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A Limit Cycle Synthesis Method of Multi-Modal and 2-Dimensional Piecewise Affine Systems

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***Abstract*— This paper is devoted to a synthesis problem of a multi-modal and 2-dimensional piecewise affine system that generates a desired stable limit cycle. By using the proposed synthesis method, we can obtain a multi-modal and 2-dimensional piecewise affine system from given data on a desired limit cycle trajectory. In addition, the existence and the uniqueness of desired stable limit cycle for the obtained system can be proven. It also turns out that we can determine the rotation direction and the period of the limit cycle by adjusting the parameters of the system. In order to show the effectiveness of our new synthesis method, we illustrate various simulation results.**

I. INTRODUCTION

A limit cycle has been attracted a lot of researchers' interest as a specific phenomenon for nonlinear systems along with chaos, fractal and solitons [1]. Limit cycles in real world can be found in various research fields, for example, stable walking or gaits of humanoid robots in robotic engineering, oscillator circuits in electronic engineering, catalytic hypercycles in chemistry, circadian rhythms in biology, boom-bust cycles in economics and so on. Researches on limit cycles have been actively done from mathematical and engineering perspectives so far [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]. Especially, some conditions for nonlinear systems that generate periodic solutions and some applications were shown in [2], and in [7], a synthesis method of hybrid systems whose solution trajectories converge to desired trajectories was proposed. In these studies, it is guaranteed that solution trajectories of the systems converges to a desired closed curve, and the existence of limit cycles was confirmed by numerical simulations. However, the mathematical guarantee of the existence of limit cycles was not indicated.

In this paper, we consider a synthesis problem of multi-modal and 2-dimensional piecewise affine systems that generate desired limit cycles, and give the mathematical guarantee of the existence and uniqueness of limit cycles. The outline of this paper is as follows. We first give the problem formulation on piecewise affine systems in Section II. Next, in Section III, we derive a specific form of the piecewise affine system. In Section IV, we then prove the existence and uniqueness of limit cycle for the piecewise affine system. Moreover, theoretical analysis on rotational directions and periods of limit cycles is obtained. Finally, we show some numerical simulations in order to confirm the effectiveness of our new synthesis method.

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II. PROBLEM SETTING

In this section, we give the problem formulation which is dealt with throughout this paper. Consider the 2-dimensional Euclidian space: $\mathbf{R}^2$, its coordinate: $x = [\, x_1 \;\; x_2 \,]^\top \in \mathbf{R}^2$, and the origin of $\mathbf{R}^2$: O. Let us set N $(N > 3)$ points $P_i \neq O \, (i = 1, \cdots, N)$ in $\mathbf{R}^2$ and denote the vector from O to P_i by $p_i = [\, p_i^1 \;\; p_i^2 \,]^\top$. We also denote the angle between the half line OP_i and the x_1-axis by θ_i. Now, without loss of generality, we assume that the N points $P_i \; (i = 1, \cdots, N)$ are located in the counterclockwise rotation from the x_1-axis, that is, $\theta_1 < \theta_2 < \cdots < \theta_N$ holds.

Next, we define the semi-infinite region D_i which is sandwiched by the half lines OP_i and OP_{i+1} and the line segment C_i joining P_i and P_{i+1}, where $P_{N+1} = P_1$. Set a polygon that is a union of $C_i \; (i = 1, \cdots, N)$ as

$$C := \bigcup_{i=1}^{N} C_i. \tag{1}$$

Fig. 1 shows an example of a Polygonal Closed Curve for $N = 5$.

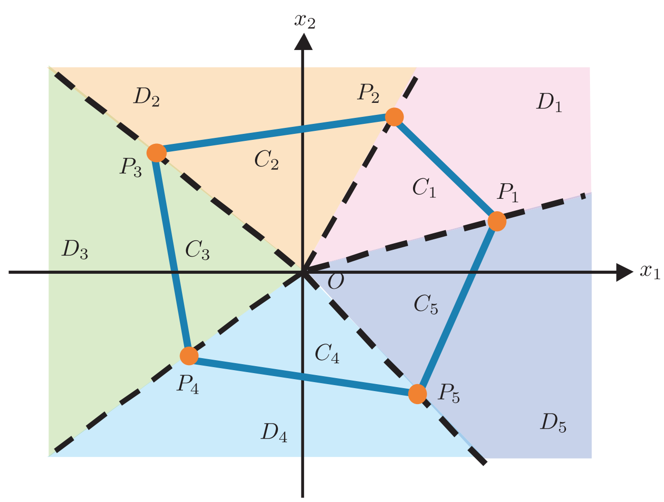

Fig. 1 : Example of Polygonal Closed Curve $(N = 5)$

We then consider the next affine system defined in D_i:

$$\dot{x} = a_i + A_i x, \tag{2}$$

where $x = [\, x_1 \;\; x_2 \,]^\top \in \mathbf{R}^2$ is the state variable, and $a_i \in \mathbf{R}^2$, $A_i = \mathbf{R}^{2\times 2}$ are the affine term and the coefficient matrix, respectively. Consequently, we treat the N-modal and 2-dimensional piecewise affine system that consists of N regions $D_i \, (i = 1, \cdots, N)$ and N affine systems (2). In this paper, we consider the following synthesis problem on limit cycles.

