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Commutativity and asymptotic stability for linear switched DAEs

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Abstract— **For linear switched ordinary differential equations with asymptotically stable constituent systems, it is well known that commutativity of the coefficient matrices implies asymptotic stability of the switched system under arbitrary switching. This result is generalized to linear switched differential algebraic equations (DAEs). Although the solutions of a switched DAE can exhibit jumps it turns out that it suffices to check commutativity of the "flow" matrices. As in the ODE case we are also able to construct a common quadratic Lyapunov function.**

I. Introduction

In this paper we study *switched differential algebraic equations (switched DAEs)* of the form

$$E_\sigma \dot{x} = A_\sigma x, \tag{1}$$

where $\sigma : \mathbb{R} \to \{1, 2, \ldots, \mathtt{p}\}$, $\mathtt{p} \in \mathbb{N}$, is the switching signal, and $E_p, A_p \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$, are constant matrices for each parameter $p \in \{1, 2, \ldots, \mathtt{p}\}$. For this system class we will show that stability for the constituent systems $E_p\dot{x} = A_p x$, $p \in \{1, 2, \ldots, \mathtt{p}\}$ and commutativity of the associated "flow matrices", which describe the dynamic part of the system, is sufficient for the existence of a common quadratic Lyapunov function. This implies in particular uniform exponential stability with respect to arbitrary switching signals.

To compare the situation with the case of linear ordinary differential equations ([1], see also [2], [3]), recall that for

$$\dot{x} = A_\sigma x\,,$$

where σ is the switching signal as above and $A_p \in \mathbb{R}^{n\times n}$ is Hurwitz for $p = 1, \ldots, \mathtt{p}$, the commutativity condition

$$[A_p, A_q] := A_pA_q - A_qA_p = 0\,, \quad \forall\ p, q \tag{2}$$

implies asymptotic stability of the switched ODE for arbitrary measurable switching signals. Since a multiplication of a DAE from the left with an invertible matrix doesn't change the solution behavior but does change the commutativity property it is obvious that the condition (2) is not appropriate for the switched DAE (1). In fact, Example 6 shows that even for commuting A-matrices and stable constituent systems the switched DAE can have unstable solutions.

Switched linear DAEs occur for instance in the modeling of electrical circuits. For such systems it is natural that switching occurs, so that a deeper analysis of switched DAEs is desirable.

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There are only a few recent papers which study stability of switched DAEs in the general form (1) (see e.g. [4], [5]). In [6] the positive realness of transfer functions associated to DAEs are studied and a characterization of the existence of a common quadratic Lyapunov function for a pair of DAEs with a rank-one difference are obtain. Commutativity and the switched DAE (1), but with constant E-matrix, are studied in [7], [8], however, the authors do not discuss possible jumps in the state trajectories.

It is well known that all solutions of each individual DAE $E_p\dot{x} = A_px$, $p \in \{1, \ldots, \mathtt{p}\}$, evolve within a so-called consistency space, which is a linear subspace of $\mathbb{R}^n$. In general, at a switching time $t \in \mathbb{R}$ there does not exist a continuous extension of the solution into the future, because the value $x(t-)$ need not be within the consistency space corresponding to the DAE after the switch. Therefore, it is necessary to allow for solutions with jumps. However, this leads to difficulties in evaluating the derivative of the solutions. To resolve this problem we adopt the distributional framework introduced in [9], [10], i.e. as solutions of the switched DAE (1) distributions (generalized functions), in particular Dirac impulses, are considered. For this, we have to assume that the switching signal has only a locally finite set of switching times. Furthermore, to ensure existence and uniqueness of solutions we have to assume that each matrix pair (E_p, A_p) is regular, i.e. the polynomial $\det(E_ps - A_p)$ is not identically zero. Finally, we will make one more assumption which ensures that no impulses[1] occur in the solutions of the switched DAE (1); for details see Section III and Theorem 5. A consequence of these assumptions is that although a distributional solution framework is necessary as a theoretical basis for treating switched DAEs of the form (1), the only solutions that arise in this paper are piecewise-smooth functions with locally finitely many discontinuities.

Because of the existence of jumps, the generalization of the commutativity results is not straightforward, as it is not immediately clear how the jumps have to be incorporated. It is shown in [4] that the jumps play an essential role for the overall stability of the switched DAE (1). Surprisingly, it is sufficient to assume commutativity of the "flow matrices" for the stability of the switched DAE, given that the constituent systems are asymptotically stable. In this situation the jumps induced by the consistency projectors need not be checked.

The paper is organized as follows. In Section II we recall the solution theory for linear DAEs, define asymptotic stability for such systems and provide the solution theory for linear switched DAEs with respect to piece-wise constant switching signals. In Section III we discuss commutativity for DAEs

[1] With impulses we mean Dirac impulses or derivatives thereof, hence impulse freeness of solutions does not exclude jumps in the solution.

and show that commutativity of the flow matrices implies asymptotic stability by a direct commutativity argument. In Section IV we strengthen this result by showing that in the case of commuting flow matrices it is even possible to construct a common quadratic Lyapunov function.

