2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Open Stochastic Systems

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Abstract— **The problem of giving an adequate definition of an open stochastic system is addressed and motivated using examples. A stochastic system is defined as a probability triple on the outcome space. The collection of events is an essential part of a stochastic model and it is argued that for phenomena with as outcome space a finite dimensional vector space, the framework of classical random vectors with the Borel σ-algebra as events is inadequate even for elementary applications. A stochastic system is linear if the events are cylinders with fibers parallel to a linear subspace of a vector space. We also address interconnection of stochastic systems.**
Keywords: **Stochastic system, linearity, gaussian system, interconnection.**

I. Introduction

This CDC presentation is a short version of a more extensive article [1] that has been submitted to the *IEEE Transactions on Automatic Control*.

We discuss stochastic systems from the point of view explained in [2] for deterministic systems. This setting goes under the name of the 'behavioral approach'. The two main underlying ideas are the following.

The first point is that the best way to think of a model is as a em relation between variables, rather than a map from some variables to some others (which is the idea underlying input/output thinking). The second point is that the best way to think of a model is as an *open* system and that a mathematical theory of modeling should reflect this aspect from the very beginning. A model should incorporate the influence of the environment, as an unmodeled feature. Open systems are the building blocks for modeling. They allow to construct models of complex systems from models of subsystems, through 'tearing, zooming, and linking' (see [2, Figure 1]).

The aim of this presentation is to put forward some of these ideas in the context of stochastic systems. We deal mainly with phenomena whose outcomes take their value in a finite dimensional real vector space.

Whereas the mathematical definition of a probability space and the concepts used in stochastic analysis can accommodate a very wide variety of phenomena, usually attention is focussed on situations in which the events for which the probability is defined form a very rich σ-algebra. This means that for phenomena that take their values in a countable set, all subsets of outcomes are assumed to be events, while for phenomena that take their values in a finite dimensional vector space, the probability is defined for the Borel σ-algebra, leading to models which can be described by their probability distribution or density. The theme underlying the present article is that this emphasis on rich σ-algebras (as the Borel σ-algebra for random vectors) is unduly restrictive, even for elementary applications. Stochastic models with the Borel σ-algebra as events are basically *closed* systems. These models do not incorporate the influence of the environment.

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We start with two motivational examples, a noisy resistor and the price/demand and price/supply elasticities of an economic good. We argue that in these examples the Borel σ-algebra requires stochastic modeling of many more events than justified.

Motivated by these examples, we define a stochastic system as a general probability triple on an outcome space and incorporate notions as linearity and gaussian systems in this context. Next, we discuss interconnection of stochastic systems. Interconnection of two stochastic systems means that two distinct probabilistic laws are *simultaneously* imposed on an outcome space. We deal with interconnection in terms of complementarity. This part is the main mathematical novelty of the paper, and we feel that the notion of interconnection should be of considerable interest in the field.

II. Motivational examples

Example 1: A noisy resistor. Consider a resistor with thermal noise. Model this as an Ohmic resistor in series with

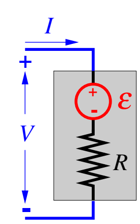

Fig. 1. **Noisy resistor**

a voltage source, as shown in Figure 1, often referred to as a *Johnson-Nyquist resistor*. This leads to the following relation between the current I through the resistor and the voltage V across it

$$V = RI + \varepsilon \tag{1}$$

with R the value of the resistor and ε the voltage generated by the noisy resistor. The noise ε is zero mean, wide band, and gaussian with standard deviation proportional to $\sqrt{RT}$ with T the temperature of the resistor.

Assume therefore that in (1) ε is a gaussian random variable with zero mean and variance σ. It then follows that

$V - RI$ is a random variable. But $\left[\begin{smallmatrix}V\\I\end{smallmatrix}\right]$ is not a 2-dimensional random vector in the usual sense of the term, since it is not implied that all the Borel sets in $\mathbb{R}^2$ are events. In other words, not all open and closed subsets of the $\left[\begin{smallmatrix}V\\I\end{smallmatrix}\right]$-space is assigned a probability. In particular, we cannot speak of the distribution of the vector $\left[\begin{smallmatrix}V\\I\end{smallmatrix}\right]$. How should we then view the 2-dimensional vector $\left[\begin{smallmatrix}V\\I\end{smallmatrix}\right]$ as a stochastic object? This is the first question which we deal with in this article.