Problem 1 : For the N-modal and 2-dimensional piecewise affine system (2), design a_i, A_i $(i = 1, \cdots, N)$ such that a given polygonal closed curve C (1) is a unique and stable limit cycle. ∎

III. SYNTHESIS OF PIECEWISE AFFINE SYSTEMS

In this section, we will derive a candidate of a solution for Problem 1. First, we focus on a behavior of a solution trajectory of (2) in D_i. We can easily confirm that the equation of C_i is represented by

$$(p_i^2 - p_{i+1}^2)x_1 - (p_i^1 - p_{i+1}^1)x_2 + p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1 = 0. \tag{3}$$

Using (3), we now define *a limit cycle function* V_i as

$$V_i(x) = (p_i^2 - p_{i+1}^2)x_1 - (p_i^1 - p_{i+1}^1)x_2 + p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1. \tag{4}$$

If V_i converges to 0 along a solution trajectory of (2), then the solution trajectory of (2) also converges to C_i. Hence, a_i and A_i should be determined so that V_i converges to 0 along a solution trajectory of (2).

We now show an important result derived by Green in [2] on design of nonlinear systems as follows.

Theorem 1 [2] : Consider the following 2-dimensional nonlinear system:

$$\dot{x} = f(x) + g(x), \tag{5}$$

where $x \in \mathbf{R}^2$ and $f, g \in \mathbf{R}^2 \to \mathbf{R}^2$ are vector fields defined in $\mathbf{R}^2$. In addition, consider a radial and unbounded function define on $\mathbf{R}^2$: $V : \mathbf{R}^2 \to \mathbf{R}$ such that $V(0) = 0$ and $V(x) \neq 0, \ \forall x \neq 0$ hold. We now define f and g as

$$f := U_f(x)\frac{\partial V}{\partial x}^{\mathsf{T}}, \quad g := -u_g(V)U_g(x)\frac{\partial V}{\partial x}^{\mathsf{T}}, \tag{6}$$

where a skew-symmetric matrix U_f, a positive definite matrix U_g, and u_g such that $u_g(V) > 0, \ V \neq 0$ holds. Then, for the system (5) with (6),

$$\lim_{t\to\infty} V(x(t)) = 0 \tag{7}$$

holds ∎

By Theorem 1, we can derive an affine term a_i and a coefficient matrix A_i of the affine system (2) such that V_i converges to 0 along a solution trajectory of (2). As a specific form of (5) and (6), we use

$$\dot{x} = f_i + g_i, \quad f_i := \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix} \frac{\partial V_i}{\partial x}^{\mathsf{T}}, \ g_i := -V_i(x)\frac{\partial V_i}{\partial x}^{\mathsf{T}}. \tag{8}$$

Substituting (4) into (8), we can obtain a_i and A_i of (2) as

$$a_i = \begin{bmatrix} -(p_i^2 - p_{i+1}^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) - \omega_i(p_i^1 - p_{i+1}^1) \\ (p_i^1 - p_{i+1}^1)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) - \omega_i(p_i^2 - p_{i+1}^2) \end{bmatrix},$$
$$A_i = \begin{bmatrix} -(p_i^2 - p_{i+1}^2)^2 & (p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) \\ (p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) & -(p_i^1 - p_{i+1}^1)^2 \end{bmatrix}. \tag{9}$$

However, it is only guaranteed that the system (2) with (9) satisfies the convergence property (7), that is, its solution trajectory converges to C_i in D_i. Hence, it is not guaranteed that the system (2) with (9) has a unique and stable limit cycle. In the next section, we will discuss the existence of a unique and stable limit cycle of the system (2) with (9).

IV. THEORETICAL ANALYSIS

A. Proof on Existence and Uniqueness of Limit Cycle

The main purpose of this subsection is to prove that the N-modal and 2-dimensional piecewise affine systems (2) with (9) has a unique and stable limit cycle, and it is equivalent to C. To complete the proof, we first indicate three lemmas, and then we show the main theorem by using them.

First, we give the definition on the clockwise and counterclockwise rotations of limit cycle solution trajectories of the system (2) with (9).