The following notation is used throughout the paper. $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ are the natural numbers, integers, real and complex numbers, respectively. For a matrix $M \in \mathbb{R}^{n\times m}$, $n, m \in \mathbb{N}$, the kernel (null space) of M is $\ker M$, the image (range, column space) of M is $\operatorname{im} M$, and the transpose of M is $M^\top \in \mathbb{R}^{m\times n}$. The image of a set $\mathcal{S} \subset \mathbb{R}^n$ under M is $M\mathcal{S} := \{\ Mx \in \mathbb{R}^n \mid x \in \mathcal{S}\ \}$ and the pre-image of $\mathcal{S}$ under M is $M^{-1}\mathcal{S} := \{\ x \in \mathbb{R}^n \mid \exists y \in \mathcal{S} : Mx = y\ \}$. The identity matrix is denoted by I. For a piecewise-continuous function $f : \mathbb{R} \to \mathbb{R}$ the left-sided evaluation $\lim_{\varepsilon\searrow 0} f(t-\varepsilon)$ at $t \in \mathbb{R}$ is denoted by $f(t-)$.

II. Preliminaries

A. Consistency and differential projectors

In this section we consider the non-switched (i.e. time-invariant) DAE

$$E\dot{x} = Ax$$

with $E, A \in \mathbb{R}^{n\times n}$ and classical (i.e. differentiable) solutions $x : \mathbb{R} \to \mathbb{R}^n$. We assume that the matrix pair (E, A) is regular. The following result goes back to Weierstrass [11] and is useful for the rest of the paper:

Theorem 1 (Quasi-Weierstrass form): A pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular, if and only if there exist invertible transformation matrices $S, T \in \mathbb{R}^{n\times n}$ which put (E, A) into quasi Weierstrass form

$$(SET, SAT) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right]\right), \tag{3}$$

where $N \in \mathbb{R}^{n_2\times n_2}$, $0 \le n_2 \le n$, is a nilpotent matrix and $J \in \mathbb{R}^{n_1\times n_1}$, $n_1 := n - n_2$. ◇

We call (3) *quasi* Weierstrass form because we do not assume that J and N are in Jordan canonical form. In [12] (see also [13]) it is shown that the transformation matrices S and T in (3) can be easily obtained using the *Wong sequences* [14][2]:

$$\begin{aligned} \mathcal{V}^0 &:= \mathbb{R}^n, & \mathcal{V}^{i+1} &= A^{-1}(E\mathcal{V}^i), & i &\in \mathbb{N}, \\ \mathcal{W}^0 &:= \{0\}, & \mathcal{W}^{i+1} &= E^{-1}(A\mathcal{W}^i), & i &\in \mathbb{N}. \end{aligned}$$

It is straightforward that $(\mathcal{V}_i)_{i\in\mathbb{N}}$ is a decreasing and $(\mathcal{W}_i)_{i\in\mathbb{N}}$ is an increasing sequence of subspaces. As the dimension of $\mathbb{R}^n$ is finite, the sequences become stationary after finitely many steps and reach the limits

$$\mathcal{V}^* := \bigcap_{i\in\mathbb{N}} \mathcal{V}^i \quad \text{and} \quad \mathcal{W}^* := \bigcup_{i\in\mathbb{N}} \mathcal{W}^i\,.$$

It is known that both sequences become stationary after exactly the same number of steps [12]. With the choice of any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the transformation matrices may be represented as $T = [V, W]$ and $S = [EV, AW]$, which then leads to (3).

Definition 2 (Consistency and differential projector): Consider a regular matrix pair (E, A) with transformation matrices S, T which put (E, A) into a quasi Weierstrass form (3). The *consistency projector* $\Pi_{(E,A)}$ of (E, A) is given by

$$\Pi_{(E,A)} := T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1}, \tag{4}$$

where the block sizes correspond to the block sizes in the quasi Weierstrass form (3). The *differential projector* $\Pi^{\text{diff}}_{(E,A)}$ of (E, A) is given by

$$\Pi^{\text{diff}}_{(E,A)} := T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] S$$

and the *flow matrix* A^{diff} is given by

$$A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A = T \left[\begin{smallmatrix} J & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1}. \tag{5}$$

◇

Note that, due the special structure of the consistency projector $\Pi_{(E,A)}$ and A^{diff},

$$A^{\text{diff}}\Pi_{(E,A)} = A^{\text{diff}} = \Pi_{(E,A)}A^{\text{diff}} \tag{6}$$

The consistency projector only plays a role when considering inconsistent initial values as they occur when switching between different DAEs, see the next section. Note that the differential projector is not a projector in the usual sense, because it is not idempotent, but due the similarity to the definition of the consistency projector the matrix $\Pi^{\text{diff}}_{(E,A)}$ is also called a "projector". The importance of the differential projector (and the flow matrix) becomes clear with the next result.

Lemma 3 (Role of the differential projector, [16]): Consider the regular matrix pair (E, A) and the corresponding flow matrix A^{diff}, then a classical solution x of $E\dot{x} = Ax$ also solves

$$\dot{x} = A^{\text{diff}}x.$$

◇

Note furthermore, that in the ODE-case (i.e. when E is invertible) it is easy to see that $A^{\text{diff}} = E^{-1}A$. In particular, for the classical ODE formulation (i.e. $E = I$) it follows that $A^{\text{diff}} = A$.

B. Asymptotic stability of time-invariant DAEs

The time-invariant DAE $E\dot{x} = Ax$ is called asymptotically stable, if all (classical) solutions tend to zero as $t \to \infty$. Note that for linear DAEs with classical solutions attractivity already implies stability in the sense of Lyapunov (which is actually a direct consequence of the next result). The following result characterizes asymptotic stability of the DAE $E\dot{x} = Ax$ and is an easy corollary of Theorem 1 together with the observation that the DAE $N\dot{w} = w$ for nilpotent N only has the trivial solution.

