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Enforcing Convergence in Nonlinear Economic MPC

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Abstract— **This papers proposes new ways to enforce convergence to equilibria for Economic Model Predictive Control schemes. Economic Model Predictive Control is a control technique capable of optimizing an economic performance index while enforcing state and input constraints. For nonlinear systems and/or non-convex cost functionals, performance optimization may result in non converging behaviours. While this might be acceptable in some cases (i.e. operation of chemical reactors), it may be undesirable for other types of applications. In the present paper we discuss ways of enforcing convergence to equilibrium by trading it off with asymptotic performance. Indeed, while all trajectories converging to a given equilibrium yield the same asymptotic average cost, transient costs may differ and trade-offs are naturally highlighted between the latter and speed of convergence.**

I. Introduction

Control engineers are often facing trade-offs in the design of controllers, the most typical one being the trade-off between cost and performance. Cost is usually meant in terms of some norm of the control signal, while performance is often inversely proportional to the norm of the output signal (assuming 0 as a target for the design). This view is to some extent more motivated by mathematical and physical insight than by economic considerations. In fact, the true economic cost of operating a system can often be very different from the simplistic convex and quadratic design criteria which are commonly used in control design. Economic criteria, if ever taken into account, are usually addressed only at the level of set-point planning. The so-called Real Time Optimization layer (RTO for short) determines, among all feasible *steady-state* plant operating conditions those with minimal cost; see for instance [3, 8, 7, 6, 2].

Economic Model Predictive Control [9] is motivated by the need to reconcile 'control design' and 'system economics'. In particular, previous contributions to this field have shown how improvement of performance both in terms of average or transient costs can be achieved when control design is carried out by taking into account directly the true economics of a plant while designing a receding horizon feedback controller. Due to potential nonconvexity of costs considered as well as nonlinearity of the underlying dynamics, convergent behaviors are not always optimal and/or desirable [1]. One peculiar feature of this method is to go beyond the usual static optimization layer that is adopted in standard plant operation in many industrial processes. While this is reasonable for applications in which outputs are material/physical outflows that lend themselves to the possibility of storage, in many other contexts convergence to an equilibrium is a requirement that cannot be sacrificed by trading it off with economics. For these cases, then, it makes sense to investigate ways of fulfilling convergence requirements while still optimizing transient economic performance. This is the goal of this note, in which two methods are discussed and compared to systematically enforce convergence to the best equilibrium provided by the static optimization layer while still employing an economic MPC scheme.

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II. Problem Formulation

We consider a nonlinear finite dimensional discrete time control system:

$$x^+ = f(x,u) \tag{1}$$

with state $x \in \mathbb{X} \subseteq \mathbb{R}^n$ and control $u \in \mathbb{U} \subseteq \mathbb{R}^m$, for some closed sets $\mathbb{X}, \mathbb{U}$. We seek to optimize a cost-functional

$$\sum_k \ell(x(k), u(k)) \tag{2}$$

along solutions of (1) and subject to point-wise in time constraints:

$$(x(k), u(k)) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}. \tag{3}$$

for some compact $\mathbb{Z}$. Due to the nonlinearity of the system and the fact that neither constraints nor cost functionals are required to be convex, the optimal solution need not correspond to a steady state of the system. In particular, while it is possible to define the set of best steady states (which for the sake of simplicity we assume to be a singleton):

$$(x_s, u_s) = \arg \min_{(x,u)\in\mathbb{Z}} \{\ell(x,u) \mid x = f(x,u)\} \tag{4}$$

better average performances can often be encountered along periodic or complex transients that never approach x_s. While this can be acceptable in certain types of applications (for instance in the production of chemicals, or more generally for plants in which outputs are truly outflows which can easily be stored and dispatched at some later time) in other types of applications it is instead crucial to converge to equilibrium within a specified amount of time. We investigate below how to achieve this goal by comparing different types of techniques. The first is already proposed in [1] and amounts to adjusting the stage cost in order to enforce a certain type of dissipativity that is known to imply convergence in Economic MPC schemes. The second exploits the possibility of imposing average constraints in order to impose a zero variance constraint that may enforce asymptotic convergence to the desired equilibrium. The following notion of average was introduced in [1]:

$$\begin{aligned}\mathrm{Av}[v] \quad &= \Big\{ \bar{v} \in \mathbb{R}^{n_v} \mid \exists \{t_n\}_{n=1}^{+\infty} : \\ & t_n \to +\infty \text{ and } \lim_{n\to+\infty} \tfrac{\Sigma_{k=0}^{t_n} v(k)}{t_n+1} = \bar{v} \Big\},\end{aligned} \tag{5}$$

where $\{t_n\}_{n=1}^{+\infty}$ denotes any sequence with values in $\mathbb{N}$. Moreover, an Economic MPC algorithm was proposed in order to fulfill, together with point-wise in time constraints, average output constraints for systems of the following type:

$$\begin{aligned} x^+ &= f(x,u) \\ y &= h(x,u). \end{aligned} \tag{6}$$

In particular, the output y is controlled so as to fulfill:

$$\mathrm{Av}[y] \subseteq \mathbb{Y} \tag{7}$$

where $\mathbb{Y}$ is a closed, convex set that contains $h(x_s,u_s)$. For the sake of completeness we recall here the details of the algorithm. At each time t we solve the following optimization problem:

$$\min_{\mathbf{u}} \sum_{k=0}^{N-1} \ell(z(k),v(k)) \tag{8}$$

subject to the following constraints

$$\begin{aligned} z^+ \quad &= f(z,v) \\ (z(k),v(k)) \quad &\in \mathbb{Z}, \quad k \in \mathbb{I}_{0:N-1} \\ z(N) \quad &= x_s, \quad z(0) = x(t) \\ \textstyle\sum_{k=0}^{N-1} h(z(k),v(k)) \quad &\in \mathbb{Y}_t \end{aligned} \tag{9}$$

The time-varying output constraint set is the new feature of this problem. To enforce the average constraints, we define the constraint sets recursively

$$\mathbb{Y}_{i+1} = \mathbb{Y}_i \oplus \mathbb{Y} \ominus \{h(x(i),u(i))\} \qquad \text{for } i \in \mathbb{I}_{\geq 0} \tag{10}$$

in which the symbols $\oplus$ and $\ominus$ denote set addition, and subtraction, respectively,

$$\begin{aligned} \mathbb{V} \oplus \mathbb{W} \quad &= \{z = v + w \mid v \in \mathbb{V}, w \in \mathbb{W}\} \\ \mathbb{V} \ominus \mathbb{W} \quad &= \{z \mid \{z\} \oplus \mathbb{W} \subseteq \mathbb{V}\} \end{aligned}$$

We initialize the recursion using

$$\mathbb{Y}_0 = N\mathbb{Y} + \mathbb{Y}_{00} \tag{11}$$

in which the set $\mathbb{Y}_{00} \subset \mathbb{R}^p$ is an arbitrary compact set containing $h(x_s,u_s)$. As shown in [1], this algorithm preserves feasibility and guarantees fulfillment both of pointwise in time and average constraints. For the following developments it is useful to introduce a weaker notion of asymptotic convergence.

Definition 2.1: We say that the sequence $v(t), t = 0, 1, \ldots$ is essentially converging to 0 if the following is true:

$$\forall\, \varepsilon > 0 : \quad \limsup_{T\to+\infty} \frac{\mathrm{card}(\{t \leq T : |v(t)| \geq \varepsilon\})}{T+1} = 0. \tag{12}$$

To clarify this concept and illustrate a fundamental technical subtlety when trying to impose convergence by means of average constraints, consider the following discrete-time sequence:

$$v(t) = \begin{cases} 1 & \text{if } t = 2^n \text{ for } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

It is easy to see that $v(t)$ has all average moments equal to zero:

$$\lim_{T\to+\infty} \frac{\sum_0^T v(t)^k}{T+1} = 0 \tag{14}$$

for all $k \in \mathbb{N}$; it is however not a convergent signal in the standard sense. It is worth pointing out that $v(t)$ is *essentially converging* to zero. The following Lemma shows that zero even moments are indeed enough to guarantee essential convergence.

Lemma 2.2: A sequence $v(t)$ with a zero even moment is essentially converging to its average value.