Definition 1 : For limit cycle solution trajectories of the N-modal and 2-dimensional piecewise affine system (2) with (9), one that rotates in the clockwise direction in $\mathbf{R}^2$ is called *a limit cycle solution trajectory in the clockwise rotation*. On the contrary, one that rotates in the counterclockwise direction in $\mathbf{R}^2$ is called *a limit cycle solution trajectory in the counterclockwise rotation* (see Fig. 2). ∎

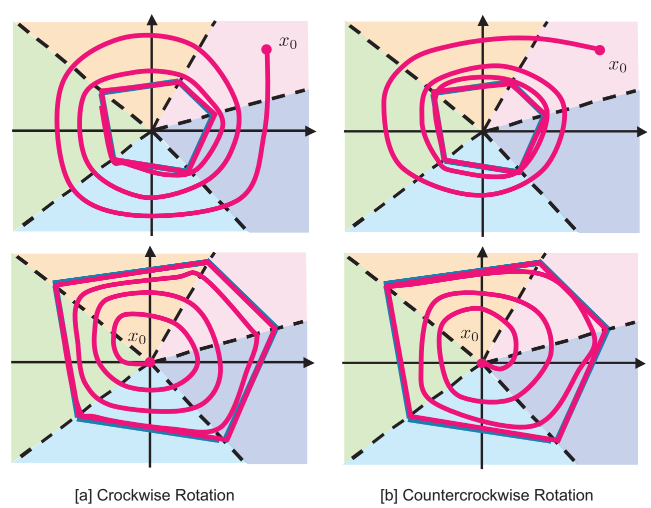

Fig. 2 : Clockwise and Counterclockwise Rotations of Limit Cycle Solution Trajectories

Let us define a subset in D_i as

$$M_i(\varepsilon_i) := \{ \, x \in D_i \mid \varepsilon_i^- \leq V_i(x) \leq \varepsilon_i^+ \, \}, \tag{10}$$

where ε_i^-, $\varepsilon_i^+ \in \mathbf{R}$ satisfies $\varepsilon_i^- < \varepsilon_i^+$ and we set $\varepsilon_i = (\varepsilon_i^-, \varepsilon_i^+)$. We also define a sum of these subsets as

$$M(\varepsilon) := \bigcup_{i=1}^{N} M_i(\varepsilon_i), \tag{11}$$

where we use the notations: $\varepsilon^- = (\varepsilon_1^-, \cdots, \varepsilon_N^-)$, $\varepsilon^+ = (\varepsilon_1^+, \cdots, \varepsilon_N^+)$, $\varepsilon = (\varepsilon^-, \varepsilon^+)$, and the parameters ε^- and ε^+

are determined such that

$$M^-(\varepsilon^-) = \bigcup_{i=1}^{N} \{ x \in D_i \mid V_i(x) = \varepsilon_i^- \}$$
$$M^+(\varepsilon^+) = \bigcup_{i=1}^{N} \{ x \in D_i \mid V_i(x) = \varepsilon_i^+ \} \tag{12}$$

form closed polygons. If $M^-(\varepsilon^-)$ and $M^+(\varepsilon^+)$ are closed polygons, that is, $M(\varepsilon)$ is a bounded and closed set, then ε is said to be *admissible* (see Fig. 3). We can derive the following proposition on $M(\varepsilon)$.

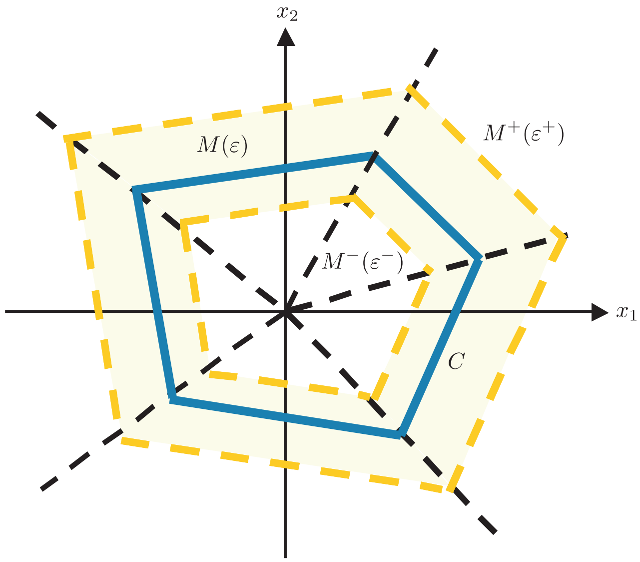

Fig. 3 : Example of $M^-(\varepsilon^-)$, $M^+(\varepsilon^+)$ and $M(\varepsilon)$

Lemma 1 : For the N-modal and 2-dimensional piecewise affine system (2) with (9), $M(\varepsilon)$ is a positively invariant, bounded and closed set for any admissible ε.