Corollary 4 (Asymptotic stability of a DAE): Consider a regular matrix pair (E, A) with quasi Weierstrass form (3). Then the DAE $E\dot{x} = Ax$ is asymptotically stable, if and only if the underlying ODE $\dot{x} = Jx$ is asymptotically stable. In particular, asymptotic stability of $E\dot{x} = Ax$ implies invertibility of A. ◇

[2]To the best knowledge of the authors, Wong was the first who explicitly considered these sequences for studying matrix pencils. Dieudonné [15] also considered similar sequences in the context of matrix pairs, however his main focus was only on the first sequence and the second sequence does not appear explicitly.

C. Solutions of switched DAEs

As motivated in the introduction we will make the following assumptions throughout the remainder of this note.

A1 Switching signals $\sigma : \mathbb{R} \to \{1, \ldots, \mathtt{p}\}$ are piecewise constant (with a locally finite set of discontinuities) and right-continuous.

A2 Each matrix pair (E_p, A_p), $p \in \{1, \ldots, \mathtt{p}\}$, is regular, i.e. $\det(sE_p - A_p) \in \mathbb{R}[s]$ is not the zero polynomial.

A3 For the consistency projectors $\Pi_p := \Pi_{(E_p,A_p)}$, $p \in \{1, \ldots, \mathtt{p}\}$, corresponding to the regular matrix pairs (E_p, A_p) from (1), it holds that

$$E_p(I - \Pi_p)\Pi_q = 0\,, \quad \forall p, q \in \{1, \ldots, \mathtt{p}\}.$$

The set of all switching signals, i.e. all functions satisfying A1, is denoted by Σ. Under these assumptions, the following result is known.

Theorem 5 ([10, Thms. 4.2.13&4.2.8]): Consider the switched DAE (1) satisfying Assumptions A1, A2 and A3. Then every distributional solution of (1) is impulse free and is represented by a piecewise-smooth function $x : \mathbb{R} \to \mathbb{R}^n$. Furthermore, for all solutions $x : \mathbb{R} \to \mathbb{R}^n$,

$$\forall t \in \mathbb{R}: \quad x(t) = \Pi_{\sigma(t)} x(t-)$$

and all solutions are uniquely determined by $x(0-)$. $\diamond$

Following [4], we call the switched DAE (1) *asymptotically stable* with respect to a fixed switching signal $\sigma \in \Sigma$, if all distributional solutions of (1) are impulse free and each solution $x : \mathbb{R} \to \mathbb{R}^n$ fulfills $x(t) \to 0$ as $t \to \infty$. Note that Assumption A3 actually is a necessary condition for impulse freeness under arbitrary switching; hence A3 is also necessary for asymptotic stability of the switched DAE (1) under arbitrary switching. We call (1) *uniformly exponentially stable* with respect to Σ, if for all $\sigma \in \Sigma$ solutions are impulse free, and if there are constants $M, \beta > 0$ such that for all $\sigma \in \Sigma$ and all corresponding solutions $x : \mathbb{R} \to \mathbb{R}^n$

$$\|x(t)\| \leq Me^{-\beta t}\|x(0-)\|\,, \quad \forall t \geq 0\,. \tag{7}$$

III. Commutativity of DAEs

In this section we introduce a concept of commutativity for switched DAEs. We start with an example which shows that a naive generalization from the ODE case to the DAE case is not working.

Example 6 (Commuting A-matrices): Consider the switched DAE (1) with matrix pairs:

$$(E_1, A_1) = \left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right]\right)$$
$$(E_2, A_2) = \left(\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\right)$$

Clearly, the A-matrices commute and both constituent systems are asymptotically stable. However, the solution behavior of the corresponding switched DAE is identical to the solution behavior of Example 1 in [4], hence sufficiently fast switching yields unbounded solutions. $\diamond$

For ODEs it is known that two matrices $A, B \in \mathbb{R}^{n\times n}$ commute if and only if the evolution operators e^{At}, e^{Bs} commute for all $s, t \in \mathbb{R}$. To get an analogous statement for DAEs we need a representation of the evolution operators. Therefor, note that Lemma 3 gives

$$x(t) = e^{A_p^{\text{diff}}(t-t_0)}x(t_0), \quad t \in [t_0, t_1)$$

for all solutions x of the switched DAE (1) where $\sigma\big|_{[t_0,t_1)}$ is constantly p. Furthermore, Theorem 5 yields that

$$x(t_0) = \Pi_p x(t_0-),$$

hence the desired evolution operator of the p-th constituent system is given by

$$\Phi_p(t) := e^{A_p^{\text{diff}}t}\Pi_p, \quad t \geq 0. \tag{8}$$

Despite the presence of the consistency projectors in the evolution operators it turns out that commutativity of the flow matrices A_p^{diff} is sufficient and necessary for commutativity of the evolution operator, provided the original A-matrices are invertible.

