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Preservation of piecewise-linear Lyapunov function under Padé discretization

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***Abstract*— In this paper we show that certain piecewise-linear Lyapunov functions are preserved for LTI systems under Padé approximations. In particular, we present a simple method to find a piecewise-linear Lyapunov function that is so preserved under the Padé discretization of any order and sampling time. This result may be of interest in the discretisation of switched linear systems for both simulation and control design.**

I. INTRODUCTION

The investigation of the properties of control systems when passing from the continuous-time analysis to the discrete-time one has been subject of particular attention in the literature of control theory. The general framework is the following: given a time-continuous control system

$$\dot{x} = f_c(x, u),$$

we look for a discrete-time control system

$$x_{k+1} = f_d(x_k, u_k)$$

that shares some properties with the original one (e.g. behavior of trajectories, or stability), and such that f^d is easily computable.

This general goal is almost completely understood for the investigation of stability of linear time-invariant (LTI) systems $\dot{x} = A_c x$. In this context, the natural choice for the discretization is to fix a sampling time $h > 0$ and define $x_{k+1} = A_d x_k$ with $A_d = e^{A_c h}$. Since the exponential of matrices is hard to compute (see [1]), it can be replaced by its diagonal Padé approximation of a given order p. The choice of the Padé approximation is very common in engineering. For example, the Tustin or bilinear approximation is a particular Padé approximation, and even the $expm$ function in MATLAB is realized by Padé approximation. It is also intensively studied from the numerical viewpoint, see [2], [3].

Our goal is to study the more general problem of good discretization of switched linear systems (SLS). This problem is new but is emerging as a topic of increasing interest in the control and simulation communities; see for example [4], [5], [6], [7]. We recall that SLS are particular cases of hybrid systems in which the dynamics f changes (i.e. it switches) between different possible linear laws $\{A_1, \ldots, A_m\}$, that are fixed a priori. The set of rules that orchestrate the switching among is the set of all possible time-dipendent laws. The continuous-time case is

$$\dot{x} = A_c^{\sigma(t)} x \qquad \text{where } \sigma : [0, T] \to \{1, \ldots, m\} \text{ measurable},$$

while the discrete-time case is

$$x_{k+1} = A_d^{\sigma_k} x_k \qquad \text{where } \sigma : \{0, \ldots, K\} \to \{1, \ldots, m\}.$$

Then, the problem of good discretization of SLS can be restated as following: find a rule for time discretization $[0, T] \to \{0, \ldots, K\}$ and a method to compute A_d^i from A_c^i. The first idea, coming from LTI system, is to fix a sampling time $h > 0$ and to compute each A_d^i as the diagonal Padé approximation of A_c^i. Surprisingly, this discretization method fails to preserve stability of SLS. Examples can be found in [8], [4].

Even though this negative result is known, it is still unclear in which cases stability is preserved when discretization is computed via the diagonal Padé approximation. Our contribution is a first step in this direction. We thus focus our attention on the preservation of piecewise-linear Lyapunov function under Padé approximation. Our contribution is to show that, given a stable LTI system, it is always possible to find a particular piecewise-linear Lyapunov that is preserved for all kind of Padé approximations, regardless to the order p and the sampling time h.

Beside being interesting directly in the context of LTI systems, the existence of such a piecewise-linear Lyapunov function can be a starting point to investigate the stability of SLS.

II. DEFINITIONS AND KNOWN RESULTS

A. *Padé discretization*

Consider a linear autonomous system

$$\dot{x}(t) = A_c x(t) \tag{1}$$

where $x(t) \in \mathcal{R}^n$ and assume that the system is asymptotically stable, i.e. matrix A_c is Hurwitz (all eigenvalues in the open left half of the complex plane). It is well known that the motion of the state, associated with an initial state $x(0) = x_0$, can be written as $x(t) = e^{A_c t} x_0$. The exponential matrix $e^{A_c t}$ can be numerically approximated in a variety of different ways. In this paper we focus on the most popular one, that is *diagonal Padé approximation* approximation of p^{th} order, see e.g. [2], [3]. This operator is well known to engineers and is commonly used by