(Proof) Calculating the time differential of $1/2V_i^2$ along a solution trajectory of the system (2) with (9), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(V_i^2) &= V_i\dot{V}_i = V_i\frac{\partial V_i}{\partial x}\dot{x} = V_i\frac{\partial V_i}{\partial x}(f_i + g_i) \\ &= V_i\frac{\partial V_i}{\partial x}\begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}\frac{\partial V_i}{\partial x}^{\mathsf{T}} - V_i^2\frac{\partial V_i}{\partial x}\frac{\partial V_i}{\partial x}^{\mathsf{T}} \\ &= -V_i^2\frac{\partial V_i}{\partial x}\frac{\partial V_i}{\partial x}^{\mathsf{T}} < 0.\end{aligned}$$

Hence, it turns out that the velocity vector field of the system (2) with (9) points to the direction of the inner side of the bounded and closed set M at any points on the boundary $M^-(\varepsilon^-) \cup M^+(\varepsilon^+)$ of M. Consequently, $M(\varepsilon)$ is a positively invariant, bounded and closed set. ∎

Next, we consider equilibrium points of the system (2) with (9). The following lemma on equilibrium points can be obtained.

Lemma 2 : Assume $\omega_i \neq 0$ $(i = 1, \cdots, N)$. Then, the N-modal and 2-dimensional piecewise affine system (2) with (9) does not have any equilibrium points in $M(\varepsilon)$ for any admissible ε.

(Proof) The unit vector which is on a parallel with C_i in D_i and points to the counterclockwise rotation is given by $(p_i - p_{i+1})/||p_i - p_{i+1}||$. By considering the inner product of this unit vector and the velocity vector field of the system (2) with (9), we have the magnitude of the velocity component to the direction of $p_i - p_{i+1}$ for a solution trajectory v_i of the system (2) with (9) in D_i as

$$v_i = (a_i + A_i x) \cdot \frac{p_i - p_{i+1}}{||p_i - p_{i+1}||}. \tag{13}$$

Now, we denote a point in D_i by $x = \alpha_i p_i + \beta_i p_{i+1}, \alpha_i, \beta_i \geq 0$. Hence, we can calculate (13) as

$$\begin{aligned} v_i &= \{a_i + A_i(\alpha_i p_i + \beta_i p_{i+1})\} \cdot \frac{p_i - p_{i+1}}{||p_i - p_{i+1}||} \\ &= -\omega_i\sqrt{(p_i^1 - p_{i+1}^1)^2 + (p_i^2 - p_{i+1}^2)^2}. \end{aligned} \tag{14}$$

From (14), we can see that the parameters α_i and β_i vanish, and hence v_i is constant at any point $x \in D_i$. Since v_i does not vanish at any point $x \in D_i$, the system (2) with (9) does not have any equilibrium points in $M(\varepsilon)$. ∎

Then, we here give a definition on the concept "traversal" for the system (2) with (9) as follows [10].

Definition 2 : Let Σ be a line segment in the positively invariant, bounded and closed set $M(\varepsilon)$. If the value of an inner product of the unit normal vector to Σ: e_Σ and the velocity vector of the N-modal and 2-dimensional piecewise affine system (2) with (9) is not equal to 0 and its sign does not change at any point in Σ, then Σ is said to be *traversal* with respect to the system (2) with (9). ∎

In addition to Lemma 2, under the condition of $\omega_i > 0 (i = 1, \cdots, N)$, a solution trajectory vector of the system (2) with (9) always has a velocity component in the counterclockwise rotation. On the other hand, under the condition of $\omega_i < 0\,(i = 1, \cdots, N)$, a solution trajectory vector of (2) with (9) always has a velocity component in the clockwise rotation. From this fact, we can derive the following lemma.

Lemma 3 : For the N-modal and 2-dimensional piecewise affine system (2) with (9), assume that $\omega_i > 0 (i = 1, \cdots, N)$ or $\omega_i < 0$ $(i = 1, \cdots, N)$ holds. Then, there exists a traversal line segment Σ at any point in $x \in M(\varepsilon)$, and it is satisfied that $x \in \Sigma$ and Σ infinitely intersects with solution trajectories of the system (2) with (9).

(Proof) We assume that $\omega_i > 0$ $(i = 1, \cdots, N)$ or $\omega_i < 0$ $(i = 1, \cdots, N)$ holds. Then, a solution trajectory of the system (2) with (9) always circles to the counterclockwise rotation or to the clockwise rotation. Now, for a point $x \in M$, we consider a half line whose origin is O and that passes through x, and define a subset $\Sigma \subset M$ as the intersection of the half line and M. Since the velocity vector field of the system (2) with (9) always has the velocity component of the counterclockwise rotation or the clockwise rotation, the inner product of a normal vector of Σ and the vector field of the system (2) with (9) at any point in Σ is not equal to 0 and its sign does not change, that is, Σ is traversal. Moreover, in each M_i $(i = 1, \cdots, N)$, since the velocity vector field of the system (2) with (9) always has the velocity component of the clockwise rotation or the counterclockwise rotation, a solution trajectory of the system (2) with (9) $x(t)$ that intersected Σ intersects Σ in a finite time again. Consequently, it turns out the solution trajectory intersects Σ infinitely. ∎

Using Lemmas 1–3, we can derive the main theorem on the existence of the limit cycle of the system (2) with (9).