Theorem 7 (Commuting evolution operators and A^{diff}): Consider the switched DAE (1) satisfying A1-A3 with corresponding matrices A_p^{diff}, consistency projectors Π_p and evolution operators $\Phi_p(\cdot)$. Assume that A_p is invertible for all $p \in \{1, \ldots, \mathtt{p}\}$. Then, for all $p, q \in \{1, \ldots, \mathtt{p}\}$,

$$[\Phi_p(t), \Phi_q(s)] = 0 \ \forall s, t \geq 0 \ \Leftrightarrow \ [A_p^{\text{diff}}, A_q^{\text{diff}}] = 0. \quad \diamond$$

Before proving Theorem 7 we formulate the important consequence for the stability of the switched DAE (1).

Corollary 8 (Commutativity and asymptotic stability): Consider the switched DAE (1) satisfying A1-A3. Assume that each constituent system $E_p\dot{x} = A_p x$, $p \in \{1, \ldots, \mathtt{p}\}$, is asymptotically stable. Then

$$[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0\,, \quad \forall\, p, q \in \{1, \ldots, \mathtt{p}\}$$

implies that (1) is asymptotically stable under arbitrary switching. $\diamond$

Proof of Corollary 8: Let $t_0 < t_1 < t_2 < \ldots$ be the switching times of σ, then, for $t \geq t_0$,

$$\begin{aligned} x(t) = \Phi_{p_k}(t - t_k)\Phi_{p_{k-1}}(t_k - t_{k-1}) \\ \cdots \Phi_{p_1}(t_2 - t_1)\Phi_{p_0}(t_1 - t_0)x(t_0-), \end{aligned} \tag{9}$$

where $k \in \mathbb{N}$ is such that $t \in [t_k, t_{k+1})$ and $\sigma(t_i) = p_i$ for $0 \leq i \leq k$. Due to Corollary 4 each matrix A_p is invertible and Theorem 7 implies commutativity of the evolution operators. Note that $\Phi_p(t + s) = \Phi_p(t)\Phi_p(s)$ for all $s, t \geq 0$ and all $p \in \{1, \ldots, \mathtt{p}\}$ because of (6) and the representation of the exponential as a power series. We obtain from (9) by commutativity

$$x(t) = \Phi_0(\Delta t_0)\Phi_1(\Delta t_1)\cdots\Phi_{\mathtt{p}}(\Delta t_{\mathtt{p}}),$$

with $\Delta t_p \to \infty$ as $t \to 0$ for at least one $p \in \{1, \ldots, \mathtt{p}\}$. This yields $x(t) \to 0$ as $t \to \infty$. ∎

The sufficiency and necessity part of the proof of Theorem 7 are each direct consequences of the following two lemmas.

Lemma 9 (Sufficiency): Consider two regular matrix pairs (E_1, A_1) and (E_2, A_2) with corresponding flow matrices A_1^{diff}, A_2^{diff} and consistency projectors Π_1, Π_2. If A_1, A_2 are invertible and $A_1^{\text{diff}}, A_2^{\text{diff}}$ commute, then

$$[\Pi_1, A_2^{\text{diff}}] \quad = \quad [\Pi_1, \Pi_2] \quad = \quad [\Pi_2, A_1^{\text{diff}}] \quad = \quad 0.$$

In particular, $[e^{A_1^{\text{diff}} s}\Pi_1, e^{A_2^{\text{diff}} t}\Pi_2] = 0$ for all $s, t \geq 0$. $\diamond$

Proof: From $[A_1^{\text{diff}}, A_2^{\text{diff}}] = 0$ it follows that

$$T_1 \left[\begin{smallmatrix} J_1 & 0 \\ 0 & 0 \end{smallmatrix}\right] T_1^{-1} T_2 \left[\begin{smallmatrix} J_2 & 0 \\ 0 & 0 \end{smallmatrix}\right] T_2^{-1} = T_2 \left[\begin{smallmatrix} J_2 & 0 \\ 0 & 0 \end{smallmatrix}\right] T_2^{-1} T_1 \left[\begin{smallmatrix} J_1 & 0 \\ 0 & 0 \end{smallmatrix}\right] T_1^{-1}.$$

Hence, with $T_1^{-1} T_2 \left[\begin{smallmatrix} J_2 & 0 \\ 0 & 0 \end{smallmatrix}\right] T_2^{-1} T_1 =: \left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right]$,

$$\left[\begin{smallmatrix} J_1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right] = \left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right] \left[\begin{smallmatrix} J_1 & 0 \\ 0 & 0 \end{smallmatrix}\right].$$

In particular,

$$J_1 M_{12} = 0 \quad \text{and} \quad M_{21} J_1 = 0.$$

By (3) invertibility of A_1 implies invertibility of J_1, so that $M_{12} = 0$ and $M_{21} = 0$. Therefore

$$\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right] = \left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right],$$

which is equivalent to

$$\Pi_1 A_2^{\text{diff}} = A_2^{\text{diff}} \Pi_1 \quad \text{or} \quad [\Pi_1, A_2^{\text{diff}}] = 0.$$

Interchanging the indices yields $[\Pi_2, A_1^{\text{diff}}] = 0$, finally, repeating the above argument with the starting point $[A_1^{\text{diff}}, \Pi_2] = 0$ (i.e. replacing J_2 by I), it also follows that $[\Pi_1, \Pi_2] = 0$. ■

