquad Q^{M^{\#2}}_{\alpha} = \begin{pmatrix} 1 & 2.3815 \\ 2.3815 & 7.15 \end{pmatrix} > 0.$$

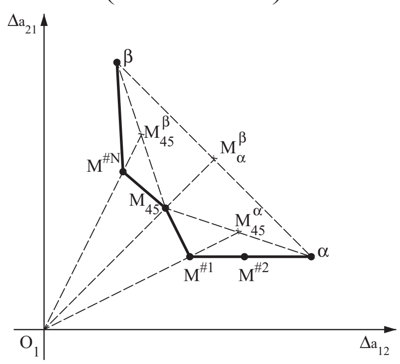

Fig. 5.1. A pictorial view of attending non-convexity, the construction of a path between two stable points.

Now, all tests are satisfactorily completed. Thus, the matrix is stable along the entire line between $M^{\#2}$ and α. We then reverse the computations and build the rest of the path starting from $M^{\#2}$ (as a new starting α) to $M^{\#1}$; followed by that to M_{45}, …, $M^{\#N}$ and finally β. Since we already have most components of our computations, for instance, $P_{M^{\#1}}$, $Q^{\alpha}_{M^{\#1}}$, *etc.,* the construction proceeds fast. We also note that with fine-tuning of segmentations, we have completed the process at $N_{\alpha}=3$ instead of 4, resulting in a more conservative path than that resulted from $N_{\alpha}\geq 4$. In the final analysis, the line connecting α to β is segmented − not because of our algorithm, but rather because of the dynamics in its underlying system for which the *SRM* is a non-convex set. Our procedure detects these segmentations when is used prudently and that is our contribution. We believe this is a very strong result. Our method can be used for systems with many varying parameters, and there is no comparable methodology for constructing such a stable path in the literature.

6. Conclusions. It is clear that not only our results bridges two stable points, but also, in many cases of interest to us including the entire class of polynomials when the coefficients are the only set of variables that change linearly and resulting in a convex perturbation matrix, all we need for constructing the *E-Box* is the set of the boundary lines that can now be generated easily. Thus, the actual repeated applications of Theorem 2.2 as depicted in Fig. 3.1 is not needed here, because of this convexity property. Clearly this is a strong result and a major advancement in this area. For extensions of our results to linear systems with multiple-nonlinear perturbations, as suggested in [6], the first step is to utilize a matrix-series expansion similar to techniques described in [14]. Similar opportunities for constructing *E-Boxes* in nonlinear dynamical systems, as well as in problems with dependencies among their various parameters are of interest.

Dedication. The author dedicates this article to his advisor Professor David L. Russell, who has continually supported this research since 1975, on the occasion of his 70th-birthday.

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