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both the control and signal processing community. To be precise, taking a sampling time h, the p^{th} order Padé discretization of $e^{A_c h}$ is defined as

$$A_d = Z(A_c h)Z(-A_c h)^{-1} \quad (2a)$$

$$Z(X) = \sum_{i=0}^{p} c_i X^i, \quad c_i = \frac{(2p-i)!p!}{(2p)!i!(p-i)!} \quad (2b)$$

Hence, it is possible to associate with system (1), its discrete approximation

$$x_{k+1} = A_d x_k \quad (3)$$

where x_k approximates $x(kh) = (e^{A_c hk})x_0$. It is well known that the Padé discretization preserves the stability properties. As a matter of fact, A_c is Hurwitz if and only if A_d is Schur stable (all eigenvalues inside the open unit disc), for any given sampling times $h > 0$. Moreover, the eigenvalues of A_c and A_d are related by the same transformation induced by (2), and the eigenstrucure is preserved. If λ is an eigenvalue of A_c associated with an eigenvector $\bar{x}$, then $z = Z(\lambda h)Z(-\lambda h)^{-1}$ is an eigenvalue of A_d associated with the same eigenvector $\bar{x}$. Even more, the transformation is basis independent, i.e.

$$\begin{aligned} &Z(TA_cT^{-1}h)Z(-TA_cT^{-1}h)^{-1} = \\ &= TZ(A_ch)Z(-A_ch)^{-1}T^{-1} = TA_dT^{-1} \end{aligned}$$

Finally, if TA_cT^{-1} is a Jordan form for A_c, then TA_dT^{-1} is a Jordan form for A_d. A particular Padé transformation is the celebrated *bilinear transformation* (or Tustin transformation), that is given by (2) with $p = 1$, i.e.

$$A_d = (I + \frac{h}{2}A_c)(I - \frac{h}{2}A_c)^{-1} \quad (4)$$

B. Piecewise-linear Lyapunov functions

Consider system (1) and its Padé discretization (3). Assume that A_c is Hurwitz stable, so that A_d is Schur stable for each $h > 0$. Also, Let X_{ij} be the entries of a square matrix X. We define the ∞-measure as

$$\mu_\infty(X) = \max_i \left(X_{ii} + \sum_{j \neq i} |X_{ij}| \right)$$

and the ∞-norm as

$$\|X\|_\infty = \max_{\mathbf{i}} \sum_j |X_{ij}|$$

The main known results about stability of LTIs that we will use are recalled below, see e.g. [10]. The same reference recalls results about existence of quadratic Lyapunov functions.

Lemma 2.1: (i) A_c is Hurwitz stable if and only if there exists a full column rank matrix $W_c \in \mathcal{R}^{N\times n}$, $N \geq n$, and Q_c such that

$$W_c A_c = Q_c W_c, \quad \mu_\infty(Q_c) < 0. \quad (5)$$

(ii) A_d is Schur stable if and only if there exists a full column rank matrix $W_d \in \mathcal{R}^{N\times n}$, $N \geq n$, and Q_d such that

$$W_d A_d = Q_d W_d, \quad \|Q_d\|_\infty < 1. \quad (6)$$

Remark 2.2: Notice that $W_cA_c = Q_cW_c$ always imply that $W_dA_d = Q_dW_d$ with $W_d = W_c$ and $Q_d = Z(Q_ch)$. However it is not true in general that $\mu_\infty(Q_c) < 0$ implies $\|Q_d\|_\infty < 1$, unless h is small. A counterexample is given in [9].

The Lyapunov functions associated with Lemma 2.1 are piecewise-linear, i.e. $\|W_cx_c\|_\infty$, resp. $\|W_dx_d\|_\infty$, where the number of vertices of the polyhedra is $2N$. In general $N > n$, and the minimal N for which (5), (6) are verified depends on the location of the eigenvalues of A_c, A_d in the complex plane. It can be proved, see [11], that for a matrix A_c with distinct eigenvalues a necessary and sufficient condition for $N = n$ is that the complex eigenvalues $\lambda = -\alpha + j\beta$, $\alpha > 0$, belong to the sector $|\beta|\alpha^{-1} < 1$.

C. Computation of piecewise-linear Lyapunov function

In this section we look for matrices W_c satisfying (5), and matrices W_d satisfying (6). We recall two results available in the literature, [12] for the continuous-time and [13] for the discrete-time case. They have been shown to be valid also in case of multiple eigenvalues. However, for the sake of simplicity, we assume that the eigenvalues are distinct.