Lemma 10 (Necessity): Consider two regular matrix pairs (E_1, A_1) and (E_2, A_2) with corresponding flow matrices A_1^{diff}, A_2^{diff}, consistency projectors Π_1, Π_2 and evolution operators $\Phi_1(\cdot)$, $\Phi_2(\cdot)$ as in (8). Then $[\Phi_1(s), \Phi_2(t)] = 0$ for all $s, t \geq 0$ implies

$$[A_1^{\text{diff}}, A_2^{\text{diff}}] = 0. \qquad \diamond$$

Proof: Commutativity of the evolution operators implies (setting $t = 0$ and/or $s = 0$)

$$[\Pi_1, \Pi_2] = [\Phi_1(s), \Pi_2] = [\Pi_1, \Phi_2(t)] = 0.$$

Hence, for all $s > 0$ and $t > 0$,

$$\left[\tfrac{\Phi_1(s) - \Pi_1}{s}, \ \tfrac{\Phi_2(t) - \Pi_2}{t} \right] = 0.$$

Since, for $i = 1, 2$,

$$\lim_{t \searrow 0} \tfrac{\Phi_i(t) - \Pi_i}{t} = \Phi_i'(0) = A_i^{\text{diff}} \Pi_i \overset{(6)}{=} A_i^{\text{diff}}$$

the claim follows. ■

Remark 11: If the assumption on the invertibility of the A-matrices does not hold only the necessity part of Theorem 7 remains true in general. This can be seen when considering a variation of Example 6 where we set $J_1 = 0$ in the quasi-Weierstrass form of (E_1, A_1). Consequently, $A_1^{\text{diff}} = 0$ and the commutativity condition $[A_1^{\text{diff}}, A_2^{\text{diff}}]$ trivially holds. But the consistency projectors are the same as before and do not commute, hence the evolution operators cannot commute either. $\diamond$

IV. Construction of a common quadratic Lyapunov function

In this section we carry the stability analysis of commuting DAEs a step further and show that it is possible to construct quadratic Lyapunov functions for commuting DAEs with asymptotically stable constituent systems. The common Lyapunov function will have the form, for some $l \in \mathbb{N}$,

$$V(x) := x^\top T^{-\top} \operatorname{diag}(P_1, P_2, \ldots, P_l) T^{-1} x.$$

Here the invertible matrix T is chosen such that it simultaneously block-diagonalizes the flow matrices A_p^{diff}, i.e. $T^{-1} A_p^{\text{diff}} T = \operatorname{diag}(A_{p1}, A_{p2}, \ldots, A_{pl})$ for all $p \in \{1, 2, \ldots, \mathtt{p}\}$. Using some specific properties of the blocks A_{pk}, the positive definite matrices P_k, $k = 1, \ldots, l$, can be chosen as common solutions of the Lyapunov inequalities $A_{pk}^\top P_k + P_k A_{pk} < -\alpha_k P_k$ unless $A_{pk} = 0$.

In order to show that such a construction of a common quadratic Lyapunov function is possible we recall the following facts about commuting matrices. Proofs are included for the convenience of the reader.

Lemma 12 (Commutativity and block-diagonal matrices): Consider $A, B \in \mathbb{R}^{n \times n}$ with $[A, B] = 0$. Assume there is an invertible transformation matrix $T \in \mathbb{R}^{n \times n}$ such that for certain $A_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, 2$, $n_1 + n_2 = n$ we have

$$T^{-1} A T = \left[\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right] \quad \text{with} \quad \operatorname{spec} A_1 \cap \operatorname{spec} A_2 = \emptyset.$$

Then

$$T^{-1} B T = \left[\begin{smallmatrix} B_1 & 0 \\ 0 & B_2 \end{smallmatrix}\right],$$

where B_i has the same size as A_i, $i = 1, 2$. $\diamond$

Proof: For any eigenvalue $\lambda \in \mathbb{C}$ of A and any $x \in \mathbb{C}^n$ and $k \in \mathbb{N}$ we have by commutativity that

$$(A - \lambda I)^k x = 0 \quad \Rightarrow \quad (A - \lambda I)^k B x = B (A - \lambda I)^k x = 0,$$

which implies that the generalized eigenspace corresponding to any eigenvalue λ of A is B-invariant. Let $T = [T_1 \ T_2]$ where $T_1 \in \mathbb{R}^{n \times n_1}$ and $T_2 \in \mathbb{R}^{n \times n_2}$. Then T_1 spans the sum of the generalized eigenspaces of the eigenvalues of A in $\operatorname{spec} A_1$ and T_2 spans the sum of the remaining generalized eigenspaces. Hence $\operatorname{im} T_1$ and $\operatorname{im} T_2$ are B-invariant. In particular, there exists $B_1 \in \mathbb{R}^{n_1 \times n_1}$ and $B_2 \in \mathbb{R}^{(n-n_1) \times (n-n_1)}$ such that

$$B T_1 = T_1 B_1 \quad \text{and} \quad B T_2 = T_2 B_2,$$

which is an equivalent formulation of the claim. ■

An important consequence of this result is the following lemma which shows that a set of commuting matrices can be simultaneously transformed to a certain block-diagonal form.