Suppose now that we interconnect the noisy resistor with another circuit, for example a voltage source with internal resistance and thermal noise. This leads to the configuration

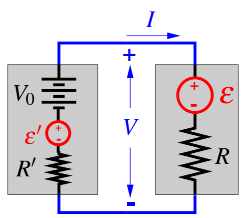

Fig. 2. **Interconnection of noisy resistors**

shown in Figure 2, with V_0 a constant voltage, R' the internal resistance, and ε' a random variable that is independent of ε. Assume that ε' is gaussian, with zero mean and standard deviation σ'. Intuitively, $\left[\begin{smallmatrix}V\\I\end{smallmatrix}\right]$ in the interconnected circuit becomes a 2-dimensional random vector in the classical sense of the term. Is this the case for any interconnecting circuit, or are there regularity conditions required on the interconnection? How is the random vector $\left[\begin{smallmatrix}V\\I\end{smallmatrix}\right]$ deduced from the mathematical specifications of the noisy resistor and the circuit it is interconnected with? This is the second question which we deal with in this article. ■

Example 2: Price/demand and price/supply. Important characteristics of an economic good are the responsiveness of the demand and of the supply to the price. Typical price/demand and price/supply characteristics are shown in Figure 3. When these characteristics pertain to the same good, we obtain the equilibrium price, demand, and supply determined by the intersection of the price/demand and price/supply characteristics.

Fig. 3. **Deterministic price/demand and price/supply characteristics**

In order to express that the demand depends on uncertain factors other than the price, randomness can be added to the price/demand characteristic. This leads to models that state in particular that the price/demand vector lies for example in the shaded region of the left part of Figure 4 with a certain probability. While it is viable to assign a probability to similar regions of the price/demand plane, it is not reasonable to assume that the price/demand is modeled as a 2-dimensional random vector in the usual sense of the term. Indeed, the uncertainty of the price/demand phenomenon does not imply a probability distribution for the price. No such probability distribution for the price is implied in the deterministic case, so why should a distribution be implied in the stochastic case? Similarly, for the price/supply it is reasonable to assume for example that the price/supply vector lies in the shaded region of the right part of Figure 4 with a certain probability, and that a probability is assigned to similar regions of the price/supply plane.

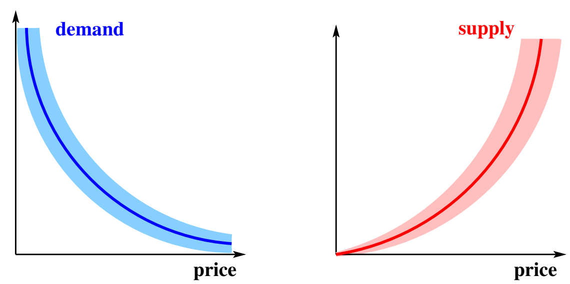

Fig. 4. **Stochastic price/demand and price/supply characteristics**

The equilibrium price/demand/supply is obtained by adding the condition demand = supply. When this equilibrium condition is imposed on the stochastic price/demand and price/supply phenomena, is it reasonable to expect that the price, demand, and supply then become random variables in the classical sense of the term? How is the randomness in the equilibrium case deduced from the original price/demand and price/supply randomness? We view the addition of the equilibrium condition as 'interconnection'. We shall see that under mild conditions the interconnection leads to a 2-dimensional probability distribution for the equilibrium price and the demand = supply.

The question how to mathematize the randomness of the price/demand and the price/supply is the main problem we deal with in this article. Further, we formalize how to combine the random models of the price/demand and of the price/supply into a random model when the equilibrium condition demand = supply is added. ■

III. STOCHASTIC SYSTEMS

Definition 1: A *stochastic system* Σ is a probability triple

$$(\mathbb{W}, \mathscr{E}, P)$$

with

- $\mathbb{W}$ a non-empty set, the *outcome space*, with elements called *outcomes*,
- $\mathscr{E}$ a σ-algebra of subsets of $\mathbb{W}$; elements of $\mathscr{E}$ are called *events*,
- $P : \mathscr{E} \to [0,1]$ a probability.

■

The construction of a stochastic model involves therefore three steps. Firstly, the phenomenon is formalized mathematically by determining the outcome space. Subsequently, the set of events to which we are willing to assign a probability is specified. Finally, we need to quantify the probability of these events. We view the specification of the events $\mathscr{E}$ as a crucial part of probabilistic modeling, contrary, as we shall see, to the classical view of probabilistic modeling.

Two important special cases are the following.