We start with the continuous-time setting. Given a stable matrix A_c, we provide a method to compute a particular W_c.

Lemma 2.3 (Existence for continuous-time LTI): Consider a Hurwitz stable matrix A_c, with distinct eigenvalues, with n_r real and $2n_c$ complex eigenvalues. For each pair of conjugate complex eigenvalue $\lambda_i = \alpha_i \pm j\beta_i$, $i = 1, 2, \cdots, n_c$, take an integer m_i such that λ_i lies in the sector $\mathcal{S}_c(m_i)$, where

$$\mathcal{S}_c(m) = \{\lambda = -\alpha + j\beta : \ \alpha > 0, \ |\beta| < \frac{\sin(\frac{\pi}{m})}{1 - \cos(\frac{\pi}{m})}\alpha\}. \quad (7)$$

Then there exist $W_c \in \mathcal{R}^{N\times n}$ and $Q_c \in \mathcal{R}^{N\times N}$, with $N = \sum_{i=1}^{k} m_i + n_r$, satisfying (5).

In Figure 1, the sectors $\mathcal{S}_c(m)$ are drawn for $m = 2$ (angle $\pi/4$), $m = 3$ (angle $\pi/3$), $m = 4$ (angle $3\pi/8$) and $m = 5$ (angle $4\pi/10$). One can remark that the minimum number m such that an eigenvalue λ lies in S_m is increasing for λ approaching the imaginary axis. Nevertheless, a finite m exists for each λ with negative real part.

Lemma 2.4 (Computation for continuous-time LTI): Consider a Hurwitz stable matrix A_c as in the previous lemma. Take T_c the state-space transformation that puts A_c in its real Jordan form, i.e.

$$T_cA_cT_c^{-1} = \begin{bmatrix} H_{c1} & 0 & \cdots & 0 & 0 \\ 0 & H_{c2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & H_{cn_c} & 0 \\ 0 & 0 & 0 & 0 & R_c \end{bmatrix}$$

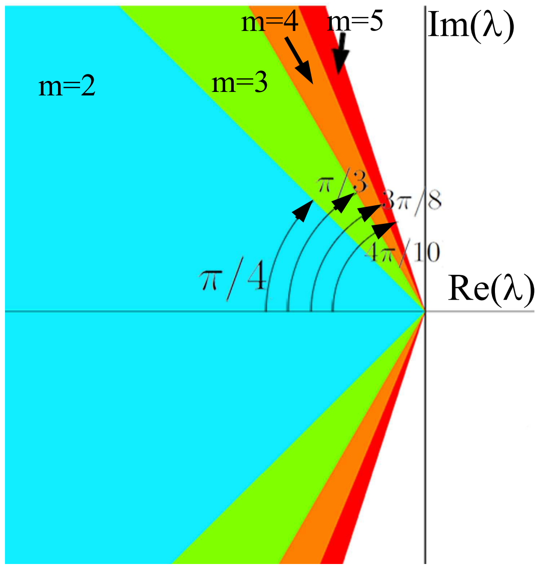

Fig. 1. The sectors $\mathcal{S}_c(m)$ for $m = 2$ (angle $\pi/4$), $m = 3$ (angle $\pi/3$), $m = 4$ (angle $3\pi/8$) and $m = 5$ (angle $4\pi/10$).

where

$$H_{ci} = \begin{bmatrix} -\alpha_i & \beta_i \\ -\beta_i & -\alpha_i \end{bmatrix}$$

and R_c is a $n_r \times n_r$ diagonal matrix accounting for the real eigenvalues. Define

$$\tilde{W}_c = \begin{bmatrix} W_{c,1} & 0 & \cdots & 0 & 0 \\ 0 & W_{c,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & W_{c,n_c} & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

with

$$W_{c,i} = \begin{bmatrix} 1 & 0 \\ cos(\frac{\pi}{m_i}) & sin(\frac{\pi}{m_i}) \\ cos(\frac{2\pi}{m_i}) & sin(\frac{2\pi}{m_i}) \\ \vdots & \vdots \\ cos(\frac{(m_i-1)\pi}{m_i}) & sin(\frac{(m_i-1)\pi}{m_i}) \end{bmatrix}.$$

Then $W_c := \tilde{W}_c T_c$ defines a Lyapunov function $F(x) = \|W_c x\|_\infty$ for the system $\dot{x} = A_c x$.