Lemma 13 (Simultaneous block-diagonalization): Consider a set $\{A_1, A_2, \ldots, A_\nu\}$ of commuting matrices, i.e. $[A_i, A_j] = 0$ for all $i, j \in \{1, 2, \ldots, \nu\}$ and assume existence of an invertible T_i such that, for all $i \in \{1, 2, \ldots, \nu\}$,

$$T_i^{-1} A_i T_i = \left[\begin{smallmatrix} J_i & 0 \\ 0 & 0 \end{smallmatrix}\right] \quad \text{with invertible } J_i.$$

Then there exists a single invertible matrix $T \in \mathbb{R}^{n \times n}$ and $l \in \mathbb{N}$ such that

$$T^{-1} A_i T = \operatorname{diag}(A_{i1}, A_{i2}, \ldots, A_{il}) \quad \forall i \in \{1, 2, \ldots, \nu\}$$

with $A_{ik} \in \mathbb{R}^{n_k \times n_k}$, $n_k \in \mathbb{N}$, $n_1 + n_2 + \ldots + n_l = n$, and, for all $i \in \{1, 2, \ldots, \nu\}$ and all $k \in \{1, 2, \ldots, l\}$,

$$A_{ik} = 0 \quad \text{or} \quad A_{ik} \text{ is invertible}.$$

Proof: By assumption

$$T_1^{-1} A_1 T_1 = \begin{bmatrix} \widehat{A}_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

and invertibility of $\widehat{A}_{11} = J_1$ allows us to use Lemma 12 to obtain, for $i = 2, 3, \ldots, \nu$

$$T_1^{-1} A_i T_1 = \begin{bmatrix} \widehat{A}_{i1} & 0 \\ 0 & \widehat{A}_{i2} \end{bmatrix}.$$

Proceeding inductively, assume that for $\widehat{\nu} < \nu$ there exists invertible $\widehat{T}$ and $\widehat{l}$ such that

$$\widehat{T}^{-1} A_i \widehat{T} = \operatorname{diag}(\widehat{A}_{i1}, \widehat{A}_{i2}, \ldots, \widehat{A}_{i\widehat{l}}) \quad \forall i \in \{1, 2, \ldots, \nu\}$$

and such that for all $j = 1, 2, \ldots, \widehat{\nu}$, either $\widehat{A}_{jk} = 0$ or $\widehat{A}_{jk}$ is invertible for each $k \in \{1, 2, \ldots, \widehat{l}\}$.

From the assumption that $T_{\widehat{\nu}+1}^{-1} A_{\widehat{\nu}+1} T_{\widehat{\nu}+1} = \begin{bmatrix} J_{\widehat{\nu}+1} & 0 \\ 0 & 0 \end{bmatrix}$ with $J_{\widehat{\nu}+1}$ invertible it follows that the algebraic and geometric multiplicity of the eigenvalue zero of $A_{\widehat{\nu}+1}$ are the same. This property is inherited by the matrices $\widehat{A}_{\widehat{\nu}+1,1}, \widehat{A}_{\widehat{\nu}+1,2}, \ldots, \widehat{A}_{\widehat{\nu}+1,\widehat{l}}$, hence we find invertible matrices $S_1, S_2, \ldots, S_{\widehat{l}}$ such that, for $k = 1, 2, \ldots, \widehat{l}$.

$$S_k^{-1} \widehat{A}_{\widehat{\nu}+1,k} S_k = \begin{bmatrix} \widetilde{A}_{\widehat{\nu}+1,2k-1} & 0 \\ 0 & 0 \end{bmatrix},$$

with $\widetilde{A}_{\widehat{\nu}+1,2k-1}$ invertible. Commutativity of A_i and A_j imply commutativity of each pair $(\widehat{A}_{ik}, \widehat{A}_{jk})$ for all $i, j \in \{1, 2, \ldots, \nu\}$ and $k \in \{1, 2, \ldots, \widehat{l}\}$. Now Lemma 12 implies, for all $i = 1, 2, \ldots, \nu$ and all $k \in \{1, 2, \ldots, \widehat{l}\}$,

$$S_k^{-1} \widehat{A}_{i,k} S_k = \begin{bmatrix} \widetilde{A}_{i,2k-1} & 0 \\ 0 & \widetilde{A}_{i,2k} \end{bmatrix},$$

where, for $i \le \widehat{\nu}$, either $\widetilde{A}_{i,2k-1}$ and $\widetilde{A}_{i,2k}$ both are zero or both are invertible. With $\widetilde{T} = \widehat{T} \operatorname{diag}(S_1, S_2, \ldots, S_{\widehat{l}})$ and $\widetilde{l} := 2\widehat{l}$ we obtain

$$\widetilde{T}^{-1} A_i \widetilde{T} = \operatorname{diag}(\widetilde{A}_{i1}, \widetilde{A}_{i2}, \ldots, \widetilde{A}_{i\widetilde{l}}) \quad \forall i \in \{1, 2, \ldots, \nu\}$$

and, for all $j = 1, 2, \ldots, \widehat{\nu} + 1$, either $\widetilde{A}_{jk} = 0$ or $\widetilde{A}_{jk}$ is invertible for all $k \in \{1, 2, \ldots, \widetilde{l}\}$. This concludes the proof. ∎