- Let $\mathbb{W}$ be a countable set and $\mathscr{E} = 2^{\mathbb{W}}$ ($2^{\mathbb{W}}$ denotes the power set of $\mathbb{W}$, that is, the class of all subsets of $\mathbb{W}$). P can then be specified by giving the probability of the outcomes, $p : \mathbb{W} \to [0,1]$, and defining P by $P(E) = \sum_{e \in E} p(e)$.
- Let $\mathbb{W} = \mathbb{R}^{\mathtt{n}}$ and $\mathscr{E} = \mathscr{B}(\mathbb{R}^{\mathtt{n}})$ ($\mathscr{B}(\mathbb{R}^{\mathtt{n}})$ denotes the Borel σ-algebra on $\mathbb{R}^{\mathtt{n}}$). P can then be specified by a probability distribution on $\mathbb{R}^{\mathtt{n}}$, or, if the distribution is sufficiently smooth, by the probability density function $p : \mathbb{R}^{\mathtt{n}} \to [0,\infty)$ leading to $P(A) = \int_A p(x)\, dx$.

We refer to these special cases as *classical* stochastic systems. For a classical stochastic system 'essentially every' subset of $\mathbb{W}$ is an event and is therefore assigned a probability. Thus for classical stochastic systems, the events are obtained from the structure of the outcome space. No probabilistic modeling enters in the specification of the events. However, in Definition 1, the event space $\mathscr{E}$ is very much a part of the stochastic model. We now illustrate the importance of specifying $\mathscr{E}$ by showing that our motivating examples are not classical stochastic systems.

Example 1: The noisy resistor. Equation (1) defines a

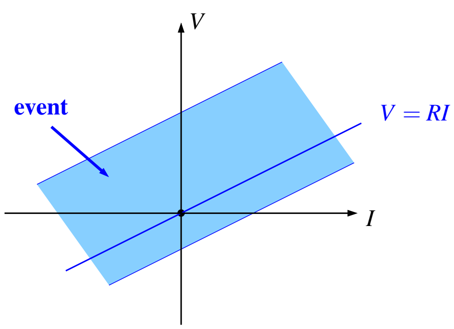

Fig. 5. **Events for the noisy resistor**

stochastic system with outcome space $\mathbb{W} = \mathbb{R}^2$ and as outcomes voltage/current vectors $\begin{bmatrix} V \\ I \end{bmatrix}$. The events are the sets

$$\left\{ \begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V - RI \in A \text{ with } A \text{ a Borel subset of } \mathbb{R} \right\}$$

(see Figure 5). The probability of this event is equal to the probability measure of $A \subseteq \mathbb{R}$, with the probability of A induced by the normal distribution with mean 0 and standard deviation σ.

Hence, whereas ε is a classical random variable, $\begin{bmatrix} V \\ I \end{bmatrix}$ is not a classical random vector in $\mathbb{R}^2$. Only cylinders with sides parallel to $V = RI$ (see Figure 5) are events and are assigned a probability. In particular, V and I are not classical random variables. Indeed, the basic model of a noisy resistor does not imply a stochastic law for V or I, in the sense that (1) does not model V and I individually as classical random variables. The model states only that $V - RI$ is a classical random variable. ■

Example 2: Price/demand and price/supply. For the price/demand, the outcome space $\mathbb{W}$ is $[0,\infty)^2$ with as outcomes price/demand vectors $\begin{bmatrix} p \\ d \end{bmatrix}$. The events are the sets for which the probability of occurrence in the set is defined, in the sense illustrated for instance by the shaded area of the left part of Figure 4.

For the price/supply, the outcome space $\mathbb{W}$ is $[0,\infty)^2$ with as outcomes price/supply vectors $\begin{bmatrix} p \\ s \end{bmatrix}$. The events are the sets for which the probability of occurrence in the set is defined, in the sense illustrated for instance by the shaded area of the right part of Figure 4.

The basic idea again is that a stochastic model of the price/demand or the price/supply characteristic of an economic good only models the relation between the price and the demand or between the price and the supply. But it does not imply that the price, the demand, and the supply are themselves classical random variables. ■

IV. Linearity

Definition 2: The $\mathtt{n}$-dimensional stochastic system $(\mathbb{R}^{\mathtt{n}}, \mathscr{E}, P)$ is said to be *linear* if there exists a linear subspace $\mathbb{L}$ of $\mathbb{R}^{\mathtt{n}}$ such that the events are the Borel subsets of the quotient space $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$, a finite dimensional real vector space of dimension $\mathtt{n} - \text{dimension}(\mathbb{L})$. The probability of an event $E \in \mathscr{E}$ is given by a Borel probability on $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$. $\mathbb{L}$ is called the *fiber* of the linear stochastic system, and $\mathtt{dimension}(\mathbb{L})$ is called *the number of degrees of freedom* of the linear stochastic system $(\mathbb{R}^{\mathtt{n}}, \mathscr{E}, P)$. An $\mathtt{n}$-dimensional stochastic system is said to be *gaussian* if it is linear and if the Borel probability on $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$ is gaussian. ■

We consider a measure that is concentrated in a point to be a gaussian measure. More generally, a gaussian probability measure may be concentrated on a linear variety.