We now state the corresponding results for the discrete-time case.

Lemma 2.5 (Existence for discrete-time LTI): Consider a Schur stable matrix A_d, with distinct eigenvalues, with n_r real and $2n_c$ complex eigenvalues. For each pair of conjugate complex eigenvalue $\lambda_i = \sigma_i \pm j\omega_i$, $i = 1, 2, \cdots, n_c$, take an integer m_i such that λ_i lies in the interior of the regular polygon $\mathcal{P}_{ol}(m_i)$, where

$$\mathcal{P}_{ol}(m) = int\ conv \left\{ e^{j\frac{p\pi}{m}} \right\}_{p=0}^{2m-1}. \tag{8}$$

Then there exists $W_d \in \mathcal{R}^{N \times n}$ and $Q_d \in \mathcal{R}^{N \times N}$, with $N = \sum_{i=1}^{k} m_i + n_r$, satisfying (6).

In Figure 2 the polygons $\mathcal{P}_{ol}(m)$ are depicted for $m = 2$ (square), $m = 3$ (hexagon), $m = 4$ (octagon), $m = 5$ (decagon).

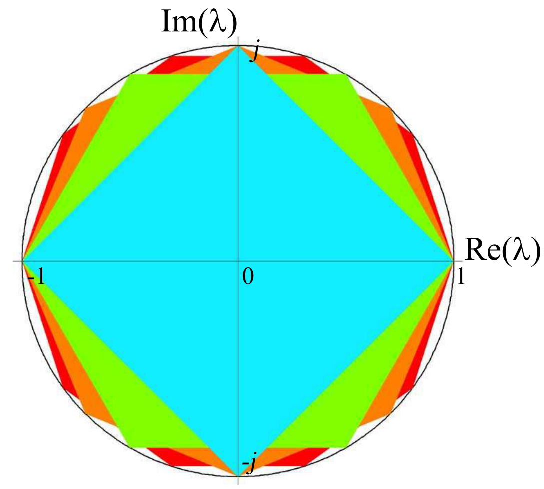

Fig. 2. The polygons for $m = 2$ (square), $m = 3$ (hexagon), $m = 4$ (octagon), $m = 5$ (decagon).

Lemma 2.6 (Computation for discrete-time LTI): Consider a Schur stable matrix A_d as in the previous lemma. Take T_d the state-space transformation that puts A_d in its real Jordan form, i.e.

$$T_d A_d T_d^{-1} = \begin{bmatrix} H_{d1} & 0 & \cdots & 0 & 0 \\ 0 & H_{d2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & H_{dn_c} & 0 \\ 0 & 0 & 0 & 0 & R_d \end{bmatrix}$$

where

$$H_{di} = \begin{bmatrix} -\alpha_i & \beta_i \\ -\beta_i & -\alpha_i \end{bmatrix}$$

and R_d is a $n_r \times n_r$ diagonal matrix accounting for the real eigenvalues. Define

$$\tilde{W}_d = \begin{bmatrix} W_{d,1} & 0 & \cdots & 0 & 0 \\ 0 & W_{d,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & W_{d,n_c} & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

with

$$W_{d,i} = \begin{bmatrix} 1 & 0 \\ cos(\frac{\pi}{m_i}) & sin(\frac{\pi}{m_i}) \\ cos(\frac{2\pi}{m_i}) & sin(\frac{2\pi}{m_i}) \\ \vdots & \vdots \\ cos(\frac{(m_i-1)\pi}{m_i}) & sin(\frac{(m_i-1)\pi}{m_i}) \end{bmatrix}.$$

Then $W_d := \tilde{W}_d T_d$ defines a Lyapunov function $F(x) = \|W_d x\|_\infty$ for the system $x_{k+1} = A_d x_k$.