We are now able to construct a common quadratic Lyapunov function for the switched DAE (1) with asymptotically stable constituent systems and commuting flow matrices $A_p^{\text{diff}} = T_p \begin{bmatrix} J_p & 0 \\ 0 & 0 \end{bmatrix} T_p^{-1}$, where J_p is Hurwitz for all $p = 1, 2, \ldots, \mathtt{p}$. To this end, invoking Lemma 13, we choose an invertible matrix T and $l \in \mathbb{N}$ such that

$$T^{-1} A_p^{\text{diff}} T = \operatorname{diag}(A_{p1}, A_{p2}, \ldots, A_{pl}) \quad \forall p \in \{1, 2, \ldots, \mathtt{p}\}$$

with $A_{pk} \in \mathbb{R}^{n_k \times n_k}$, $n_k \in \mathbb{N}$, $n_1 + n_2 + \ldots + n_l = n$, and either $A_{pk} = 0$ or A_{pk} is invertible for $p \in \{1, 2, \ldots, \mathtt{p}\}$ and $k \in \{1, 2, \ldots, l\}$. Note that $\operatorname{spec} A_{pk} \subseteq \operatorname{spec} J_p$, if A_{pk} is invertible. Hence any invertible A_{pk} is actually Hurwitz. Additionally, $[A_i^{\text{diff}}, A_j^{\text{diff}}] = 0$ implies $[A_{ik}, A_{jk}] = 0$. It is well known [1] that a finite commuting set of Hurwitz matrices admit a common quadratic Lyapunov function. Hence for each $k \in \{1, 2, \ldots, l\}$ there exists a symmetric, positive definite $P_k \in \mathbb{R}^{n_k \times n_k}$ and an $\alpha_k > 0$ such that

$$A_{pk}^\top P_k + P_k A_{pk} < -\alpha_k P_k \quad \forall p : A_{pk} \neq 0. \tag{10}$$

In case that the set of indices p such that $A_{pk} \neq 0$ is empty for some $k \in \{1, 2, \ldots, l\}$ we set $P_k := I$ and $\alpha_k := \infty$. In the sequel we denote

$$\alpha := \min_k \alpha_k > 0. \tag{11}$$

We now claim that

$$V(x) := x^\top T^{-\top} \operatorname{diag}(P_1, P_2, \ldots, P_l) T^{-1} x \tag{12}$$

is a common quadratic Lyapunov function for the switched DAE (1).

Theorem 14: Consider the switched DAE (1) satisfying A1-A3 and with flow matrices A_p^{diff} as in Definition 2. Assume that each constituent system $E_p \dot{x} = A_p x$, $p \in \{1, \ldots, \mathtt{p}\}$, is asymptotically stable and that the commutativity condition

$$[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0, \quad \forall\, p, q \in \{1, \ldots, \mathtt{p}\}$$

is satisfied. Then V as defined in (12) is a common quadratic Lyapunov function for the switched DAE (1) in the sense of [4, Thm. 9], i.e. V is a quadratic Lyapunov function for each constituent system of (1) and, in addition, V satisfies

$$V(\Pi_p x) \le V(x) \quad \forall p \in \{1, 2, \ldots, \mathtt{p}\}\ \forall x \in \mathbb{R}^n,$$

where Π_p is the consistency projector of the constituent system (E_p, A_p). ◇

Proof: Note that V is positive definite, hence it remains to show that $\dot{V}(x) < 0$, to this end we adopt the approach from [17].

Step 1: We show existence of $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $\nabla V(x) z = F(x, E_p z)$ for all $x, z \in \operatorname{im} \Pi_p$. From $T^{-1} A_p^{\text{diff}} T = \operatorname{diag}(A_{p1}, \ldots, A_{pl})$ and $T_p^{-1} A_p^{\text{diff}} T_p = \begin{bmatrix} J_p & 0 \\ 0 & 0 \end{bmatrix}$ it follows that, for $H_p := T_p^{-1} T$,

$$\begin{bmatrix} J_p & 0 \\ 0 & 0 \end{bmatrix} = H_p \operatorname{diag}(A_{p1}, \ldots, A_{pl}) H_p^{-1}.$$

Rewriting $z \in \operatorname{im} \Pi_p$ as $z = \Pi v$ for some $v \in \mathbb{R}^n$ we obtain

$$\nabla V(x) z = 2 x^\top T^{-\top} \operatorname{diag}(P_1, \ldots, P_l) T^{-1} \Pi_p v$$

and

$$\begin{aligned} \Pi_p &= T_p \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_p^{-1} \\ &= T_p S_p E_p T_p \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_p^{-1} = T_p S_p E_p \Pi_p. \end{aligned}$$

This shows the claim with

$$F(x, w) = 2 x^\top T^{-\top} \operatorname{diag}(P_1, \ldots, P_l) T^{-1} T_p S_p w.$$

Step 2: We show that $\dot{V}(x) = F(x, A_p x) < -\alpha V(x)$. First, note that with $A_p^{\text{diff}+} = T_p \begin{bmatrix} J_p^{-1} & 0 \\ 0 & 0 \end{bmatrix} T_p^{-1}$ we have

$$T^{-1} A_p^{\text{diff}+} T = \operatorname{diag}(A_{p1}^+, A_{2p}^+, \ldots, A_{pl}^+),$$

where $A_{pk}^+ = A_{pk}^{-1}$, if A_{pk} is invertible and $A_{pk}^+ = 0$, if $A_{pk} = 0$. Hence

$$\begin{aligned} T^{-1} \Pi_p &= T^{-1} T_p \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_p^{-1} \\ &= T^{-1} T_p \begin{bmatrix} J_p^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J_p & 0 \\ 0 & 0 \end{bmatrix} T_p^{-1} = T^{-1} A_p^{\text{diff}+} A_p^{\text{diff}} \\ &= \operatorname{diag}(A_{p1}^+, \ldots, A_{pl}^+) \operatorname{diag}(A_{p1}, \ldots, A_{pl}) T^{-1} \end{aligned}$$

and, therefore,

$$T^{-1}\Pi_p = \operatorname{diag}(\Pi_{p1}, \dots, \Pi_{pl})T^{-1}\,, \qquad (13)$$

with $\Pi_{pk} \in \mathbb{R}^{n_k \times n_k}$ and $\Pi_{pk} = I_{n_k}$, if A_{pk} is invertible and $\Pi_{pk} = 0$, if $A_{pk} = 0$. Fix $x \in \operatorname{im}\Pi_p$; then we obtain

$$F(x, A_p x) = 2x^\top T^{-\top} \operatorname{diag}(P_1, \dots, P_p)T^{-1}T_p S_p A_p x.$$