The idea behind Definition 2 is illustrated in Figure 6. The

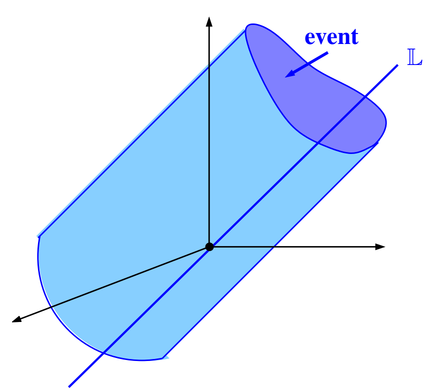

Fig. 6. **Events for a linear system**

events are cylinders in $\mathbb{R}^{\mathtt{n}}$ with sides parallel to the fiber $\mathbb{L}$. A linear $\mathtt{n}$-dimensional stochastic system is a classical $\mathtt{n}$-dimensional random vector if and only if $\mathbb{L} = \{0\}$. Therefore every classical random vector defines a linear stochastic system. At the other extreme, when $\mathbb{L} = \mathbb{R}^{\mathtt{n}}$, the σ-algebra of events trivializes to $\{\emptyset, \mathbb{R}^{\mathtt{n}}\}$.

A more concrete way of thinking about a linear $\mathtt{n}$-dimensional stochastic system is in terms of two linear subspaces $\mathbb{L}, \mathbb{M}$ of $\mathbb{R}^{\mathtt{n}}$ that are complementary, $\mathbb{L} \oplus \mathbb{M} = \mathbb{R}^{\mathtt{n}}$, and a Borel probability $P_{\mathbb{M}}$ on $\mathbb{M}$. Take as events the sets of the form

$$\mathscr{E} = \{E \subseteq \mathbb{R}^{\mathtt{n}} \mid E = \bigcup_{w \in M} (w + \mathbb{L}) \quad \text{with } M \text{ a Borel subset of } \mathbb{M}\}$$

(see Figure 7) and set $P(E)$ equal to $P_{\mathbb{M}}(M)$. A linear $\mathtt{n}$-

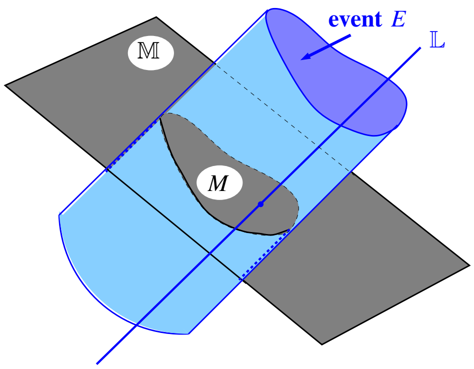

Fig. 7. **Events for a linear system**

dimensional stochastic system is thus parameterized by its fiber $\mathbb{L}$ and a Borel probability on $\mathbb{M}$.

V. Interconnection of stochastic systems

One of the central aspects of systems thinking is the possibility of combining subsystems. This feature allows to set up a model of a complex system from models of simpler subsystems. In [2] we have discussed such 'tearing, zooming, and linking' modeling procedures for deterministic systems.