Theorem 2 : For the N-modal and 2-dimensional piecewise affine system (2) with (9), assume that $\omega_i > 0 (i = 1, \cdots, N)$ or $\omega_i < 0\ (i = 1, \cdots, N)$ holds. Then, the unique and stable limit cycle of the system (2) with (9) is equivalent to C.

(Proof) By the result on the hybrid Poincare-Bendixson theorem derived in [3], [10], it turns out that sufficient conditions for the existence of stable limit cycles of the system (2) with (9) in $M(\varepsilon)$ are the following three: (i) $M(\varepsilon)$ is a positively invariant, bounded and closed set, (ii) there do not exist any equilibrium points at the boundary and in the interior of $M(\varepsilon)$ (iii) there exists a traversal line segment $\Sigma \subset M(\varepsilon)$ such that $x \in \Sigma$ and Σ infinitely intersects with solution trajectories of the system (2) with (9). Since we have confirmed these three conditions in Lemma 1, 2, and 3, we can see that there exists a stable limit cycle in $M(\varepsilon)$ for the system (2) with (9) for any admissible ε. Moreover, since $M(\varepsilon)$ converges to C as the values of ε goes to 0, it can be confirmed that C is a unique and stable limit cycle. Hence, the proof is completed. ∎

B. Analysis of Rotational Directions and Periods

This section presents theoretical analysis on rotational directions and periods of limit cycle solution trajectories of the system (2) with (9). First, we consider the relationship between rotational directions of limit cycles and the parameters in (9). We can easily derive the following result.

Proposition 1 : For the N-modal and 2-dimensional piecewise affine system (2) with (9), its limit cycle solution trajectory moves in the counterclockwise rotation for $\omega_i > 0\ (i = 1, \cdots, N)$, and conversely it moves in the clockwise rotation for $\omega_i < 0\ (i = 1, \cdots, N)$.

(Proof) The proof of this proposition is trivial from the discussion in the previous section. ∎

Next, we analyze periods of limit cycles of the system (2) with (9). It can be expected that after a solution trajectory of the system (2) with (9) converges to C, it behaves as a periodic trajectory. By calculating the velocity component of the vector of the system along C, we can derive the next proposition.

Proposition 2 : When a limit cycle solution trajectory of the N-modal and 2-dimensional piecewise affine system (2) with (9) is sufficiently close to C, the period with which it rotates around C is given by

$$T \approx \sum_{i=1}^{N} \frac{1}{|\omega_i|}. \tag{15}$$

(Proof) The velocity component of a solution trajectory v_i of (2) with (9) in D_i to the direction of $p_i - p_{i+1}$ is given by (13). The length of C_i: l_i can be calculated as

$$l_i = \sqrt{(p_i^1 - p_{i+1}^1)^2 + (p_i^2 - p_{i+1}^2)^2}. \tag{16}$$

Therefore, we can obtain the period T as

$$T \approx \sum_{i=1}^{N} \frac{l_i}{|v_i|} = \sum_{i=1}^{N} \frac{1}{|\omega_i|}. \tag{17}$$

This completes the proof of this proposition. ∎

From Proposition 1 and 2, it turns out that by turning the parameters $\omega_i\ (i = 1, \cdots, N)$ in $a_i\ (i = 1, \cdots, N)$ of (9), we can determine the rotational direction and the period of a limit cycle solution trajectory of the N-modal and 2-dimensional piecewise affine system (2) with (9).

V. SIMULATIONS

In this section, we consider an example and carry out some numerical simulations to confirm the results derived in the previous sections. We now give data of the polygon with $N = 8$ as $P_1 = (1.20,\ 0.00)$, $P_2 = (0.19,\ 0.23)$, $P_3 = (0.60,\ 1.04)$, $P_4 = (0.00,\ 0.70)$, $P_5 = (-0.60,\ 1.04)$, $P_6 = (-0.80,\ 0.00)$, $P_7 = (-0.50,\ -0.87)$, $P_8 = (0.40,\ -0.69)$, which are shown in Fig. 4. First, we use the parameters in (9) as $\omega_i = 1\ (i = 1, \cdots, 8)$. Then, the piecewise affine system is given by