As $\Pi_p x = x$ we have

$$\begin{aligned} T_p S_p A_p x &= T_p S_p A_p \Pi_p x \\ &= T_p \left[\begin{smallmatrix} J_p & 0 \\ 0 & I \end{smallmatrix}\right] T_p^{-1} T_p \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] T_p^{-1} x \\ &= T_p \left[\begin{smallmatrix} J_p & 0 \\ 0 & 0 \end{smallmatrix}\right] T_p^{-1} x = A_p^{\text{diff}} x \\ &= T \operatorname{diag}(A_{p1}, \dots, A_{pl}) T^{-1} x. \end{aligned}$$

Invoking (10) and (13), it follows that

$$\begin{aligned} &F(x, A_p x) = 2x^\top T^{-T} \operatorname{diag}(P_1 A_{p1}, \dots, P_l A_{pl}) T^{-1} x \\ &< x^\top T^{-\top} \operatorname{diag}(-\alpha \Pi_{p1} P_1 \Pi_{p1}, \dots, -\alpha \Pi_{pl} P_l \Pi_{pl}) T^{-1} x \\ &= -\alpha x^\top \Pi_p^\top T^{-\top} \operatorname{diag}(P_1, \dots, P_l) T^{-1} \Pi_p x, \\ &= -\alpha V(\Pi_p x) = -\alpha V(x), \end{aligned}$$

which shows $\dot{V}(x) = F(x, A_p x) < -\alpha V(x)$ for all $x \in \operatorname{im}\Pi_p \setminus \{0\}$.

Step 3: We show $V(\Pi_p x) \le V(x)$.

Due to (13) we have

$$\begin{aligned} V(\Pi_p x) &= x^\top \Pi_p^\top T^{-\top} \operatorname{diag}(P_1, \dots, P_l) T^{-1} \Pi_p x \\ &= x^\top T^{-\top} \operatorname{diag}(\Pi_{p1} P_1 \Pi_{p1}, \dots, \Pi_{pl} P_l \Pi_{pl}) T^{-1} x \\ &\le x^\top T^{-\top} \operatorname{diag}(P_1, \dots, P_l) T^{-1} x = V(x) \end{aligned}$$

where the last inequality follows from positive definiteness of each P_k and because Π_{pk} is the identity or 0. ■

We note the following obvious consequence. In the statement we denote with slight abuse of notation the maximal and minimal eigenvalue of the symmetric positive definite matrix defining V in (12) by $\lambda_{\max}(V)$, resp. $\lambda_{\min}(V)$.

Corollary 15: Under the assumptions of Theorem 14, let α be defined by (11) and V be given by (12). Then for all $\sigma \in \Sigma$ and all solutions $x : \mathbb{R} \to \mathbb{R}^n$ of (1) we have

$$\|x(t)\| \le \tfrac{\lambda_{\max}(V)^{1/2}}{\lambda_{\min}(V)^{1/2}} e^{-(\alpha/2)t} \|x(t-)\|\,. \qquad (14)$$

In particular, the switched DAE (1) is uniformly exponentially stable. ◇

Proof: The proof of Theorem 14 shows that $\dot{V}(x) \le -\alpha V(x)$ along the differential part of any trajectory x and $V(x(t)) \le V(x(t-))$ for all $t \ge 0$. It follows by a standard argument that

$$V(x(t)) \le e^{-\alpha t} V(x(0-))\,,$$

for all $\sigma \in \Sigma, t \ge 0$. The claim now follows by the usual comparison of $V(x)$ with the Euclidean norm. See, e.g., [18, Prop. 5.5.33]. ■

Remark 16 (Orthogonality of consistency spaces): Under the assumptions of Theorem 14 define the inner product

$$\langle x, y\rangle_V := x^\top T^{-\top} \operatorname{diag}(P_1, P_2, \dots, P_l) T^{-1} y$$

with induced norm $\|x\|_V = \sqrt{\langle x, x\rangle_V} = \sqrt{V(x)}$. Since, for $p \in \{1, \dots, \mathtt{p}\}$, $V(\Pi_p x) \le V(x)$ for all $x \in \mathbb{R}^n$ it follows that

$$\|\Pi_p\|_V \le 1 \quad \forall p \in \{1, \dots, \mathtt{p}\},$$

where $\|\cdot\|_V$ denotes the induced matrix norm. Hence each consistency projector Π_p is in fact an *orthogonal* projector with respect to the inner product $\langle\cdot,\cdot\rangle_V$. Under the assumption of commutativity of the projection operators it follows that *all consistency spaces are orthogonal* to each other (modulo the intersections) with respect to the inner product induced by V.

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