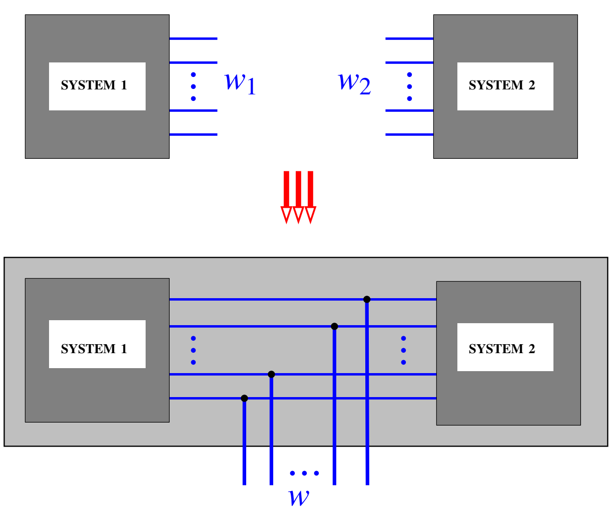

Fig. 8. **Interconnection of systems**

In this section we discuss interconnection of stochastic systems. We start by considering the situation shown in Figure 8 with the assumptions that the two interconnected systems are stochastically independent before interconnection. Note that interconnection comes down to imposing $w = w_1 = w_2$, hence imposing*two distinct* probabilistic laws on the same set of variables. Is it possible to define one law which respects both laws? Clearly we cannot simply state $P(E) = P_1(E) = P_2(E)$ if E is an event that belongs to both interconnected system and if $P_1(E) \neq P_2(E)$. We call the required regularity condition 'complementarity'.

Definition 3: The stochastic systems $\Sigma_1 = (\mathbb{W}, \mathscr{E}_1, P_1)$ and $\Sigma_2 = (\mathbb{W}, \mathscr{E}_2, P_2)$ are said to be *complementary* if for all $E_1, E_1' \in \mathscr{E}_1$ and $E_2, E_2' \in \mathscr{E}_2$ there holds

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![P_1(E_1)P_2(E_2) = P_1(E_1')P_2(E_2')]\!].$$

■

Two σ-algebras $\mathscr{E}_1$ and $\mathscr{E}_2$ on a set $\mathbb{W}$ are said to be *complementary* if for all nonempty sets $E_1, E_1' \in \mathscr{E}_1, E_2, E_2' \in \mathscr{E}_2$ there holds

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![E_1 = E_1' \text{ and } E_2 = E_2']\!].$$

In words, system complementarity requires that the intersection of two evens from each of the σ-algebras determines the product of the probabilities of the events uniquely, while complementarity of the σ-algebras requires that the intersection of two sets from each of the σ-algebras determines the intersecting sets uniquely.

Complementarity of two stochastic systems is implied by the complementarity of the associated σ-algebras. On the other hand, it is easy to construct examples that show that complementarity of two stochastic systems does not imply complementarity of the associated σ-algebras. The problem is that the stochastic systems may have too many zero probability events. Complementarity of the event σ-algebras is a more primitive condition that is convenient for proving complementarity of stochastic systems.

Definition 4: Let $\Sigma_1 = (\mathbb{W}, \mathscr{E}_1, P_1)$ and $\Sigma_2 = (\mathbb{W}, \mathscr{E}_2, P_2)$ be stochastic systems and assume that they are complementary. Then the *interconnection* of Σ_1 and Σ_2, assumed stochastically independent, denoted by $\Sigma_1 \wedge \Sigma_2$, is defined as the stochastic system

$$\Sigma_1 \wedge \Sigma_2 := (\mathbb{W}, \mathscr{E}, P)$$

with $\mathscr{E}$ the σ-algebra generated by $\mathscr{E}_1 \cup \mathscr{E}_2$, and with the probability P defined through rectangles $\{E_1 \cap E_2 \mid E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2\}$ by

$$P(E_1 \cap E_2) := P_1(E_1)P_2(E_2)$$

for $E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2$. ■

The definition of the probability P for rectangles uses complementarity in an essential way. Note that $\mathscr{E}$ is the σ-algebra generated by the rectangles. It is readily seen that the class of subsets of $\mathbb{W}$ that consist of the union of a finite number of disjoint rectangles forms an algebra of sets, that is, a class of subsets of $\mathbb{W}$ that is closed under taking the complement, under intersection, and under union. The probability of rectangles defines the probability on the subsets of $\mathbb{W}$ that consist of a union of a finite number of disjoint rectangles. By the Hahn-Kolmogorov extension theorem, this leads to a unique probability measure on $\mathscr{E}$. This construction of the probability measure is completely analogous to the construction of a product measure in measure theory.

Note that $\mathscr{E}_1$ and $\mathscr{E}_2$ are both sub-σ-algebras of $\mathscr{E}$. Moreover, for $E_1 \in \mathscr{E}_1$ and $E_2 \in \mathscr{E}_2$, $P(E_1 \cap E_2) = P_1(E_1)P_2(E_2) = P_1(E_1 \cap \mathbb{W})P_2(\mathbb{W} \cap E_2) = P(E_1)P(E_2)$. Hence $\mathscr{E}_1$ and $\mathscr{E}_2$ are stochastically independent sub-σ-algebras of $\mathscr{E}$. This expresses that Σ_1 and Σ_2 model phenomena that are stochastically independent.

We illustrate interconnection by our two examples.