Proof: Let $\varepsilon > 0$ be arbitrary. The following equality follows from the definition of cardinality:

$$\mathrm{card}(\{t \leq T : |v(t) - \bar{v}| \geq \varepsilon\}) = \sum_{t \leq T : |v(t)-\bar{v}| \geq \varepsilon} 1 \tag{15}$$

Consequently, for any even positive integer n:

$$\begin{aligned} \varepsilon^n \mathrm{card}(\{t \leq T \quad &: \quad |v(t) - \bar{v}| \geq \varepsilon\}) \\ &= \sum_{\{t \leq T:\ |v(t)-\bar{v}| \geq \varepsilon\}} \varepsilon^n \\ &\leq \sum_{\{t \leq T:\ |v(t)-\bar{v}| \geq \varepsilon\}} |v(t) - \bar{v}|^n \\ &\leq \sum_{t=0}^{T} |v(t) - \bar{v}|^n \end{aligned} \tag{16}$$

Thanks to (16), and choosing $\bar{v}$ such that the n-th moment of v centered at $\bar{v}$ is zero, we may conclude:

$$\begin{aligned} & \limsup_{T\to+\infty} \tfrac{\mathrm{card}(\{t \leq T : |v(t)-\bar{v}| \geq \varepsilon\})}{T+1} \\ \leq \quad & \limsup_{T\to+\infty} \tfrac{1}{\varepsilon^n} \tfrac{\Sigma_{t=0}^T |v(t)-\bar{v}|^n}{T+1} = 0. \end{aligned} \tag{17}$$

■

Notice that the recursion in (10) can be solved explicitly, giving:

$$\mathbb{Y}_t = (t+N)\mathbb{Y} \oplus \mathbb{Y}_{00} \ominus \{\sum_{\tau=0}^{t-1} h(x(\tau),u(\tau))\} \tag{18}$$

We propose the following variant to (18) for reasons which will be clearer later:

$$\mathbb{Y}_t = (t+N)\mathbb{Y} \oplus (1+t^\alpha)\mathbb{Y}_{00} \ominus \{\sum_{\tau=0}^{t-1} h(x(\tau),u(\tau))\} \tag{19}$$

with $\alpha \in [0,1)$. It will be shown that (19) retains the capability of enforcing average constraints when used in conjunction with the Economic MPC algorithm.

Proposition 2.3: Consider the MPC algorithm described in (8) and (9) where the set $\mathbb{Y}_t$ is defined according to equation (19) for some convex compact sets $\mathbb{Y}$ and $\mathbb{Y}_{00}$ (the latter containing the origin). Then, if $x(0)$ is a feasible initial condition, all subsequent values $x(t)$ are feasible and constraints satisfaction is guaranteed, namely: $(x(t), u(t)) \in \mathbb{Z}$ and $\text{Av}[y] \subseteq \mathbb{Y}$.

Proof: The proof is as usual by induction.

1) *Feasibility*: Let $z^\star(0), z^\star(1), \ldots, z^\star(N)$, and $v^\star(0), v^\star(1), \ldots, v^\star(N-1)$ fulfill all the constraints in (9), including the average constraint. Then, applying the control $u(t) = v^\star(0)$ we have $x(t+1) = z^\star(1)$ and the shifted sequence: $z^\star(1), z^\star(2), \ldots, z^\star(N), x_s$, $v^\star(1), \ldots, v^\star(N-1), u_s$ is again feasible. In particular it fulfills the constraint:

$$\begin{aligned} & \sum_{k=1}^{N-1} h(z^\star(k), v^\star(k)) + h(x_s, u_s) \\ = & \sum_{k=0}^{N-1} h(z^\star(k), v^\star(k)) + h(x_s, u_s) - h(x(t), u(t)) \\ \in & \ \mathbb{Y}_t \oplus \mathbb{Y} \ominus \{h(x(t), u(t))\} \qquad (20) \\ = & \ (t+N)\mathbb{Y} \oplus (1+t^\alpha)\mathbb{Y}_{00} \oplus \mathbb{Y} \\ & \ominus \{\sum_{\tau=0}^{t-1} h(x(\tau), u(\tau))\} \ominus \{h(x(t), u(t))\} \\ \subseteq & \ (t+1+N)\mathbb{Y} \oplus (1+(t+1)^\alpha)\mathbb{Y}_{00} \\ & \ominus \{\sum_{\tau=0}^{t} h(x(\tau), u(\tau))\} = \mathbb{Y}_{t+1} \end{aligned}$$