We refer to the original papers [12], [13] for the proofs of the results. Nevertheless, we point out the key idea of these proofs. Given A_c, pass to the real Jordan form via T_c. For each submatrix H_{ci}, find W_{ci} and Q_{ci}. Then find the global W_c and Q_c by applying the inverse change of coordinate T_c^{-1} (being careful about covariant and contrvariant matrices). The same idea is applied in the discrete-time setting, for which it is a bit harder to find Q_d.

III. Main result

In this section we state the main result of the paper. Given a matrix A_c and its diagonal Padé approximation A_d of a given order p and sampling time h, the function F computed in Lemma 2.4 is a Lyapunov function both for A_c and A_d. As already stated, this implies that F is a Lyapunov function for A_c and all its Padé approximations.

We first prove that, given an integer m, the image of $\mathcal{S}_c(m)$ under a Padé approximation is contained in $\mathcal{P}_{ol}(m)$. We only give a graphical idea of the proof, since it is rather technical. The complete proof is given in [14].

Lemma 3.1: Let m be a positive integer number, $\mathcal{S}_c(m)$ defined in (7) and $\mathcal{P}_{ol}(m)$ defined in (8). Fix a sampling time $h > 0$ and consider $\mathcal{S}_d(m, h)$ the image of $\mathcal{S}_c(m)$ under the Padé transformation (2), i.e.

$$\mathcal{S}_d(m, h) = \{z = Z(\lambda h)Z(-\lambda h)^{-1}, \lambda \in \mathcal{S}_c(m)\}$$

Then

$$\mathcal{S}_d(m, h) \subseteq \mathcal{P}_{ol}(m).$$

Sketch of the proof. Take the two half-lines $\Lambda_m, \overline{\Lambda_m}$ that are the boundaries of $\mathcal{S}_c(m)$. Their precise expression is

$$\Lambda_m = \left\{-\alpha + j\beta \mid \beta = \frac{\sin(\frac{\pi}{m})}{1 - \cos(\frac{\pi}{m})}\alpha\right\}$$

and $\overline{\Lambda_m} = \{\overline{z} \mid z \in \Lambda_m\}$. First prove that $\mathcal{S}_d(m, h) \subseteq \mathcal{P}_{ol}(m)$ if and only if the image of $\Lambda_m, \overline{\Lambda_m}$ under the Padé approximation is contained in $\mathcal{P}_{ol}(m)$. Due to invariance of $\mathcal{P}_{ol}(m)$ under complex conjugation, it is equivalent to prove that $Z(\Lambda_m h)Z(-\Lambda_m h)^{-1} \in \mathcal{P}_{ol}(m)$. Due to invariance of Λ_m with respect to rescaling, it is equivalent to prove that $Z(\Lambda_m)Z(-\Lambda_m)^{-1} \in \mathcal{P}_{ol}(m)$. This result being technical, we only show some images of Λ_2 under Padé approximations of order $p = 1, 2, 3$ and images of $\Lambda_2, \Lambda_3, \Lambda_4$ under Padé approximation of order $p = 2$. For a complete proof, see [14]. ■

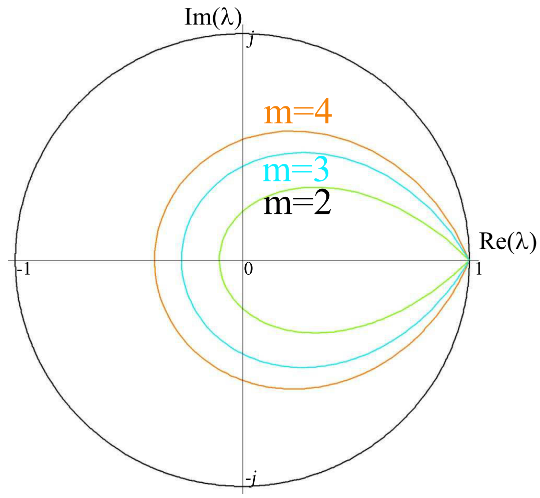

Fig. 3. The curve $z = Z(\Lambda_m)Z(-\Lambda_m)^{-1}$ for $m = 2, 3, 4$ and $p = 2$.