$$\begin{aligned}
a_1 &= \begin{bmatrix} -0.94652 \\ 0.50876 \end{bmatrix},\ A_1 = \begin{bmatrix} -0.0529 & -0.2323 \\ -0.2323 & -1.0201 \end{bmatrix}, \\
a_2 &= \begin{bmatrix} 0.458276 \\ 0.785564 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.6561 & 0.3321 \\ 0.3321 & -0.1681 \end{bmatrix}, \\
a_3 &= \begin{bmatrix} -0.7428 \\ -0.088 \end{bmatrix},\ A_3 = \begin{bmatrix} -0.1156 & 0.204 \\ 0.204 & -0.36 \end{bmatrix}, \\
a_4 &= \begin{bmatrix} -0.4572 \\ -0.592 \end{bmatrix},\ A_4 = \begin{bmatrix} -0.1156 & -0.204 \\ -0.204 & -0.36 \end{bmatrix}, \\
a_5 &= \begin{bmatrix} -1.06528 \\ -0.8736 \end{bmatrix},\ A_5 = \begin{bmatrix} -1.0816 & 0.208 \\ 0.208 & -0.04 \end{bmatrix}, \\
a_6 &= \begin{bmatrix} -0.30552 \\ -1.0788 \end{bmatrix},\ A_6 = \begin{bmatrix} -0.7569 & -0.261 \\ -0.261 & -0.09 \end{bmatrix}, \\
a_7 &= \begin{bmatrix} 1.02474 \\ -0.4437 \end{bmatrix},\ A_7 = \begin{bmatrix} -0.0324 & 0.162 \\ 0.162 & -0.81 \end{bmatrix}, \\
a_8 &= \begin{bmatrix} 1.37132 \\ 0.0276 \end{bmatrix},\ A_8 = \begin{bmatrix} -0.4761 & 0.552 \\ 0.552 & -0.64 \end{bmatrix}.
\end{aligned} \tag{18}$$

The initial state is set as $x_0 = [\,0,\ 0\,]^{\mathsf{T}}$. The simulation results are illustrated in Figs. 5–7. Fig. 5 shows the solution trajectory on the x_1x_2-plane. In Figs. 6 and 7, the time series of x_1 and x_2 are shown, respectively. From these results, we can see that the solution trajectory that starts from x_0 behaves as a limit cycle for the desired polygonal closed curve C, and hence Theorem 1 holds. Since we use the parameters $\omega_i = 1\ (i = 1, \cdots, 8)$, the solution trajectory moves in the counterclockwise rotation, and this result is coincident with Proposition 1. In addition, we can derive the estimated period as $T \approx 8$ by Proposition 2, and it is mostly agree about the simulation result from Figs. 6 and 7.