2) *Average constraints*: We show next that average constraints are fulfilled (asymptotically). To this end, at any time t it holds:

$$\begin{aligned} & \textstyle\sum_{k=0}^{t-1} h(x(k), u(k)) + \sum_{k=0}^{N-1} h(z(k), v(k)) \\ \in & \ (t+N)\mathbb{Y} \oplus (1+t^\alpha)\mathbb{Y}_{00} \end{aligned} \qquad (21)$$

where **z** and **v** denote respectively an arbitrary virtual state and control sequence that is feasible at time t. Notice that, due to compactness of the set $\mathbb{Z}$, for any sequence of feasible virtual states and control moves it holds:

$$\lim_{t \to +\infty} \frac{\sum_{k=0}^{N-1} h(z(k), v(k))}{t} = 0 \qquad (22)$$

In particular then, taken any time sequence $t_n \to +\infty$ such that

$$\lim_{n \to +\infty} \frac{\sum_{k=0}^{t_n-1} h(x(k), u(k))}{t_n} \quad \text{exists,} \qquad (23)$$

it holds:

$$\begin{aligned} & \lim_{n \to +\infty} \frac{\sum_{k=0}^{t_n-1} h(x(k), u(k))}{t_n} \\ \in & \lim_{n \to +\infty} \frac{(t_n+N)\mathbb{Y} \oplus (1+t_n^\alpha)\mathbb{Y}_{00}}{t_n} = \mathbb{Y}. \end{aligned}$$

This concludes the proof of the claim.

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As anticipated, one way of enforcing convergence is to ensure a zero variance constraint:

$$y = h(x, u) = |x - x_s|^2 \qquad \text{Av}[y] \subseteq \{0\}, \qquad (24)$$

where x_s is the best feasible equilibrium as defined in (4).

We remark that when definition (19) is adopted to declare $\mathbb{Y}_t$, feasible initial states give rise to essentially converging trajectories, by virtue of Lemma 2.2 and Proposition 2.3. When $\mathbb{Y}_t$ is defined as in (18), instead, more can be expected, as shown in the following Proposition.

Proposition 2.4: Consider the algorithm described in (8), where y is defined as in equation (24), while $\mathbb{Y}_t$ is updated according to equations (10) and (11), with $\mathbb{Y} = \{0\}$. Then, given a feasible initial state $x(0)$, the corresponding closed-loop solution is defined for all subsequent times, fulfills point-wise in time constraints, and $x(t) \to x_s$ as $t \to +\infty$.

Proof: Feasibility and satisfaction of point-wise in time constraints are proven in [1]. Convergence to x_s can be shown considering that

$$\mathbb{Y}_t = \mathbb{Y}_{00} \ominus \sum_{\tau=0}^{t-1} |x(t) - x_s|^2.$$

As a consequence, for every t:

$$\sum_{k=0}^{N-1} |z(k) - x_s|^2 + \sum_{\tau=0}^{t-1} |x(\tau) - x_s|^2 \in \mathbb{Y}_{00}. \qquad (25)$$

In particular then:

$$\sum_{t=0}^{+\infty} |x(t) - x_s|^2 < +\infty \qquad (26)$$

and this implies $x(t) \to x_s$ asymptotically. ■

Remark 2.5: It is worth pointing out that Proposition (2.4) is still valid provided one defines a vector output constraint:

$$y = \begin{bmatrix} \vdots \\ |x^i - x_s^i|^2 \\ \vdots \end{bmatrix} : \qquad \text{Av}[y] \subseteq \{0_n\}. \qquad (27)$$

III. EXAMPLE: CSTR WITH PARALLEL REACTIONS

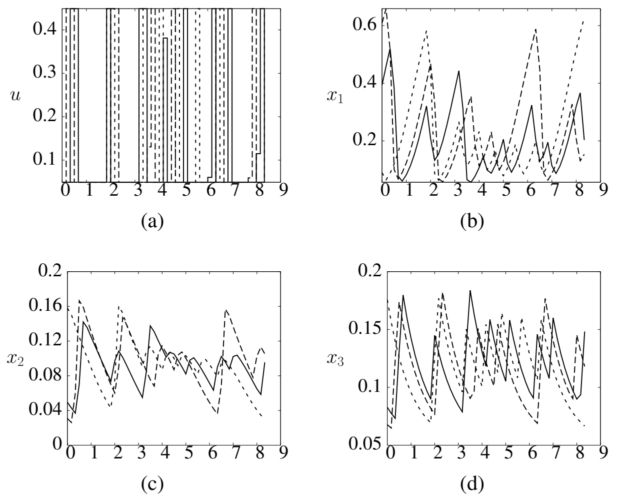
We consider the control of a nonlinear continuous flow stirred-tank reactor with parallel reactions [4].