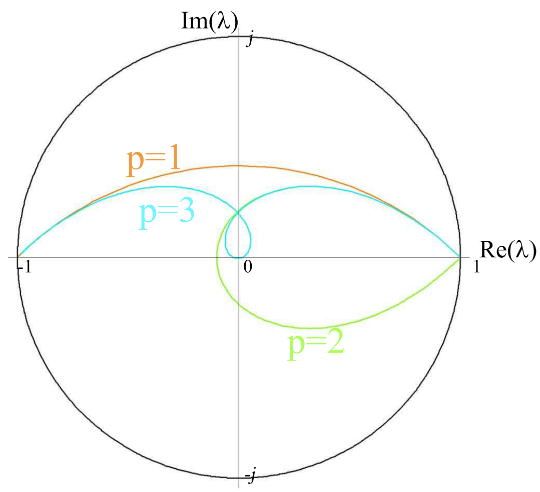

Fig. 4. The curve $z = Z(\Lambda_m)Z(-\Lambda_m)^{-1}$ for $m = 2$ and $p = 1, 2, 3$.

We now state precisely the main result of the paper.

Theorem 3.2: Consider a Hurwitz stable matrix A_c of dimension n and its Padé discretization A_d of order p and sampling time $h > 0$. Let n_r be the number of real negative eigenvalues, and $2n_c$ be the number of pairs of complex eigenvalues $-\alpha_i \pm j\beta_i$, $i = 1, 2, \cdots, n_c$. For each pair of complex eigenvalues, let m_i be an integer greater than one such that $-\alpha_i \pm j\beta_i$ belongs to the sector

$$|\beta_i| < \frac{sin(\frac{\pi}{m_i})}{1 - cos(\frac{\pi}{m_i})}\alpha_i.$$

Then there exist $W = W_c = W_d \in \mathcal{R}^{N \times n}$, with $N = \sum_{i=1}^{k} m_i + n_r$ such that $F(x) = \|Wx\|_\infty$ is a Lyapunov function both for $\dot{x} = A_c x$ and $x_{k+1} = A_d x_k$. Moreover, W can be computed as in Lemma 2.4.

Proof. First recall that the Padé transformation preserves the Jordan form of A_c and A_d. Now take T such that both $J_c = TA_cT^{-1}$ and TA_dT^{-1} are in the real Jordan form. Compute W_c as in Lemma 2.4. For each pair of complex eigenvalue $\lambda_i = -\alpha_i \pm \beta_i$, the expression of W_{ci} is uniquely determined by m_i such that $-\alpha_i \pm \beta_i \in \mathcal{S}_c(m_i)$.

Now consider the expression of W_d, computed as in Lemma 2.6. Applying Lemma 3.1, we have that μ_i lies in the interior of the regular polygon $\mathcal{P}_{ol}(m_i)$, with the same m_i of the eigenvalue λ_i of the continuous system. As a consequence, the expression of W_{di} can be chosen to be identical to W_{ci}. Thus $W_d = W_c T_c^{-1} T_d$. Since $T_c = T_d$, we have the conclusion. ■

Comment : The result above says that there always exists a common piecewise-linear Lyapunov function $\|Wx\|_\infty$ for A_c and A_d. Moreover, the construction of W is based on the matrix A_c only, and the previous theorem shows $\|Wx\|_\infty$ is a piecewise-linear Lyapunov function for A_d computed with a Padé approximation of any order p. As a consequence, the piecewise-linear Lyapunov function is common to A_c and all Padé approximants, of any order p and with any sampling time h. In all these cases, it is the matrix Q_d that changes its expression.

IV. Examples

In this section, we give two examples that highlight some implications of our main result. The first, positive result, comes from [9].

Example 1: Consider the Hurwitz matrix

$$A_c = \begin{bmatrix} -1 & 0 \\ -2.4 & -3 \end{bmatrix}$$

In [9], the authors show that a piecewise linear Lyapunov function is given by choosing $W_c = I$, with $Q_c = A_c$. Now, take A_d given by the 1^{st} order Padé approximation with $h = 2$, namely

$$A_d = (I + A_c)(I - A_c)^{-1} = \begin{bmatrix} 0 & 0 \\ -0.6 & -0.5 \end{bmatrix}$$

and notice that $Q_d = (I+Q_c)(I-Q_c)^{-1}$ satisfies $A_d W_d = W_d Q_d$, with $W_d = W_c$. However, $\|Q_d\|_\infty > 1$. From this the authors in [9] concluded that $W_c = I$ is not preserved.