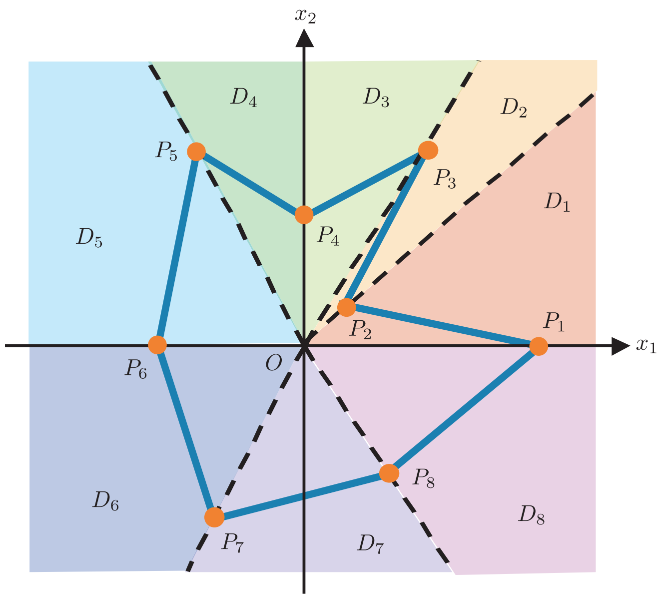

Fig. 4 : Polygonal Closed Curve of Example

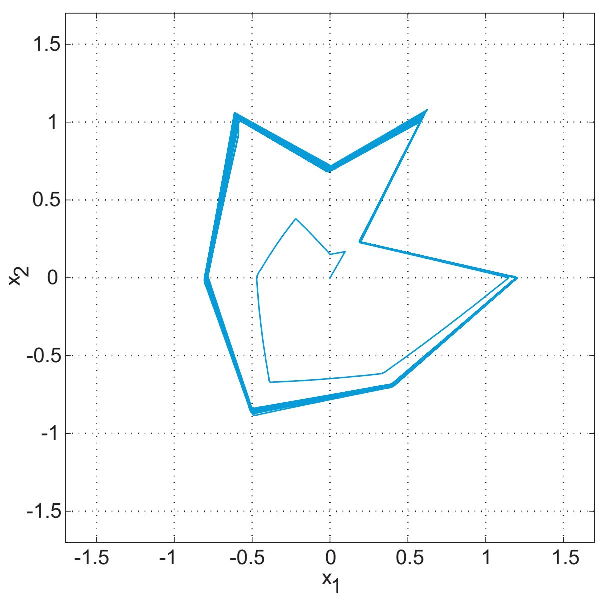

Fig. 5 : Solution Trajectory on x_1x_2-Plane

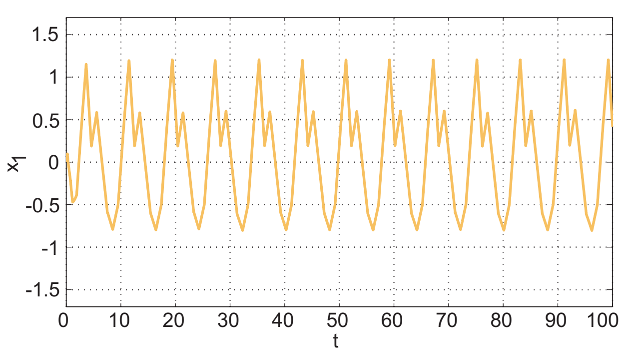

Fig. 6 : Time Series of x_1

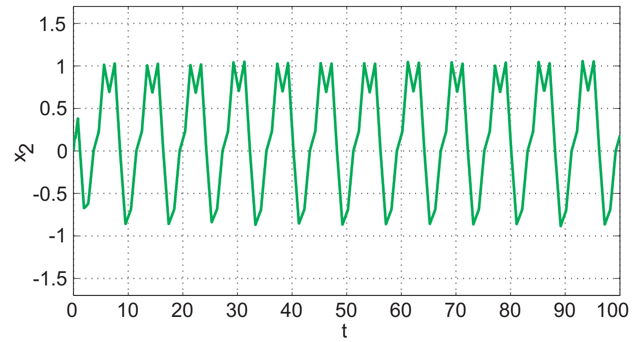

Fig. 7 : Time Series of x_2

Next, we use another set of parameters in (9) as $\omega_i = -2 \ (i = 1, \cdots, 8)$. The piecewise affine system for these parameters is given by

$$a_1 = \begin{bmatrix} 2.08348 \\ -0.18124 \end{bmatrix}, \ a_2 = \begin{bmatrix} -0.77172 \\ -1.64444 \end{bmatrix}, \ a_3 = \begin{bmatrix} 1.0572 \\ 0.932 \end{bmatrix},$$
$$a_4 = \begin{bmatrix} 1.3428 \\ -0.428 \end{bmatrix}, \ a_5 = \begin{bmatrix} -0.46528 \\ 2.2464 \end{bmatrix}, \ a_6 = \begin{bmatrix} -1.20552 \\ 1.5312 \end{bmatrix},$$
$$a_7 = \begin{bmatrix} -1.67526 \\ -0.9837 \end{bmatrix}, \ a_8 = \begin{bmatrix} -1.028682 \\ -2.0424 \end{bmatrix}. \qquad (19)$$

and $A_i \ (i = 1, \cdots, 8)$ are the same as (18).

We set the initial state as $x_0 = [-1.5, \ 0]^{\mathsf{T}}$. The simulation results are depicted in Figs. 8–10. Fig. 8 illustrates the solution trajectory on the x_1x_2-plane. Figs. 9 and 10 show the time series of x_1 and x_2, respectively. From these results, we can see that the solution trajectory that starts from x_0 behaves as a limit cycle for the desired polygonal closed curve C, and hence Theorem 1 holds. Since we use the parameters $\omega_i = -2 \ (i = 1, \cdots, 8)$, the solution trajectory moves in the clockwise rotation, and this result is agree with Proposition 1. Moreover, the estimated period can be obtained as $T \approx 2$, and it is mostly consistent with the simulation result from Figs. 9 and 10.