$$\begin{aligned} R &\to P_1 \\ R &\to P_2 \end{aligned}$$

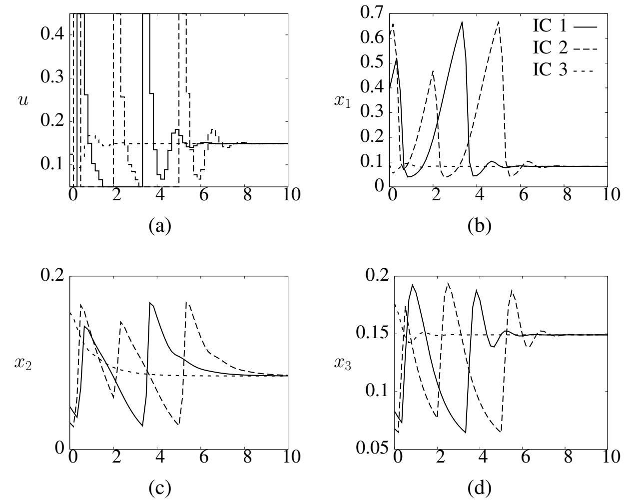
The primary objective of such processes is a desirable distribution of products in the effluent. The dimensionless heat and mass balances for this problem are

$$\begin{aligned} \dot{x}_1 &= 1 - 10^4 x_1^2 e^{-1/x_3} - 400 x_1 e^{-0.55/x_3} - x_1 \\ \dot{x}_2 &= 10^4 x_1^2 e^{-1/x_3} - x_2 \\ \dot{x}_3 &= u - x_3 \end{aligned}$$

where x_1 is the concentration of the component R, x_2 is the concentration of the desired product P_1 and x_3 is the temperature of the mixture in the reactor. P_2 is the waste product. u, which is the heat flux through the reactor wall is the manipulated variable, and is constrained to lie between 0.049 and 0.449, while x is considered non-negative. The



1: Closed-loop input (a) and state (b), (c), (d) profiles for economic MPC with different initial states



2: Closed-loop input (a) and state (b), (c), (d) profiles for economic MPC with a convex term, with different initial states.

primary objective of the process is to maximize the amount of P_1 ($\ell(x,u) = -x_2$). Previous analysis [4] has clearly highlighted that periodic operation can outperform steady-state operation. The steady-state problem has a solution $x_s = \begin{bmatrix}0.0832 & 0.0846 & 0.1491\end{bmatrix}'$ and $u_s = 0.1491$.

We solve the dynamic regulation problem using the simultaneous approach [5]. The time axis of the control horizon is divided into a fixed number of finite elements covering the control horizon. The state profiles are approximated by a family of polynomials on the finite elements. The input is parametrized according to zero order hold with the input value constant across a finite element. A terminal state constraint is used in all the simulations.

A control horizon of $N = 150$ is chosen with a sample time $T_s = 1/6$. The system is initialized at three different initial states. The closed loop system under the economic control is seen to jump between the input bounds and hence is unstable (Figure 1).

To enforce convergence, we first add a convex term in the stage cost as prescribed by Angeli et al. [1]

$$V_N(x, \mathbf{u}) = \sum_{k=0}^{N} -x_2(k) + |u(k) - u(k-1)|_S^2$$

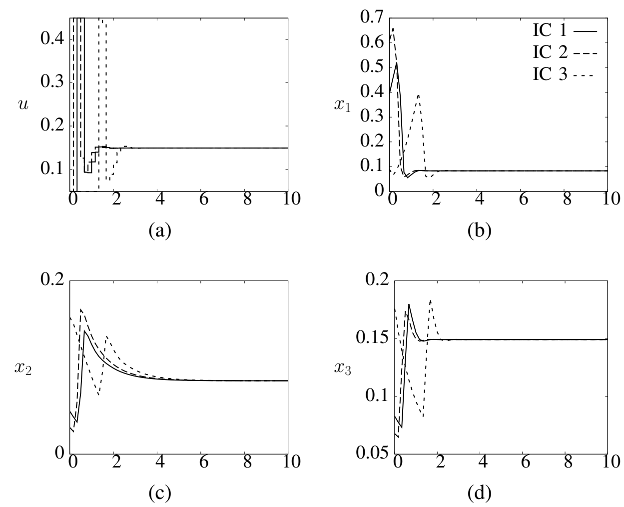
For $S = 0.17$, we observe a stable solution (Figure 2), and the closed loop system converges to the optimal steady state.