Example 1: The noisy resistor. Consider the interconnection of the noisy resistor and a voltage source with an internal resistance and thermal noise (see Figure 2). System 1 corresponds to the noisy resistor and is described by equation $V = RI + \varepsilon$. System 2 correspond to the voltage source, and is described by equation $V = V_0 - R'I + \varepsilon'$. An associated rectangular event is shown in Figure 9. It is easily seen

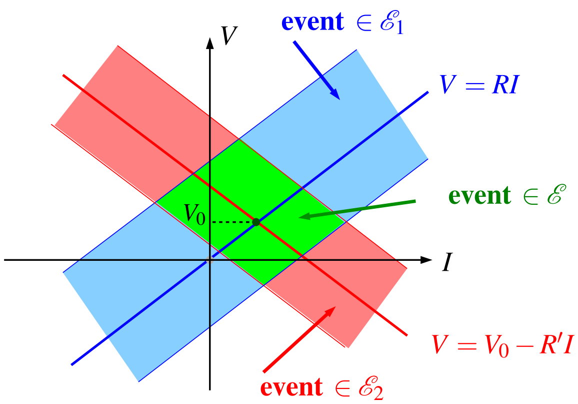

Fig. 9. **Events for the interconnected circuits**

that the corresponding σ-algebras are complementary if and only if $R + R' \neq 0$. The σ-algebra corresponding to the interconnection then becomes the Borel σ-algebra on $\mathbb{R}^2$, and $\left[\begin{smallmatrix} V \\ I \end{smallmatrix}\right]$ in the interconnected circuit is a classical 2-dimensional random vector. ∎

Example 2: Price/demand/supply. We start with system $\Sigma_1 = ([0,\infty)^2, \mathscr{E}_1, P_1)$ that models the price/demand, and a system $\Sigma_2 = ([0,\infty)^2, \mathscr{E}_2, P_2)$ that models the price/supply. The elements of $\mathscr{E}_1$ and $\mathscr{E}_2$ are those to which a probability is assigned (see the discussion of this example in Section III). Call the variables in Σ_1, $\left[\begin{smallmatrix} p_1 \\ d \end{smallmatrix}\right]$ and those in Σ_2, $\left[\begin{smallmatrix} p_2 \\ s \end{smallmatrix}\right]$. Interconnection of Σ_1 and Σ_2 means $p_1 = p_2 = p$ (expressing that the prices pertain to the same good), and $d = s$ (expressing the equilibrium condition demand = supply). Under reasonable conditions (related, for example, to the cardinality, shape, and monotonity of the price/demand and price/supply events) the associated σ-algebras $\mathscr{E}_1$ and $\mathscr{E}_2$ are complementary, and the interconnection σ-algebra consists of the Borel subsets of $[0,\infty)^2$. A rectangular event for the interconnected stochastic system is shown in Figure 10.

Note that for this example, independence of Σ_1 and Σ_2 is acceptable, since the probabilities in Σ_1 depend on things

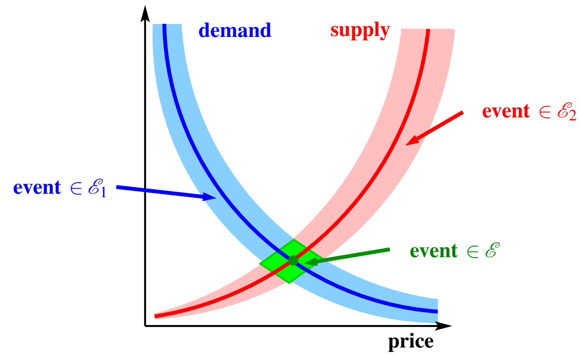

Fig. 10. **Price/demand/supply event**

like consumer preferences, while the probabilities in Σ_2 depend on things like the production technology. It is not unreasonable to assume these to be independent phenomena. ∎

VI. Open versus closed systems

Definition 4 shows that it is possible to impose two distinct probabilistic laws on an outcome space if the stochastic systems are complementary.