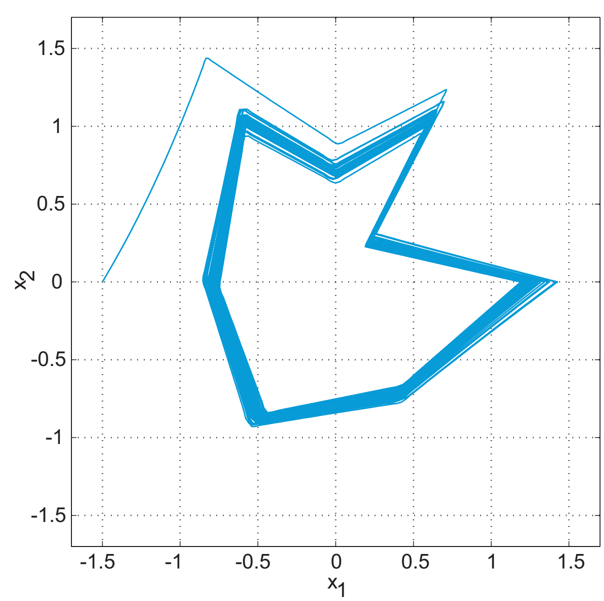

Fig. 8 : Solution Trajectory on x_1x_2-Plane

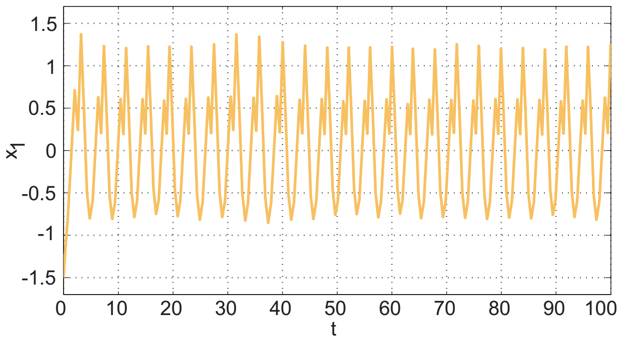

Fig. 9 : Time Series of x_1

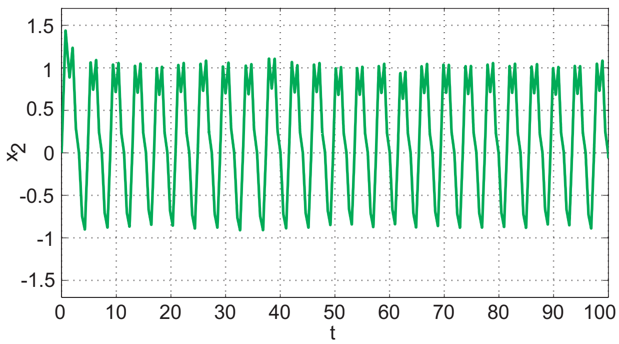

Fig. 10 : Time Series of x_2

In Fig. 11, some simulation results for some polygons are illustrated. By approximating arbitrary 2-dimensional continuous curve by a polygon, we can use the synthesis method proposed in this paper for various types of curves.

VI. CONCLUSION

In this paper, we have developed a synthesis method of a multi-modal and 2-dimensional piecewise affine system with a unique and stable limit cycle which is consistent with a desired polygonal closed curve. We have also shown the mathematical guarantee for the existence and uniqueness of the limit cycle. Moreover, we have given theoretical analysis on rotational directions and periods of limit cycles, and derives the relationships between these characteristics and the parameters of the system.

Our future work includes applications of the proposed synthesis method to real systems, a new limit cycle synthesis method for piecewise affine control systems, and extensions to multi-dimensional piecewise affine systems.

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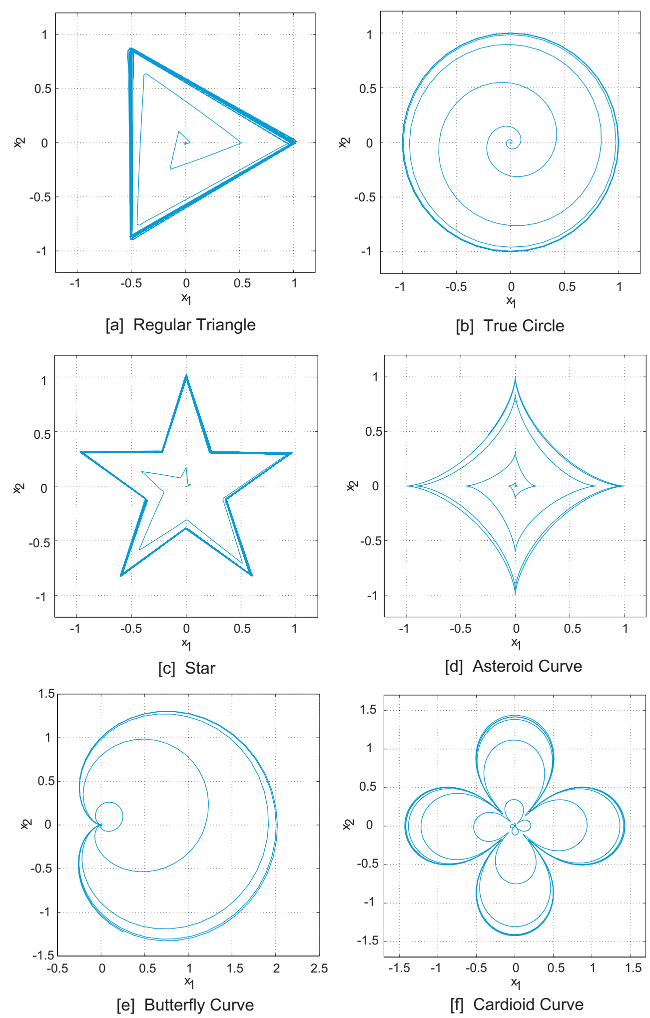

Fig. 11 : Applications to Various Closed Curves