Nevertheless, using our result, it is possible to find another $W = W_c = W_d$ that is preserved. A choice is given by

$$W = \begin{bmatrix} -1 & 0 \\ 1.2 & 1 \end{bmatrix},$$

$$Q_c = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \; Q_d = \begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix}.$$

Remark that W has be computed as follows: first compute $\tilde{A}_c = T_c A_c T_c^{-1}$ the real Jordan form of A_c. Then compute the minimum m such that S_m contains the eignevalues of A_c (in this case, $m = 2$). Finally, define $\tilde{W}$ via Lemma 2.4 and $W = \tilde{W} T_c$.

Our first example highlights the fact that some polyhedral Lyapunov functions are preserved by Padé transformations. This result is interesting as it says that Padé transformations preserve all quadratic functions, and some polyhedral Lyapunov functions. This observation is most interesting in the context of switched systems as it implies that Padé methods will preserve the stability of certain switched systems even if they are not quadratically stable to begin with. That much work remains to be done is illustrated by the following example. We start from an example given by [8], with the parameter $a = 7$.

Example 2: The switching system is given by two matrices

$$A_{c1} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \; A_{c2} = \begin{pmatrix} -1 & 1/7 \\ -7 & -1 \end{pmatrix}.$$

As already stated in [8], the switching system is asymptotically stable. One can compute explicitly the Padé approximation of order 1 of A_{ci} as a function of the sampling time h, that are

$$A_{d1} = \begin{pmatrix} \frac{2-h^2}{h^2+2h+2} & \frac{2h}{h^2+2h+2} \\ -\frac{2h}{h^2+2h+2} & \frac{2-h^2}{h^2+2h+2} \end{pmatrix},$$
$$A_{d2} = \begin{pmatrix} \frac{2-h^2}{h^2+2h+2} & \frac{2h}{7(h^2+2h+2)} \\ -\frac{14h}{h^2+2h+2} & \frac{2-h^2}{h^2+2h+2} \end{pmatrix}.$$

One can observe that for small h the system is stable, while for $h = 1$ we have instability, since one of the eigenvalues of $A_{d2}A_{d1}$ is

$$-\frac{93 + 16\sqrt{29}}{175} < -1.$$

To study stability for $h \in [0, 1]$, one can look for a piecewise linear Lyapunov function for the continuous-time system A_{ci} that is preserved under Padé approximation. Since the eigenvalues of both A_{c1} and A_{c2} lie in $\mathcal{S}_c(3)$, our method provides the following matrices W_{ci}, one for each matrix A_{ci}:

$$W_{c1} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix},$$
$$W_{c2} = \begin{pmatrix} 7 & 1 \\ \frac{7}{2} - \frac{7\sqrt{3}}{2} & \frac{1}{2} + \frac{\sqrt{3}}{2} \\ -\frac{7}{2} - \frac{7\sqrt{3}}{2} & -\frac{1}{2} + \frac{\sqrt{3}}{2} \end{pmatrix}.$$

One can then check if one of the two, say W_{c1}, defines a Lyapunov function for the other system, say A_{c2}. It is easy to see that it is not the case. Then one can check for a linear combination of the two, but also in this case we are unable to find a common piecewise linear Lyapunov function. Then, since $\mathcal{S}_c(3) \subset \mathcal{S}_c(4) \subset \ldots$, one can increase the dimension of W_{ci} and use the method to find other candidate piecewise linear Lyapunov functions. We do not go further in this direction, since the method becomes increasingly hard from the computational point of view.

It is immediately clear from the above example that the fact that some Lyapunov functions are preserved is not enough to guarantee that the discrete time system will be stable. Thus, Padé, while being well suited to discretising LTI systems, is somewhat lacking when applied to switched systems. Future work will look at the problem of developing discretisation methods that are suited for the discretisation of switched linear systems.

V. Conclusions

In this paper we present a method to compute a piecewise-linear Lyapunov function F for a continuous-time LTI system $\dot{x} = A_c x$. F is moreover preserved under Padé approximation of A_c of any order and sampling time. This result is a first step in the context of switching linear systems. Some examples show applications of the method, as well as some negative results.

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