Assume that $\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{B}(\mathbb{R}^{\mathtt{n}}), P_1)$ is a classical random vector and that $\Sigma_2 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_2, P_2)$ is a stochastic system with $\mathscr{E}_2 \subseteq \mathscr{B}(\mathbb{R}^{\mathtt{n}})$. Then the σ-algebras associated with Σ_1 and Σ_2 can only be complementary provided $\mathscr{E}_2$ is trivial, that is, $\mathscr{E}_2 = \{\emptyset, \mathbb{R}^{\mathtt{n}}\}$. More generally, if the stochastic systems Σ_1 and Σ_2 are complementary then for all $E \in \mathscr{E}_2$, we have $P_1(E) = P_1(E)P_2(E) = P_2(E)$. Therefore the following zero-one law must hold:

$$[\![E \in \mathscr{E}_2]\!] \Rightarrow [\![P_1(E) = P_2(E) = 0 \text{ or } 1]\!].$$

This is a very restrictive condition on Σ_2. For example, if $\texttt{support}(P_1) = \mathbb{R}^{\mathtt{n}}$, then $\mathscr{E}_2$ cannot contain sets E such that both E and $E^{\text{complement}}$ have a non-empty interior.

We conclude that basically *classical random vectors are models of closed systems*. These systems cannot be interconnected with other systems. Open systems require a coarse σ-algebra. This shows a serious limitation of the classical stochastic framework, since interconnection is one of the basic tenets of model building.

VII. Functions of stochastic systems

In this section we discuss functions of a random system. Consider the equation

$$f(w) = w' \tag{2}$$

with w governed by the stochastic system $(\mathbb{W}, \mathscr{E}, P)$ and f a map from $\mathbb{W}$ into $\mathbb{W}'$. We want to construct the stochastic system $(\mathbb{W}', \mathscr{E}', P')$ that governs the outcomes of the variables $w' \in \mathbb{W}'$. A special case of (2) is a projection $(w_1, w_2) \mapsto w_1$, which we have referred in Section V as ‘elimination’.

In classical probability theory, with for example $\mathbb{W} = \mathbb{R}^{\mathtt{n}}$ and $\mathbb{W}' = \mathbb{R}^{\mathtt{n}'}$, the assumption is usually made that the

σ-algebras $\mathscr{E}$ and $\mathscr{E}'$ are given, for example as Borel σ-algebras, and that the map f is measurable, for example continuous, leading to a definition of P'. In this case the events $\mathscr{E}$ and $\mathscr{E}'$ are obtained from the topological structure of the outcome spaces $\mathbb{W}$ and $\mathbb{W}'$ and therefore the construction of $\mathscr{E}$ and $\mathscr{E}'$ does not involve the probabilistic laws. However, the main theme of the present article is that the events are an essential part of a stochastic model and must therefore be constructed in accordance with the sets whose probability the model wishes to assign. The question therefore emerges how to choose $\mathscr{E}'$ from $\mathscr{E}$ so that as many subsets of $\mathbb{W}'$ as possible are events with a well-defined probability and such that $f:(\mathbb{W},\mathscr{E})\to(\mathbb{W}',\mathscr{E}')$ is measurable.

We start with some facts about σ-algebras and pullbacks. Let $f:\mathbb{W}\to\mathbb{W}'$. Denote by f^{-1} the set theoretic inverse of f, that is, for $E'\subseteq\mathbb{W}'$, define $f^{-1}(E'):=\{w\in\mathbb{W}\mid f(w)\in E'\}$. The pullback f^{-1} satisfies $f^{-1}(\mathbb{W}')=\mathbb{W}$, $f^{-1}\left(E'^{\text{complement}}\right)=f^{-1}\left(E'\right)^{\text{complement}}$, and $f^{-1}\left(\bigcup_{k\in\mathbb{N}}E'\right)=\bigcup_{k\in\mathbb{N}}f^{-1}\left(E'_k\right)$. These relations show that f^{-1} takes σ-algebras into σ-algebras, in both directions. More concretely, if $\mathscr{E}'$ is a σ-algebra of subsets of $\mathbb{W}'$, then the class of subsets $\mathscr{E}$ of $\mathbb{W}$ defined by

$$[\![E\in\mathscr{E}]\!]:\Leftrightarrow[\![E=f^{-1}(E')\text{ for some }E'\in\mathscr{E}']\!]\qquad(3)$$

is a σ-algebra of subsets of $\mathbb{W}$. Conversely, if $\mathscr{E}$ is a σ-algebra of subsets of $\mathbb{W}$, then the class of subsets $\mathscr{E}'$ of $\mathbb{W}'$ defined by

$$[\![E'\in\mathscr{E}']\!]:\Leftrightarrow[\![f^{-1}(E')\in\mathscr{E}]\!]\qquad(4)$$

is a σ-algebra of subsets of $\mathbb{W}'$.