We can also penalize the distance from the steady state for the convex term in the objective

$$V_N(x, \mathbf{u}) = \sum_{k=0}^{N} -x_2(k) + |u(k) - u_s|_R^2$$

For $R = 0.15$, we again observe that the closed loop system converges to the optimal steady state (Figure 3).

Next we enforce convergence without modifying the objective function, by enforcing the zero variance constraint (27) using the iteration scheme (10). Figure 4 shows



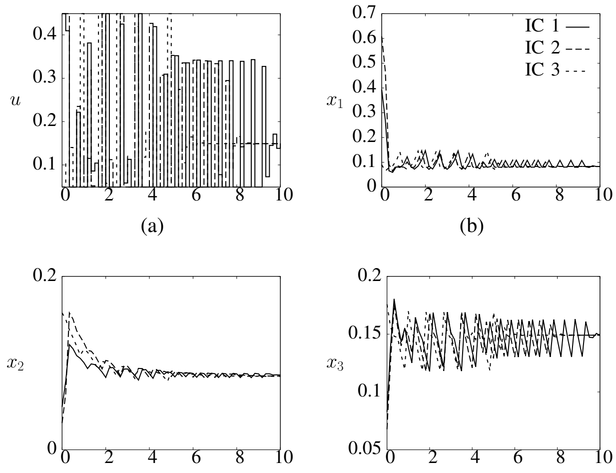
3: Closed-loop input (a) and state (b), (c), (d) profiles for economic MPC with a convex term, with different initial states.

closed loop profiles with the tuning parameter $\mathbb{Y}_{00}$ defined as

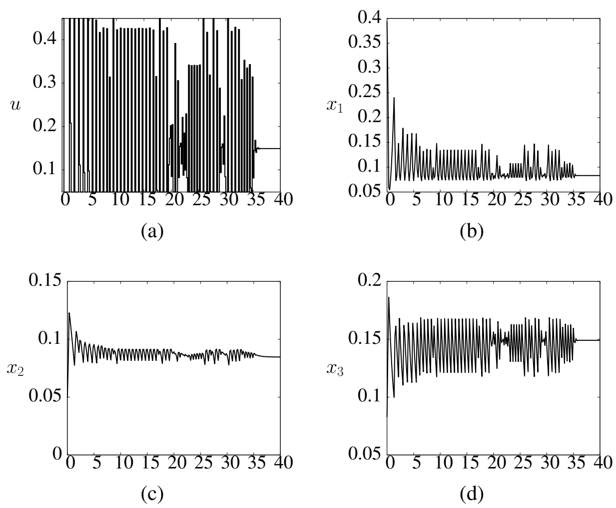
$$\mathbb{Y}_{00} = \{y \mid -w \leq y \leq w\} \tag{28}$$

in which $w = \begin{bmatrix}0.5 & 0.05 & 0.018\end{bmatrix}'$. The solution is also seen to converge to the optimal steady state.

Also note that the rate of convergence depends on the tuning parameter $\mathbb{Y}_{00}$, which is the initial variance allowance for the system. If the initial allowance is larger, iteration scheme (10) takes a longer time to converge and hence the system is in transient for a longer time, slowing down the rate of convergence. Figure 5 shows closed loop profiles for $\mathbb{Y}_{00}$ defined by (28) and $w = \begin{bmatrix}0.5 & 0.07 & 0.07\end{bmatrix}'$.



4: Closed-loop input (a) and state (b), (c), (d) profiles for economic MPC with a convergence constraint, with different initial states.



5: Closed-loop input (a) and state (b), (c), (d) profiles for economic MPC with a convergence constraint.

IV. Conclusions

Two strategies for enforcing convergence to equilibria in Economic Model Predictive Control schemes are discussed and compared by means of a simulation example taken from the chemical process control literature. The process would naturally exhibit non-converging behavior if the standard Economic MPC feedback law was to be applied. The trade-off between rate of convergence and economic performance is highlighted in both approaches. In particular, by using average constraints it is possible to enforce convergence without modifying the stage cost. The transient duration can be adjusted by suitably choosing the initial set $\mathbb{Y}_{00}$. The larger the set the slower the convergence to equilibrium. On the other hand, slower convergence yields in such cases better transient performance as the system is free to indulge a little longer in proximity of some optimal (or at least cheaper) oscillating solution.

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