Let $(\mathbb{W},\mathscr{E},P)$ be a stochastic system and $f:\mathbb{W}\to\mathbb{W}'$. Define $\mathscr{E}'$ by (4). Then $f:(\mathbb{W},\mathscr{E})\to(\mathbb{W}',\mathscr{E}')$ is measurable, leading to to the probability

$$P'(E'):=P\left(f^{-1}(E')\right)\text{ for }E'\in\mathscr{E}'.$$

$(\mathbb{W}',\mathscr{E}',P')$ is the stochastic system induced by (2).

The construction of $\mathscr{E}'$ defined by (4) leads to the largest class of subsets of $\mathbb{W}'$ for which the probability can be defined from the probability of events in $\mathscr{E}$. Note in particular that not all subsets of the form $f(E)$ for $E\in\mathscr{E}$ have a well-defined probability.

For the noisy resistor with $R\neq 0$, the maps $\left[\begin{smallmatrix}V\\I\end{smallmatrix}\right]\mapsto V$ and $\left[\begin{smallmatrix}V\\I\end{smallmatrix}\right]\mapsto I$ both generate the trivial stochastic system with events $\{\emptyset,\mathbb{R}\}$. In particular, V and I are therefore not classical random variables. The only linear functional that generates a non-trivial stochastic system is the map $\left[\begin{smallmatrix}V\\I\end{smallmatrix}\right]\mapsto V-RI$ which generates a classical gaussian random variable.

We end with some general comments regarding stochastic modeling. A common way in which probability enters into a system is that some of the variables are modeled as random and influence other related variables, and the aim is to describe the stochastic behavior of these related variables. As a typical example think of modeling the terminal current/voltage behavior of an electrical circuit that contains stochastic sources.

We explained how to construct the stochastic laws governing w' from the stochastic laws of w when w and w' are related by (2). When w and w' are related by an equation like $f(w')=w$, then the construction of (3) shows how to define the w'-events from the w-events.

The definition of the w'-events from the w-events is more involved when w and w' are related by an implicit equation like $f(w,w')=0$. An example of a system governed by an equation of this sort is $y=f(u,\varepsilon),w'=(u,y)$, with ε random playing the role of w in the equation $f(w,w')=0$. When ε a classical random vector $y=f(u,\varepsilon)$ can be dealt with by considering u as an input 'parameter' which together with ε generates the 'output' y. It is possible approach this situation by viewing u as random, and that u together with ε generates the random y. For example, for the noisy resistor, one could assume I is a random variables which together with ε generates the random variable V through (1). There are several drawbacks of dealing with the noisy resistor in this way, the main ones being that it does not put I and V a priori on equal footing, but, more importantly, that the physics simply does not specify a random distribution for I or for V. From the physical point of view our way of dealing with $\left[\begin{smallmatrix}V\\I\end{smallmatrix}\right]$ in terms of a coarse σ-algebra appears more satisfying conceptually.

Other ways of specifying random systems is by giving the 'conditional' probability of the output 'parametrized' by the input variable. Such situations occur frequently in engineering applications, for example as models for noisy communication channels. The probabilistic structure of the variables w' defined by $f(w,w')=0$, in particular or (u,y) defined by $y=f(u,\varepsilon)$, and its relation with the concepts put forward in Definitions 1, 2, and 3 form a topic of ongoing research.

VIII. Conclusion

The mathematical specification of a stochastic system involves the events on an equal footing as the probability measure. The need to have not all Borel sets as events is essential even for elementary applications. Interconnection of stochastic systems requires suitable properties of the event space, as complementarity of the stochastic systems or the associated σ-algebras.

Ackowledgement. I would like to thank Tzvetan Ivanov from the UC Louvain for some very enlightening discussions.

The SISTA research program of the K.U. Leuven is supported by the KUL Research Council under projects: GOA AMBioRICS, CoE EF/05/006 Optimization in Engineering (OPTEC), IOF-SCORES4CHEM; by the Flemish Government: FWO: projects G.0452.04 (new quantum algorithms), G.0499.04 (Statistics), G.0211.05 (Nonlinear), G.0226.06 (co-operative systems and optimization), G.0321.06 (Tensors), G.0302.07 (SVM/Kernel), G.0320.08 (convex MPC), G.0558.08 (Robust MHE), G.0557.08 (Glycemia2), G.0588.09 (Brain-machine) research communities (ICCoS, ANMMM, MLDM); G.0377.09 (Mechatronics MPC) and by IWT: McKnow-E, Eureka-Flite+, SBO LeCoPro, SBO Climaqs; by the Belgian Federal Science Policy Office: IUAP P6/04 (DYSCO, Dynamical systems, control and optimization, 2007-2011) ; by the EU: ERNSI; FP7-HD-MPC (INFSO-ICT-223854); and by several contract research projects.

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